PROCEEDINGS OF SCIENCE

Relativistic Landau levels via Feynman-Gell-Mann formulation

Vitor Bentes Mendrot* **and Antonio Soares de Castro**

Department of Physics, State University of São Paulo - Unesp
Guaratinguetá, Brazil

E-mail: vitor.mendrot@unesp.br, antonio.castro@unesp.br

Relativistic energy levels for a spin-1/2 fermion in a stationary and homogeneous magnetic field, namely the Landau levels, are obtained via the Feynman-Gell-Mann formulation of the Dirac equation using the axially symmetric gauge, such that the radial functions obey a similar equation to the one of a singular harmonic oscillator in the nonrelativistic theory.

XV International Workshop on Hadron Physics (XV Hadron Physics) 13 -17 September 2021
Online, hosted by Instituto Tecnológico de Aeronáutica, São José dos Campos, Brazil

*Speaker

PoS(XVHadronPhysics)073

1. Introduction

In the realm of relativistic quantum mechanics, the Dirac equation appears as a first order equation with a four-component spinor as a wave function, while the Klein-Gordon equation, being of second order, has only a single-component wave function. The pertinent question immediately arises: why must the Dirac equation possess a four-component wave function? One may promptly answer that the four components exist to account for the two possible spin projections, and two more for the antiparticle. However, the Klein-Gordon equation, while not having to account for spin projection, houses both particle and antiparticle states in a single component, but on the other hand, being a second order equation.

In a 1958 paper [1], Feynman and Gell-Mann propose, in this spirit, a second-order formulation of the Dirac equation, having two-component spinors as wave functions, which is now known as the Feynman–Gell-Mann formulation [2–9]. In this work, we look into the relativistic energy levels of a particle immersed in a stationary and homogeneous magnetic field using this formulation. The levels are obtained by expressing the spinor as an eigenstate of operators such that the radial functions obey uncoupled second order differential equations, which can be mapped into the singular harmonic oscillator in the non-relativistic theory.

2. The Feynman–Gell-Mann formulation

The Dirac equation for a fermion of mass m and electric charge q in 3+1 dimensions with a minimally coupled vector interaction is written (with $\hbar = c = 1$) as

$$[\gamma^\mu(p_\mu - qA_\mu) - m]\Psi = 0\,, \quad (\mu = 0, 1, 2, 3)\,, \tag{1}$$

where $p_\mu = i\partial_\mu$ is the canonical momentum operator, $A_\mu = (A^0, -\boldsymbol{A})$ is the potential, and γ^μ are the gamma matrices which satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is the Minkowski metric tensor.

To construct the second order equation, we exploit the convenient properties of the chirality operator, which in the standard representation is defined as $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. We now construct the chiral projections $\Psi^{(\lambda)} \equiv P_\lambda\Psi$, where

$$P_\lambda = \frac{1 + \lambda\gamma^5}{2} \quad , \quad \lambda = \pm 1 \quad , \quad P_\lambda\gamma^\mu = \gamma^\mu P_{-\lambda}. \tag{2}$$

One may easily discover that the chiral projection of a Dirac spinor is exactly the eigenstate of the chiral operator, which has a convenient relation between its upper and lower components

$$\gamma^5\Psi^{(\lambda)} = \lambda\Psi^{(\lambda)} \quad , \quad \Psi^{(\lambda)} = \begin{pmatrix} \Phi^{(\lambda)} \\ \lambda\Phi^{(\lambda)} \end{pmatrix}. \tag{3}$$

Now we are set to build the second order equation. Applying $P_{-\lambda}$ to (1), we obtain a relationship between both chiral projetctions, allowing us to write Ψ in terms of $\Psi^{(\lambda)}$ only. Then, we simply substitute it in (1), obtaining a second order equation for the chiral projection, which itself leads to the sought equation for a two-component spinor

$$[(p^\mu - qA^\mu)(p_\mu - qA_\mu) - m^2 + q\boldsymbol{\sigma} \cdot (\boldsymbol{B} - i\lambda \boldsymbol{E})]\Phi^{(\lambda)} = 0, \tag{4}$$

such that

$$\Psi = \begin{pmatrix} \Phi^{(\lambda)} + \Phi^{(-\lambda)} \\ \lambda(\Phi^{(\lambda)} - \Phi^{(-\lambda)}) \end{pmatrix}. \tag{5}$$

3. A stationary and homogeneous magnetic field

Using the cylindrical coordinates system (r, θ, x_3), we are interested in investigating the magnetic field $\boldsymbol{B} = (0, 0, B)$. For this field, the axially symmetric gauge will serve, such that our potential will be $A_0 = 0$, $\boldsymbol{A} = (0, Br/2, 0)$. Actually, the radial component of $\boldsymbol{A}$ could be any function as it does not contribute to the magnetic field. Considering a non-zero value for it would only add a phase factor to the spinor, which doesn't affect the physics, so the easiest choice for it is simply zero.

Since the potential is time-independent, we can factorize the time dependence in the spinor, and as the particle is free in the x_3-axis, the spinor can be written as eigenstate of the third component of momentum, spin and total angular momentum operators, respectively:

$$\begin{aligned} p_3\Phi^{(\lambda)} &= k_3\Phi^{(\lambda)} \quad , \quad k_3 \in \mathbb{R} \\ s_3\Phi^{(\lambda)} &= \frac{s}{2}\Phi^{(\lambda)} \quad , \quad s = \pm 1 \\ j_3\Phi^{(\lambda)} &= \kappa\Phi^{(\lambda)} \quad , \quad \kappa = l + \frac{s}{2} = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots \end{aligned} \tag{6}$$

where l is the third-component of the angular momentum L_3 eingenvalue.

We can now build the spinor by using the eigenstates of L_3 and s_3. Since s_3 is Hermitian, its eigenstates form a basis for the two-component spinors. Since there are not any electric fields involved, from (4), we already know that $\Phi^{(\lambda)} = \Phi^{(-\lambda)}$, therefore there is no need to keep carrying the λ label anymore. Finally, we arrive at the explicit form of the two-component spinor which will lead us to

$$\varphi_\kappa^{(s)}(r, \theta) = \frac{1}{2\sqrt{r}} \begin{pmatrix} \frac{1+s}{2} f_\kappa^{(+)}(r)\Theta_{\kappa-\frac{1}{2}}(\theta) \\ i\frac{1-s}{2} f_\kappa^{(-)}(r)\Theta_{\kappa+\frac{1}{2}}(\theta) \end{pmatrix} \tag{7}$$

$$\frac{d^2 f_\kappa^{(s)}}{dr^2} + \left[-\left(\frac{qB}{2}\right)^2 r^2 - \frac{\kappa(\kappa - s)}{r^2} + \varepsilon_\kappa^{(s)^2} \right. \\ \left. -m^2 + qB\left(\kappa + \frac{s}{2}\right)\right] f_\kappa^{(s)} = 0, \tag{8}$$

with radial probability density equal to $\rho = \left|f_\kappa^{(s)}\right|^2 / r$. Equation (8) is similar to the radial equation for a spherical harmonic potential in Schrödinger's equation [10]

$$\frac{d^2 U}{dr^2} + \left[-\left(\frac{M\omega}{\hbar}\right)^2 r^2 - \frac{S^2 - 1/4}{r^2} + \frac{2ME}{\hbar^2}\right] U = 0, \tag{9}$$

where M, ω are positive parameters and $S \geq 0$. The eigenfunctions and respective eigenvalues, considering $\int_0^\infty dr\,|U|^2 < \infty$, are

$$U_{n_r S} = A_{n_r S} r^{\frac{1}{2}+S} exp\left(-\frac{M\omega}{2\hbar} r^2\right) L_{n_r}^{(S)}\left(\frac{M\omega}{\hbar} r^2\right), \tag{10}$$

$$E_{n_r S} = \hbar\omega\,(2n_r + 1 + S), \tag{11}$$

in which $n_r = 0, 1, 2, 3, \ldots$ Since U is square integrable, we can directly map its eigensolutions to our problem, leading to

$$f_{n_r \kappa}^{(s)} = A_{n_r \kappa}^{(s)} r^{\frac{1}{2} + \left|\kappa - \frac{s}{2}\right|} exp\left(-\frac{qBr^2}{4}\right) L_{n_r}^{\left(\left|\kappa - \frac{s}{2}\right|\right)}\left(\frac{qB}{2} r^2\right) \tag{12}$$

$$\varepsilon_{n_r \kappa}^{(s)^2} = m^2 + qB\left(2n_r + 1 - \kappa - \frac{s}{2} + \left|\kappa - \frac{s}{2}\right|\right), \tag{13}$$

where $L_{n_r}^{\left(\left|\kappa - \frac{s}{2}\right|\right)}$ are generalized Laguerre polynomials. The solutions obtained allow positive and negative eigenenergies given by

$$\varepsilon_{n_r \kappa}^{(s)} = \pm\sqrt{m^2 + qB\left(2n_r + 1 - \kappa - \frac{s}{2} + \left|\kappa - \frac{s}{2}\right|\right)}. \tag{14}$$

4. Concluding remarks

As desired, we were able to obtain the Dirac spinor by just calculating a two-component spinor single radial equation, whose form allow us to map it quite simply into the non-relativistic theory.

The obtained spectrum is in conformity with results avaiable in the literature, as can be seen for example in the reference [9] — although in it there is only the spectrum for $\kappa \geq 1/2$. Furthermore, it is valid to cite that the spectrum is analogous to the same system in 2+1 dimensions, as can be seem in [11]. The spectrum here obtained, though, contradicts results avaiable in the literature [6]. Certainly, ours is favourable, since in the reference cited, the spectrum has absurd possibilities, like $\varepsilon^2 < 0$.

PoS(XVHadronPhysics)073

Acknowledgments

Grant 2019/06734-2, São Paulo Research Foundation (FAPESP). Grant 09126/2019-3, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

References

[1] R. P. Feynman and M. Gell-Mann, "Theory of the Fermi interaction," *Physical Review*, vol. 109, no. 1, pp. 193 – 198, 1958.

[2] A. Barut and G. Mullen, "Quantization of two-component higher order spinor equations," *Annals of Physics*, vol. 20, no. 2, pp. 184 – 202, 1962.

[3] L. C. Biedenharn, "Some remarks on the Feynman-Gell-Mann formulation of the Fermi interaction," *Nuovo Cimento*, vol. 10, no. 22, pp. 4797–4798, 1961.

[4] S. Depaquit, P. Guéret, and J. P. Vigier, "Classical spin variables and classical counterpart of the Dirac-Feynman-Gell-Mann equation," *International Journal of Theoretical Physics*, vol. 4, no. 25, pp. 19 – 32, 1971.

[5] F. Gallone, "On the Feynman–Gell-Mann equation," *Annals of Physics*, vol. 57, no. 1, pp. 65 – 78, 1970.

[6] J. Walter T. Grandy, *Relativistic quantum mechanics of leptons and fields*. Dordrecht: Kluwer Academic Publishers, 2012.

[7] E. Merzbacher, *Quantum mechanics*. New York: John Wiley & Sons, Inc, 1998.

[8] N. C. Petroni, P. Guéret, J. P. Vigier, and A. Kyprianidis, "Second-order wave equation for spin-(1/2) fields," *Physical Review D*, vol. 31, pp. 3157–3161, 1985.

[9] P. Strange, *Relativistic quantum mechanics with applications in condensed matter and atomic physics*. Cambridge: Cambridge University Press, 1998.

[10] P. H. F. Nogueira and A. S. de Castro, "From the generalized Morse potential to a unified treatment of the D-dimensional singular harmonic oscillator and singular Coulomb potentials," *Journal of Mathematical Chemistry*, vol. 54, no. 9, pp. 1783–1791, 2016.

[11] B. Thaller, *The Dirac equation*. Berlin: Springer-Verlag, 1992.