

## $\epsilon'/\epsilon$ in the Standard Model and Beyond: 2021

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The present status of the ratio  $\epsilon'/\epsilon$  in the Standard Model (SM) is summarized. In particular we stress the differences between three approaches that attempt to calculate  $\epsilon'/\epsilon$  within the SM: Lattice QCD, Dual QCD and Chiral Perturbation Theory. As presently there is still a significant room left for new physics (NP) contributions, we summarize the present status of the technology that has been recently developed with the goal to analyze  $\epsilon'/\epsilon$  and its correlations with other observables within the Weak Effective Theory (WET) and the Standard Model Effective Field Theory (SMEFT). We also make a few comments on the interpretation of the main dynamics behind the  $\Delta I = 1/2$  rule in  $K \rightarrow \pi\pi$  decays.

*CKM Workshop 2021, 22-26 November 2021, Melbourne, Australia*

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## 1. Overture

The ratio  $\varepsilon'/\varepsilon$  measuring the size of the direct CP violation in  $K_L \rightarrow \pi\pi$  decays ( $\varepsilon'$ ) relative to the indirect one described by  $\varepsilon$  is very sensitive to new sources of CP violation. As such it played a prominent role in particle physics already for 45 years [1].

Due to the smallness of  $\varepsilon'/\varepsilon$  its measurement required heroic efforts in the 1980s and the 1990s on both sides of the Atlantic with final results presented by NA48 and KTeV collaborations at the beginning of this millennium. On the other hand, even 45 years after the first calculation of  $\varepsilon'/\varepsilon$  we do not know to which degree the Standard Model (SM) agrees with this data and how large is the room left for new physics (NP) contributions to this ratio. This is due to significant non-perturbative (hadronic) uncertainties accompanied by partial cancellations between the QCD penguin contributions and electroweak penguin contributions. In addition to the calculation of hadronic matrix elements of the relevant operators including isospin breaking effects and QED corrections, it is crucial to evaluate accurately the Wilson coefficients of the relevant operators. While the significant control over the latter short distance effects has been achieved already in the early 1990s, with several improvements since then, different views on the non-perturbative contributions to  $\varepsilon'/\varepsilon$  have been expressed by different authors over last thirty years. In fact even at the dawn of the 2020s the uncertainty in the room left for NP contributions to  $\varepsilon'/\varepsilon$  is still very significant, which I find to be very exciting.

As in addition to [1], this topic is presented in my recent book [2] and in my recent contribution to Kaon 2019 [3] some overlap with these reviews is unavoidable but I will attempt to reduce it as far as it is possible without losing most important information.

Let us then look first at the effective Hamiltonian relevant for  $K \rightarrow \pi\pi$  decays at the low energy scales. It has the following general structure:

$$\mathcal{H}_{\text{eff}} = \sum_i C_i \mathcal{O}_i^{\text{SM}} + \sum_j C_j^{\text{NP}} \mathcal{O}_j^{\text{NP}}, \quad C_i = C_i^{\text{SM}} + \Delta_i^{\text{NP}}, \quad (1.1)$$

where

- $\mathcal{O}_i^{\text{SM}}$  are local operators present in the SM and  $\mathcal{O}_j^{\text{NP}}$  are new operators having typically new Dirac structures, in particular scalar-scalar and tensor-tensor ones.
- $C_i$  and  $C_j^{\text{NP}}$  are the Wilson coefficients of these operators. NP effects modify not only the Wilson coefficients of SM operators but also generate new operators with non-vanishing  $C_j^{\text{NP}}$ .

The amplitude for the transition  $K \rightarrow \pi\pi$  can now be written as follows

$$\mathcal{A}(K \rightarrow \pi\pi) = \sum_i C_i \langle \pi\pi | \mathcal{O}_i^{\text{SM}} | K \rangle + \sum_j C_j^{\text{NP}} \langle \pi\pi | \mathcal{O}_j^{\text{NP}} | K \rangle. \quad (1.2)$$

The coefficients  $C_i$  and  $C_j^{\text{NP}}$  can be calculated in the renormalization group (RG) improved perturbation theory. The status of these calculations is by now very advanced, as reviewed in 2014 in [4] including also rare meson decays and quark mixing. I plan to update this review this year. The complete NLO corrections to  $K \rightarrow \pi\pi$  have been calculated almost 30 years ago [5–10]. The dominant NNLO QCD corrections to electroweak penguin (EWP) contributions have been presented in [11] and those to QCD penguins (QCDP) in [12–14]. These NNLO corrections

reducing various scale and renormalization scheme dependences will play a significant role when the calculation of hadronic matrix elements will be brought under control. On the whole, the status of present short distance (SD) contributions to  $\epsilon'/\epsilon$  is satisfactory.

The evaluation of the hadronic matrix elements is a different story. In  $K \rightarrow \pi\pi$  decays, we have presently three approaches to our disposal:

- **Lattice QCD (LQCD)**. It is a sophisticated numerical method with very demanding calculations lasting many years, even decades. But eventually in the case of  $K \rightarrow \pi\pi$  decays and  $K^0 - \bar{K}^0$  mixing this method is expected to give the ultimate results for  $\epsilon'/\epsilon$ ,  $\Delta I = 1/2$  rule and  $K^0 - \bar{K}^0$  mixing, both in the SM and beyond it. For  $K \rightarrow \pi\pi$  only results for the SM operators are known and they are far from being satisfactory. The ones for BSM  $K^0 - \bar{K}^0$  matrix elements are already known with respectable precision and interesting results have been obtained for long distance contributions to  $\Delta M_K$  [15, 16].
- **Dual QCD (DQCD)** proposed already in the 1980s [17], significantly improved in the last decade [18] and also very recently as reported below. This approach, which opposed to LQCD is fully analytical, allows to obtain results for  $K \rightarrow \pi\pi$  decays and  $K^0 - \bar{K}^0$  mixing much faster than it is possible with the LQCD, typically within a few months. Therefore several relevant results have been obtained already in the 1980s and confirmed within uncertainties by LQCD in the last decade. While not as accurate as the expected ultimate LQCD calculations, it allowed already to calculate hadronic matrix elements for all BSM operators entering  $K \rightarrow \pi\pi$  decays [19] and  $K^0 - \bar{K}^0$  mixing [20]. Very importantly this approach allows to get the insight into the QCD dynamics at low energy scales which is not possible using a purely numerical method like LQCD. A good example are four hadronic matrix elements of BSM operators entering  $K^0 - \bar{K}^0$  mixing [20]. For a detailed exposition of this approach see [2, 18, 21]. More about it below.
- **Chiral Perturbation Theory (ChPT)** developed since 1978 [22–28] and discussed recently in [29, 30] in the context of  $\epsilon'/\epsilon$ . It is based on global symmetries of QCD with the QCD dynamics parametrized by low-energy constants  $L_i$  that enter the counter terms in meson loop calculations.  $L_i$  can be extracted from the data or calculated by LQCD. Presently this framework as applied to non-leptonic transitions has serious difficulties in matching long distance (LD) and short distance (SD) contributions, a problem admitted by the authors of [30, 31]. For instance the expression for the matrix element of the QCDP operator  $Q_6$  in terms of  $L_5$  is only valid in the large  $N$  limit, that is using factorization. On the other hand the dominant QCD dynamics in Wilson coefficients is given by non-factorizable contributions. This problem is absent in LQCD and DQCD as we will discuss below. Therefore, while the ChPT approach is very suitable for leptonic and semi-leptonic Kaon decays, it can only provide partial information on  $\epsilon'/\epsilon$  and the  $\Delta I = 1/2$  rule in the form of isospin breaking effects and final state interactions (FSI). Yet, in the case of isospin breaking contributions to  $\epsilon'/\epsilon$  the difficulties in the matching in question imply a rather significant error as we will see below.

This writing is arranged as follows. In Section 2 I will briefly describe the DQCD approach. In Section 3 I will summarize my view on the present status of  $\epsilon'/\epsilon$  in the SM. In Section 4 I

will review the recent progress in the evaluation of  $\varepsilon'/\varepsilon$  within the Weak Effective Theory (WET) and the Standard Model Effective Field Theory (SMEFT). In Section 5 several comments on the dynamics behind the  $\Delta I = 1/2$  rule in  $K \rightarrow \pi\pi$  decays will be made. A brief outlook in Section 6 ends this presentation.

## 2. The Dual QCD Approach: A Grand View

This analytic approach to  $K \rightarrow \pi\pi$  decays and  $K^0 - \bar{K}^0$  mixing in [17, 18, 32] is based on the ideas of 't Hooft and Witten [33–36] who studied QCD with a large number  $N$  of colours. In this limit QCD is dual to a theory of weakly interacting mesons with the coupling  $\mathcal{O}(1/N)$  and in particular in the strict large  $N$  limit it becomes a free theory of mesons, simplifying the calculations significantly. With non-interacting mesons the factorization of matrix elements of four-quark operators into matrix elements of quark currents and quark densities, used adhoc in the 1970s and early 1980s, is automatic and can be considered as a property of QCD in this limit [37]. But the factorization cannot be the whole story as the most important QCD effects related to asymptotic freedom are related to non-factorizable contributions generated by exchanges of gluons. In DQCD this role is played by meson loops that represent dominant non-factorizable contributions at the very low energy scales. Calculating these loops with a momentum cut-off  $\Lambda$  one finds then that the factorization in question does not take place at values of  $\mu \geq 1$  GeV at which Wilson coefficients are calculated, but rather at very low momentum transfer between colour-singlet currents or densities.

Thus, even if in the large  $N$  limit the hadronic matrix elements factorize and can easily be calculated, in order to combine them with the Wilson coefficients, loops in the meson theory have to be calculated. In contrast to chiral perturbation theory, in DQCD a physical cut-off  $\Lambda$  is used in the integration over loop momenta. As discussed in detail in [17, 18] this allows to achieve a much better matching with short distance contributions than it is possible in ChPT, which uses dimensional regularization. The cut-off  $\Lambda$  is typically chosen around 0.7 GeV when only pseudoscalar mesons are exchanged in the loops [17] and can be increased up to 0.9 GeV when contributions from lowest-lying vector mesons are taken into account as done in [18]. These calculations are done in a momentum scheme, but as demonstrated in [18], they can be matched to the commonly used naive dimensional regularization (NDR) scheme. Once this is done it is justified to set  $\Lambda \approx \mu$ .

The application of DQCD to weak decays consists in any NP model of the following steps:

**Step 1:** At  $\Lambda_{\text{NP}}$  one integrates out the heavy degrees of freedom and performs the RG evolution including Yukawa couplings and all gauge interactions present in the SM down to the electroweak scale. This evolution involves in addition to SM operators also beyond the SM (BSM) operators. This is the SMEFT.

**Step 2:** At the electroweak scale  $W$ ,  $Z$ , top quark and the Higgs are integrated out and the SMEFT is matched onto the WET with only SM quarks except the top-quark, the photon and the gluons. Subsequently QCD and QED evolution is performed down to scales  $\mathcal{O}(1 \text{ GeV})$ . The presence of many new operators makes this evolution rather involved. Fortunately as we will report below this evolution is known by now at NLO level in QCD.

**Step 3:** Around scales of  $\mathcal{O}(1 \text{ GeV})$  the matching to the theory of mesons is performed and the quark evolution is followed by *the meson evolution* down to the factorization scale.

**Step 4:** At the appropriate hadronic scale between the kaon and pion masses, the matrix elements of all operators are calculated in the large  $N$  limit, that is using factorization of matrix elements into products of currents or densities.

I do not claim that these are all QCD effects responsible for non-leptonic transitions, but these evolutions based entirely on non-factorizable QCD effects, both at short distance and long distance scales, appear to be the main bulk of QCD dynamics responsible for the  $\Delta I = 1/2$  rule,  $\varepsilon'/\varepsilon$  and  $K^0 - \bar{K}^0$  mixing. Past successes of this approach have been reviewed in [2,21,38]. They are related in particular to the non-perturbative parameter  $\hat{B}_K$  in  $K^0 - \bar{K}^0$  mixing and  $\Delta I = 1/2$  rule [18]. In fact DQCD allowed for the first time to identify already in 1986 the dominant mechanism behind this rule [17]. It is simply the quark evolution from short distance scales, possibly involving NP, down to scales of  $\mathcal{O}(1\text{ GeV})$ , followed by meson evolution down to the factorization scale around the kaon and pion masses.

In summary DQCD turns out to be an efficient approximate method for obtaining results for non-leptonic decays, years and even decades, before useful results from numerically sophisticated and demanding lattice calculations could be obtained. This will be particularly clear in Section 4, where we will turn our attention to the results in the WET and the SMEFT which are to date available only within DQCD approach.

### 3. $\varepsilon'/\varepsilon$ in the SM

The situation of  $\varepsilon'/\varepsilon$  as of March 2022 can be summarized as follows.

The experimental world average from NA48 [39] and KTeV [40,41] collaborations reads

$$\boxed{(\varepsilon'/\varepsilon)_{\text{exp}} = (16.6 \pm 2.3) \times 10^{-4}}. \quad (3.1)$$

The most recent result from LQCD, that is from the RBC-UKQCD collaboration [42], reads

$$\boxed{(\varepsilon'/\varepsilon)_{\text{SM}} = (21.7 \pm 8.4) \times 10^{-4}, \quad (\text{RBC-UKQCD} - 2020)}, \quad (3.2)$$

where statistical, parametric and systematic uncertainties have been added in quadrature. The central value is by an order of magnitude larger than the central 2015 value presented by this collaboration but is subject to large systematic uncertainties which dominate the quoted error. It is based on the improved values of the hadronic matrix elements of QCDP, includes the Wilson coefficients at the NLO level but does not include isospin breaking effects, charm contributions and NNLO QCD effects.

The most recent estimate of  $\varepsilon'/\varepsilon$  in the SM from ChPT [30,31] reads

$$\boxed{(\varepsilon'/\varepsilon)_{\text{SM}} = (14 \pm 5) \times 10^{-4}, \quad (\text{ChPT} - 2019)}. \quad (3.3)$$

The large error is related to the problematic matching of LD and SD contributions in this approach which can be traced back to the absence of meson evolution in this approach. Here the isospin breaking corrections are included but as discussed below they are subject to significant uncertainties again related to the problematic matching of LD and SD contributions.

Finally, based on the insight from DQCD obtained in collaboration with Jean-Marc Gérard [43,44] one finds [1]

$$\boxed{(\varepsilon'/\varepsilon)_{\text{SM}} = (5 \pm 2) \cdot 10^{-4}, \quad (\text{DQCD} - 2020)}. \quad (3.4)$$

While the results in (3.2) and (3.3) are fully consistent with the data in (3.1), the DQCD result in (3.4) if confirmed by other groups would one day imply a significant anomaly and NP at work.

Let us then have a look at the phenomenological expression for  $\varepsilon'/\varepsilon$  in the SM in order to understand the origin of these differences. This formula presented in [45] reads

$$\left(\frac{\varepsilon'}{\varepsilon}\right)_{\text{SM}} = \text{Im}\lambda_t \cdot \left[ (1 - \hat{\Omega}_{\text{eff}}) (-2.9 + 15.4 B_6^{(1/2)}(\mu^*)) + 2.0 - 8.0 B_8^{(3/2)}(\mu^*) \right]. \quad (3.5)$$

It includes NLO QCD corrections to the QCD penguin (QCDP) contributions and NNLO contributions to electroweak penguins (EWP). The coefficients in this formula and the parameters  $B_6^{(1/2)}$  and  $B_8^{(3/2)}$ , conventionally normalized to unity at the factorization scale, are scale dependent. Their values for different scales are collected in Table 1 of [45]. Here we will set  $\mu^* = 1 \text{ GeV}$  because at this scale it is most convenient to compare the values for  $B_6^{(1/2)}$  and  $B_8^{(3/2)}$  obtained in the three non-perturbative approaches in question. The four contributions in (3.5) are dominated by the following operators:

- The terms involving the non-perturbative parameters  $B_6^{(1/2)}$  and  $B_8^{(3/2)}$  contain only the contributions from the dominant QCDP operator  $Q_6$  and the dominant EWP operator  $Q_8$ , respectively. There are two main reasons why  $Q_8$  can compete with  $Q_6$  here despite the smallness of the electroweak couplings relative to the QCD one. In the basic formula for  $\varepsilon'/\varepsilon$  its contribution is enhanced relative to the  $Q_6$ 's one by the factor  $\text{Re}A_0/\text{Re}A_2 = 22.4$  with  $A_{0,2}$  being isospin amplitudes. In addition its Wilson coefficient is enhanced for the large top-quark mass which is not the case of the  $Q_6$ 's one [46,47]. The expressions for these two operators and the remaining operators mentioned below are well known and can be found in [1,2].
- The term  $-2.9$  is fully dominated by the QCDP operator  $Q_4$ .
- The term  $+2.0$  is fully dominated by EWP operators  $Q_9$  and  $Q_{10}$ .
- The quantity  $\hat{\Omega}_{\text{eff}}$  represents the isospin breaking corrections and QED corrections beyond EWP contributions. It is not in the ballpark of a few percent as one would naively expect, because in  $\varepsilon'/\varepsilon$  it is again enhanced by the factor  $\text{Re}A_0/\text{Re}A_2 = 22.4$ .
- $\text{Im}\lambda_t$  is the CKM factor that within a few percent is in the ballpark of  $1.45 \cdot 10^{-4}$ .

Now at  $\mu = 1 \text{ GeV}$  the values of  $B_6^{(1/2)}$  and  $B_8^{(3/2)}$  in the three non-perturbative approaches are found to be:

$$B_6^{(1/2)}(1 \text{ GeV}) = 1.49 \pm 0.25, \quad B_8^{(3/2)}(1 \text{ GeV}) = 0.85 \pm 0.05, \quad (\text{RBC-UKQCD} - 2020). \quad (3.6)$$

$$B_6^{(1/2)}(1 \text{ GeV}) = 1.35 \pm 0.20, \quad B_8^{(3/2)}(1 \text{ GeV}) = 0.55 \pm 0.20, \quad (\text{ChPT} - 2019). \quad (3.7)$$

$$B_6^{(1/2)}(1\text{ GeV}) \leq 0.6, \quad B_8^{(3/2)}(1\text{ GeV}) = 0.80 \pm 0.10, \quad (\text{DQCD} - 2015). \quad (3.8)$$

Let us begin with the good news. There is a very good agreement between LQCD and DQCD as far as EWP contribution to  $\varepsilon'/\varepsilon$  is concerned. This implies that this contribution to  $\varepsilon'/\varepsilon$ , that is unaffected by leading isospin breaking corrections, is already known within the SM with acceptable accuracy:

$$\boxed{(\varepsilon'/\varepsilon)_{\text{SM}}^{\text{EWP}} = -(7 \pm 1) \times 10^{-4}, \quad (\text{LQCD and DQCD}).} \quad (3.9)$$

Because both LQCD and DQCD can perform much better in the case of EWPs than in the case of QCDDs I expect that this result will remain with us for the coming years.

On the other hand the value from ChPT  $B_8^{(3/2)} \approx 0.55$  [30] implies on the basis of the last two terms in (3.5) the EWP contribution roughly by a factor of 2 below the result in (3.9). This lower value originates in the suppression caused by FSI which seems to be not supported by the results from the RBC-UKQCD collaboration. Yet, the uncertainty in the ChPT estimate is large. It would be good to clarify this difference in the future.

The case of QCDDs is a different story. Here the LQCD value of  $B_6^{(1/2)}$  overshoots the DQCD one by more than a factor of two and consequently despite the agreement on EWP contribution the result in (3.2) from RBC-UKQCD differs by roughly a factor of four from the one of DQCD in (3.4).

This difference will surely decrease when the RBC-UKQCD collaboration will include isospin breaking and QED corrections. As far as these corrections are concerned LQCD, ChPT, DQCD use in their estimates respectively

$$\hat{\Omega}_{\text{eff}} = 0, \quad \hat{\Omega}_{\text{eff}}^{(8)} = (17 \pm 9) 10^{-2}, \quad \hat{\Omega}_{\text{eff}}^{(9)} = (29 \pm 7) 10^{-2}, \quad (3.10)$$

where the index “(9)” indicates that the full nonet of pseudoscalars has been taken into account in the case of DQCD. This means that also  $\eta - \eta'$  mixing has been taken explicitly into account [48]. In the octet scheme, necessarily used in ChPT, this mixing is buried in a poorly known low-energy constant  $L_7$  and I expect that including the effect of this mixing in  $\varepsilon'/\varepsilon$  will remain a challenge for ChPT for some time. Yet, it is evident from the analysis in [48] that this mixing enhances significantly  $\hat{\Omega}_{\text{eff}}$ , the fact known already for 35 years [49, 50].

There is still another reason why the DQCD result in (3.4) is much lower than the ones in LQCD and ChPT. We include NNLO QCD corrections to both QCDD and EWP contributions that together provide a downward shift of  $\varepsilon'/\varepsilon$  in the ballpark of  $3 \times 10^{-4}$ . While ChPT experts criticized us for ignoring FSI, this is really a fake news. We devoted a full paper to FSI [44]. While we agreed that in DQCD precise estimate of FSI is difficult, we gave arguments why the effect of the enhancement of  $B_6^{(1/2)}$  by FSI is likely to be smaller than its suppression by the meson evolution. In fact FSI are included in (3.4) with  $B_6^{(1/2)} \approx 1.0$  corresponding to the upper limit when subleading FSI would fully cancel the suppression by the leading meson evolution.

In Table 1 I summarize the main SM dynamics which in my view is responsible for the strong suppression of  $\varepsilon'/\varepsilon$  below the experimental value. The question marks in the case of ChPT mean that presently it is not clear how well the low energy constants  $L_5$  and  $L_7$  will describe meson evolution and  $\eta - \eta'$  mixing, respectively. To be fair one should add that while RBC-UKQCD and

	DQCD	RBC-UKQCD	ChPT
Large $m_t$	*	*	*
Meson evolution	*	*	$L_5(?)$
QCDP at NNLO	*	—	—
EWP at NNLO	*	—	—
IB (Octet)	*	—	*
$\eta - \eta'$ mixing	*	—	$L_7(?)$

**Table 1:** Comparison of various suppression mechanism of  $\varepsilon'/\varepsilon$  in the SM taken (\*) or not taken (—) into account in a given approach. See text for more details.

ChPT experts claim to have a satisfactory treatment of FSI for penguin operators, in the case of DQCD it can only be argued at present that the subleading FSI can at best cancel the suppression of QCDP contribution by the leading meson evolution [44].

Table 1 summarizes my view of the controversy between the three approaches. In other words my message for unbiased non-experts to take home is the following

- RBC-UKQCD collaboration and ChPT experts do not claim that there is no NP in  $\varepsilon'/\varepsilon$ . But as of March 2022 their methods are not sufficiently powerful to see the anomaly in  $\varepsilon'/\varepsilon$ . I expect that RBC-UKQCD collaboration will see this anomaly after they reached their goals for coming years summarized recently in [51].
- DQCD approach, even if approximate, can much faster see the underlying dynamics responsible for possible anomaly in  $\varepsilon'/\varepsilon$  because it is an analytical approach and important QCD dynamics is not hidden in the numerical values of hadronic matrix elements evaluated by LQCD nor hidden in low energy constants  $L_i$  of ChPT.

I am truly delighted that at least one group strongly believes in NP in  $\varepsilon'/\varepsilon$  at work. How boring would be a situation of  $\varepsilon'/\varepsilon$  if also DQCD found  $\varepsilon'/\varepsilon$  the ballpark of the values presented by RBC-UKQCD and ChPT. Simultaneously I understand that both groups are, in contrast to me, not motivated to study NP in  $\varepsilon'/\varepsilon$ . I wish them therefore luck in improving their calculations and will now go beyond the SM.

#### 4. $\varepsilon'/\varepsilon$ beyond the SM

A new era for  $\varepsilon'/\varepsilon$  began in 2015, when the RBC-UKQCD collaboration [52, 53] presented their first results for  $\varepsilon'/\varepsilon$  with the central value by roughly a factor of 15 below their 2020 value in (3.2). This inspired many authors to search for NP which would explain this anomaly. I will not review these analyzes again as I have done it already in [1–3]. But I would like to stress that all these analyses with a few exceptions involved only the SM operators, that is the first term in (1.2) in which NP modifies the Wilson coefficients of the SM operators, in particular the one of  $Q_8$ .

Here I would like to confine first the discussion to the second term in (1.2) and subsequently to  $\varepsilon'/\varepsilon$  in the framework of the WET and the SMEFT which include all operators contributing to  $\varepsilon'/\varepsilon$  in these effective theories.

In fact since 2018 a significant progress towards the general search for NP in  $\varepsilon'/\varepsilon$  with the help of DQCD and more generally has been made. I will just list all papers written by us in this context and describe in more details only the most recent ones. Here we go.

- The first to date results for the  $K \rightarrow \pi\pi$  matrix elements of the chromo-magnetic dipole operators [54] that are compatible with the LQCD results for  $K \rightarrow \pi$  matrix elements of these operators obtained earlier in [55].
- The first to date calculation of  $K \rightarrow \pi\pi$  matrix elements of *all* four-quark BSM operators, including scalar and tensor operators, by DQCD [19].
- The derivation of a master formula for  $\varepsilon'/\varepsilon$  [56], which can be applied to any theory beyond the SM in which the Wilson coefficients of all contributing operators have been calculated at the electroweak scale. The relevant hadronic matrix elements of BSM operators used in this formula are from the DQCD, as lattice QCD did not calculate them yet, and the SM ones from LQCD.
- This allowed to perform the first to date model-independent anatomy of the ratio  $\varepsilon'/\varepsilon$  in the context of the  $\Delta S = 1$  effective theory with operators invariant under QCD and QED and in the context of the SMEFT with the operators invariant under the full SM gauge group [57].
- Finally the insight from DQCD [20] into the values of BSM  $K^0 - \bar{K}^0$  elements obtained by LQCD made sure that the meson evolution is hidden in lattice calculations.

The main messages from these papers are as follows:

- The inclusion of the meson evolution in the phenomenology of any non-leptonic transition like  $K^0 - \bar{K}^0$  mixing and  $K \rightarrow \pi\pi$  decays with  $\varepsilon'/\varepsilon$  and the  $\Delta I = 1/2$  rule is mandatory!
- Meson evolution is hidden in LQCD results, but among analytic approaches only DQCD takes this important QCD dynamics into account. Whether meson evolution is present in the low energy constant  $L_5$  of ChPT is an interesting question, still to be answered.
- Most importantly, the meson evolution turns out to have the pattern of operator mixing, both for SM and BSM operators, to agree with the one found perturbatively at short distance scales. This allows for a satisfactory, even if approximate, matching between Wilson coefficients and hadronic matrix elements.

The BSM part of  $\varepsilon'/\varepsilon$  in our 2018 papers was performed at LO in QCD RG quark evolution. In the more recent papers we made progress by including NLO QCD corrections and in this context we derived master formulae not only for  $\varepsilon'/\varepsilon$  and  $K \rightarrow \pi\pi$  but generally for non-leptonic  $\Delta F = 1$  and  $\Delta F = 2$  transitions. Let me briefly summarize these papers.

In [58] we reconsidered the complete set of four-quark operators in the WET for non-leptonic  $\Delta F = 1$  decays that govern  $s \rightarrow d$  and  $b \rightarrow d, s$  transitions in the SM and beyond, at NLO in QCD. In particular we discussed the issue of transformations between operator bases beyond leading order to facilitate the matching to high-energy completions or the SMEFT at the electroweak scale. As a first step towards a SMEFT NLO analysis of  $K \rightarrow \pi\pi$  and non-leptonic  $B$ -meson decays, we calculated

the relevant WET Wilson coefficients including two-loop contributions to their renormalization group running, and expressed them in terms of the Wilson coefficients in a particular operator basis, the so-called JMS basis [59] for which the one-loop matching to SMEFT is already known [60].

In [61] we have presented for the first time the NLO master formula for the BSM part of  $\epsilon'/\epsilon$  expressed in terms of the Wilson coefficients of all contributing operators of WET evaluated at the electroweak scale. To this end we use again JMS basis and the results from [58]. The relevant hadronic matrix elements of BSM operators at the electroweak scale are taken from Dual QCD approach and the SM ones from lattice QCD. It includes the renormalization group evolution and quark-flavour threshold effects at NLO in QCD from hadronic scales, at which these matrix elements have been calculated, to the electroweak scale.

This updated master formula for the BSM part of  $\epsilon'/\epsilon$  reads reads [61]

$$\left(\frac{\epsilon'}{\epsilon}\right)_{\text{BSM}} = \sum_b P_b(\mu_{\text{ew}}) \text{Im}[C_b(\mu_{\text{ew}}) - C'_b(\mu_{\text{ew}})] \times (1 \text{ TeV})^2, \quad (4.1)$$

with the sum over  $b$  running over the Wilson coefficients  $C_b$  of all operators in the JMS basis and their chirality-flipped counterparts denoted by  $C'_b$ . The relative minus sign accounts for the fact that their  $K \rightarrow \pi\pi$  matrix elements differ by a sign. Among the contributing operators are also operators present already in the SM, but their WCs in (4.1) are meant to include only BSM contributions. The numerical values of  $P_b(\mu_{\text{ew}})$  are collected in [61]. Calculating  $C_b(\mu_{\text{ew}})$  and  $C'_b(\mu_{\text{ew}})$  in a given BSM scenario this master formula allows in no time to calculate the BSM contribution to  $\epsilon'/\epsilon$ .

Another master formula, this time for  $\Delta F = 2$  transitions in the SMEFT has been presented in [62]. We illustrated it on a number of simplified models containing colourless heavy gauge bosons ( $Z'$ ) and scalars and models with coloured heavy gauge bosons ( $G'_d$ ) and scalars. Also the cases of vector-like quarks and leptoquarks are briefly considered. See brief summary of this work in [63].

The next important step is the inclusion of NLO QCD corrections to the RG evolution in the SMEFT. The important benefit of these new contributions is that they allow to remove renormalization scheme dependences present both in the one-loop matchings between the WET and SMEFT and also between SMEFT and a chosen UV completion. For  $\Delta F = 2$  transitions this step has been already made very recently in [64].

## 5. Comments on the $\Delta I = 1/2$ Rule

I have described the history of this rule, the ratio  $\text{Re}A_0/\text{Re}A_2 = 22.4$ , in Section 7 of [1] and in particular in Section 7.2.3 of my recent book [2]. From this it is evident that the credit for the identification of the basic dynamics behind this rule should go to the authors of [17] who demonstrated that the current-current operators and not QCDP operators are dominantly responsible for this rule. This has been confirmed more than 30 years later by the RBC-UKQCD collaboration [42, 65] basically ignoring our work.

Leaving this issue here aside what is really puzzling for me is there interpretation of the dynamics behind this rule as they state from first principles. According to them this rule originates in the relation between two leading contractions

$$\boxed{2} = -K \boxed{1} \quad (5.1)$$

induced by QCD. The factor  $K$  depends on the scale at which these contractions are evaluated. In 2012 they were evaluated at  $\mu = 2.15 \text{ GeV}$ , in 2020 at  $\mu = 4.0 \text{ GeV}$ .

In order to make my point let us follow the exercise performed by us in Section 9 of [18] and extract the values of these contractions from the experimental values for  $\text{Re}A_0$  and  $\text{Re}A_2$  [66]:

$$\text{Re}A_0 = 27.04(1) \times 10^{-8} \text{ GeV}, \quad \text{Re}A_2 = 1.210(2) \times 10^{-8} \text{ GeV}. \quad (5.2)$$

Adjusting the expressions for the isospin amplitudes in [65] to our normalization we have

$$(\text{Re}A_0) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left( \frac{\sqrt{2}}{3} \right) \left[ z_1 (2 \boxed{2} - \boxed{1}) + z_2 (2 \boxed{1} - \boxed{2}) \right], \quad (5.3)$$

$$(\text{Re}A_2) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left( \frac{2}{3} \right) (z_1 + z_2) (\boxed{1} + \boxed{2}), \quad (5.4)$$

where  $z_{1,2}$  are the real parts of the Wilson coefficients of the current-current operators  $Q_{1,2}$ . With the  $\overline{\text{MS}}$  values  $z_1 = -0.199$  and  $z_2 = 1.090$  at  $\mu = 4.0 \text{ GeV}$ , the data in (5.2) can be reproduced with

$$\boxed{1} = 0.0865 \text{ GeV}^3, \quad \boxed{2} = -0.0752 \text{ GeV}^3, \quad \boxed{2} = -0.87 \boxed{1}. \quad (5.5)$$

Thus at  $\mu = 4.0 \text{ GeV}$  we need  $K = 0.87$  to reproduce the data. In 2012 RBC-UKQCD found  $K \approx 0.7$  at  $\mu = 2.15 \text{ GeV}$  which was not sufficient to reproduce the data but in 2020 their value of  $K$  is in the ballpark of 0.85 with a significant error. Certainly it is an important result.

So far so good. Now using this relation in (5.4) the RBC-UKQCD finds strong suppression of  $\text{Re}A_2$  and concludes that this suppression is the main origin of the  $\Delta I = 1/2$  rule.

This is a surprising interpretation because until the 2012 RBC-UKQCD paper [65], the particle community thought that it is the strong enhancement of  $\text{Re}A_0$  and some moderate suppression of  $\text{Re}A_2$  responsible for this rule.

This very original interpretation from first principles, also repeated at this workshop, differs totally from the one of DQCD as presented in our papers, in particular in [17, 18] but also recently in [1, 2]. I do not want to repeat this interpretation again because it is easy to demonstrate in a few lines without using super computers or even calculating meson loops that the RBC-UKQCD interpretation is incorrect.

The point is that in order to find out the impact of the QCD dynamics on these two amplitudes, that is responsible for this rule, one has to ask first the question what would be the values of these two amplitudes and of the corresponding contractions in the absence of QCD. Of course we cannot fully switch off QCD because we want to have quarks confined in mesons but we can switch off non-factorizable contributions both in SD and LD by going to the strict large  $N$  limit. In this limit only the operator  $Q_2$  contributes, its coefficient is  $z_2 = 1$  and its matrix element can be calculated precisely using factorization valid in QCD in this limit. One finds then<sup>1</sup>:

$$\text{Re}A_0 = 3.59 \times 10^{-8} \text{ GeV}, \quad \text{Re}A_2 = 2.54 \times 10^{-8} \text{ GeV}, \quad \frac{\text{Re}A_0}{\text{Re}A_2} = \sqrt{2}, \quad (5.6)$$

in plain disagreement with the experimental value of 22.4. It should be emphasized that the explanation of the missing enhancement factor of 15.8 through some dynamics must simultaneously

<sup>1</sup>see [2, 18] for a detailed presentation.

give the correct values for the experimental values of  $\text{Re}A_0$  and  $\text{Re}A_2$  in (5.2). This means that these dynamics should suppress  $\text{Re}A_2$  by a factor of 2.1, not more, and enhance  $\text{Re}A_0$  by a factor of 7.5. This tells us that while the suppression of  $\text{Re}A_2$  is an important ingredient in the  $\Delta I = 1/2$  rule, it is *not* the main origin of this rule. It is the enhancement of  $\text{Re}A_0$ , as already emphasized in [67], even if, in contrast to this paper, as demonstrated in [17], the current-current operators are responsible dominantly for this rule and not QCD penguins.

One can also do this exercise using contractions to find in the factorization limit [18]

$$\boxed{1} = 0.0211 \text{ GeV}^3, \quad \boxed{2} = 0, \quad (\mu \approx 0) \quad (5.7)$$

which drastically differ from the values of contractions in (5.5). In other words the main impact of QCD is to change the values of contractions from (5.7) to the ones in the ballpark of the ones in (5.5) and this change have significantly larger effect on  $\text{Re}A_0$  than on  $\text{Re}A_2$ .

The simple discussion above shows that the RBC-UKQCD interpretation of the dynamics behind  $\Delta I = 1/2$  rule cannot be correct. But working numerically at 4 GeV and being not able to switch-off non-factorizable QCD interactions in LQCD one simply cannot see properly what is really going on. Yet, their numerical result is very important and one should congratulate them for it. In particular because their central value for the ratio in question,  $19.9 \pm 5.0$ , is closer to the data than present one from DQCD,  $16 \pm 2$  [18]. In the latter approach an additional enhancement could come from FSI which this time are not compensated by meson evolution as was the case of  $\varepsilon'/\varepsilon$ . It could also come from NP [68].

Let me then repeat. The dominant dynamics behind the  $\Delta I = 1/2$  rule for  $K \rightarrow \pi\pi$  isospin amplitudes is our beloved QCD as also confirmed by RBC-UKQCD. But the correct physical interpretation differs drastically from the one provided by them. It is simply the *quark evolution* from  $M_W$  down to scale  $\mathcal{O}(1 \text{ GeV})$  as analysed first by Altarelli and Maiani [69] and Gaillard and Lee [70], followed by the *meson evolution* down to very low scales at which QCD becomes a theory of weakly interacting mesons and free theory of mesons in the strict large  $N$  limit [33–36]. This non-perturbative evolution within the Dual QCD approach dominates by far the enhancement of  $\text{Re}A_0$  over  $\text{Re}A_2$  as demonstrated by Bardeen, Gérard and myself in [17, 18]. The relation of contractions in (5.5) cannot provide this interpretation because at  $\mu = 4 \text{ GeV}$  the computer only tells us what the values of these contractions are.

## 6. Outlook

Let me finish this presentation by stating what should be improved in the three non-perturbative approaches in order to clarify the situation of  $\varepsilon'/\varepsilon$  in the SM.

**RBC-UKQCD:** Here a very important progress would be the calculation of isospin breaking and QED corrections including also  $\eta - \eta'$  mixing. Another important issue are charm contributions still missing in RBC-UKQCD calculations. The prospects for improvements in the coming years are good [51].

**ChPT:** First of all matching to short distance which would amount, as far as I understand, at least to the determination of  $L_5$ . Better inclusion of  $\eta - \eta'$  mixing with the help of  $L_7$ .

**DQCD:** Here certainly one should have a look at the subleading FSI.

It would also be very important if other LQCD groups contributed to these investigations and in particular several LQCD groups calculated BSM hadronic matrix elements. Here some promising results are signalled by two LQCD groups [71, 72].

However, in order to identify possible NP also correlations of  $\epsilon'/\epsilon$  with other processes, in particular rare Kaon decays, have to be studied as summarized recently in [73]. Definitely there are exciting times ahead of us!

## Acknowledgements

I would like to thank Jean-Marc Gérard for invaluable comments on the manuscript and him, Jason Aebischer, Christoph Bobeth, Jacky Kumar, Mikolaj Misiak and David Straub for exciting time we spent together analyzing the topics discussed in this talk. I would like to thank the organizers of WG3 for inviting me to present these results. The research presented in this report was dominantly financed and done in the context of the ERC Advanced Grant project “FLAVOUR” (267104). It was also partially supported by the DFG cluster of excellence “Origin and Structure of the Universe”.

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