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# Finite Planck-scale-modified relativistic framework in Finsler geometry

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Using the Finsler functions, we construct deformed Lorentz transformations that preserve the modified dispersion relation inspired by the $\kappa$-Poincaré algebra in the bicrossproduct basis. The relativistic framework is completed by the derivation of the corresponding modified composition laws with back-reaction parameters. We use these results to describe some equation that govern the kinematics of the two-body decay in deformed relativity.

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## 1. Introduction

The search for quantum gravity has been one of the main research topics in the current scenario of physics, whose experimental verification has been the goal of systematic research in recent years [1]. Investigations into the effect of Lorentz symmetry violation/deformation on the propagation of massless astroparticles have been one of the main sources of constraints on the parameters of quantum gravity at the Planck scale due to the cosmological distances these particles propagate, which works as natural amplifiers of effects that are otherwise typically on the order of the small ratio of the particle energy to the quantum gravity energy scale parameter, where it is assumed to be on the order of the Planck energy $E_P$ [2–4]. Therefore, one of the greatest difficulties in the search for quantum gravity effects is that the energy scale that expected to be relevant, is in the Planck energy regime $E_P$ of order $10^{19}$ GeV, or respectively at a tiny distance scale of the order of the Planck length $\ell$ of order $10^{-35}$m. In this way, the search for amplifiers plays an important role in the development of new theories of quantum gravity. Therefore, it is important to have other types of observables that may involve different types of messengers in order to examine the quantum gravity scale as broadly as possible (we refer the reader to the review for a collection of proposals [1] and to [5] for this recent review focused on multimessenger astronomy signatures).

Recently, some approaches based on Planck-scale modifications of a particle's lifetime due to non-commutative geometry have emerged as a signature of CPT violation [6]. In those cases, the lifetime of accelerated particles and antiparticles would differ by an amount proportional to the lifetime of the particle at rest and the dimensionless quantity $E_P^{-1}\, p^2/m$, where $E_P$ is a quantum gravity energy scale parameter and $p$ and $m$ are the particle's momentum and mass, respectively. Curiously, the energy dimensionful quantity that couples to the quantum gravity parameter is not the energy of the particle, but instead is the square of its momentum divided by its mass. This means that the lightest the particle, the more prominent is the effect. The nature of this kind of amplifier is responsible for placing this correction just a few orders of magnitude away from Planck scale sensitivity at an optimistic setup involving the properties of the muon in particle accelerators.

So, we wonder whether this kind of correction could emerge in other approaches to quantum gravity and obeying a power-law correction based on the energy $E$ and mass $m$ of the particle as

$$E_P^{-1}\, E^n/m^{n-1}\,. \tag{1}$$

The case $n = 2$ corresponds to the previous case, but we would like highlight that cases with $n \geq 3$ are compelling from the experimental and theoretical sides. From the experimental point of view, each increment in $n$ could enhance the effect by a factor up to $E/m \sim 10^4$ (assuming energies of 1 TeV and, for instance, the muon with mass of $\approx 106\,$MeV), without the need of going to higher orders in the quantum gravity energy scale.[1] From the theoretical point of view, in this paper we demonstrate that a case $n = 3$ follows from the preservation of a fundamental axiom of special and general relativity when a particle propagates in a quantum spacetime: *the clock postulate*. In fact, as we shall verify, this amplification is responsible for bringing us precisely to the Planck scale at a very optimistic setup, but with interesting prospects for the future.

In this paper, we analyze the effects of this kind of amplifier in the framework that deforms rather than violates Lorentz invariance, by investigating the modified kinematics of the two-body decay.

[1]Notice that even by raising $n$, the parameter $E_P^{-1}$ is still contributing linearly.

In section 2, we review the relation between Finsler geometry and quantum gravity phenomenology, by showing how relativistic principles can be accommodated in this formalism. In section 3, we construct finite (in the boost parameter) Lorentz transformations between frames that are in relative motion in 1 + 1D. In section 4, we construct the associate modified composition law in this deformed relativistic set-up. In section 5, we derive some equations for the two-body decay. We conclude in section 6 with some final remarks. We assume units in which $c = \hbar = 1$.

## 2. Finsler geometry, modified dispersion relations and deformed relativity

Finsler geometry emerges naturally when we seek to describe modified relativistic kinematics in terms of the configuration space of particle that obeys a modified dispersion relation [7]. When we perform a transformation from Hamiltonian to Lagrangian formalism, it is verified that the trajectories of these massive particles are defined from the extremization of a functional that generalizes the one that describes the propagation in a Riemannian spacetime

$$S[x] = m \int F(x, \dot{x}) d\mu, \tag{2}$$

where $m$ is the particle's mass, $\dot{x}^a = dx^a/d\mu$ and $F$ is a 1-homogeneous function of $\dot{x}$. In fact, a modified dispersion relation is perturbatively defined as

$$H(x, p) = g^{ab}(x) p_a p_b + \epsilon h^{a_1 a_2 \dots a_n}(x) p_{a_1} p_{a_2} \dots p_{a_n}, \tag{3}$$

where $g^{ab}$ is a Riemannian metric, $p_a$ is the particle's momentum, $h^{a_1 a_2 \dots a_n}$ are parameters that describe the specific MDR and $\epsilon$ is an inverse $n$-th power of energy parameter. Following the prescription of [8, 9], when transforming from the action in the Hamiltonian formalism to the Lagrangian one, we find functional (2), with

$$F(x, \dot{x}) = \sqrt{g_{ab}\dot{x}^a \dot{x}^b} - \frac{\epsilon\, m^{n-1}}{2} \frac{h_{a_1 a_2 \dots a_n}(x) \dot{x}^{a_1} \dot{x}^{a_2} \dots \dot{x}^{a_n}}{[g_{ab}\dot{x}^a \dot{x}^b]^{\frac{n-1}{2}}} \,. \tag{4}$$

When $\epsilon \to 0$, we recover the usual result of general relativity regarding the propagation of particles in a curved spacetime. The curves that extremize functionals of the form (2) have been analyzed in some recent papers [10, 11], which show that these particles follow geodesic equations of a non-Riemannian geometry, a Finsler spacetime, whose metric is determined when one defines the arc-length functional of this geometry as $s[x] \doteq S[x]/m$ from Eq.(2), which allows us to identify the Finsler metric from the Hessian of the function $F(x, \dot{x})$. These geodesic equations are of the form

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc}(x, \dot{x}) \frac{dx^b}{ds} \frac{dx^c}{ds} = 0\,, \tag{5}$$

where $\Gamma^a_{bc}$ are Christoffel symbols of the Finsler metric, $s$ is the arc-length parameter, and these curves are a generalization of the Riemannian geodesics.

Since it is the arc-length parameter which shall be extremized in order to furnish the trajectories and it is also the one in which the geodesic equation assumes the usual sourceless form given by Eq.(5), it seems natural to extend one of the most fundamental axioms upon which special and general relativity are based:

- **Clock postulate**:

  *The proper time an observer, or massive particle, experiences between events A and B along a time-like curve (her worldline) in a Finsler spacetime $(\mathcal{M}, F)$ is the length of this curve between events A and B:*

$$\Delta\tau_{AB} \doteq m^{-1}\int_{\mu_A}^{\mu_B} F(x,\dot{x})d\mu\,. \tag{6}$$

If an unstable particle is created at an event $A$ and decays at an event $B$, $\Delta\tau_{AB}$ shall be a measure of its lifetime as if it were at rest, since the proper time is a measure of time lapse in a reference frame that is comoving to the propagating particle. As we shall see, the dilation of particles lifetimes can be expressed from (6). Besides that, as we want to study the decay of fundamental particles in accelerators, we can disregard the pure gravitational effects, and analyze the Finsler deformations of Minkowski spacetime, since it is possible to mathematically justify the existence of special coordinates, which allow us to neglect the effects of curvature at small coordinate distances around each point and a given direction in Finsler spacetimes [12]. Thus, for zero order, we consider the Riemannian $g(\dot{x},\dot{x})$ in Eq.(4) as the usual Minkowski metric, which we label $\eta(\dot{x},\dot{x})$. In Cartesian coordinates we simply write

$$\eta(\dot{x},\dot{x}) = (\dot{x})^2 - \delta_{ij}(\dot{x}^i)(\dot{x}^j). \tag{7}$$

Now, knowing that the arc length is invariant by reparametrization, we can perform transformations on the parameter $\mu$ for the coordinate time of the laboratory frame, $x^0 \doteq t$ in (6). Using (4) we have the following modification of the proper time between the events with parameters $(x^0)_A = t_A$ to $(x^0)_B = t_B$ (we omit the label "$AB$" in $\Delta\tau_{AB}$):

$$\Delta\tau = \int_{t_A}^{t_B} dt\left[\gamma^{-1} - \frac{\epsilon}{2}m^{n-2}\gamma^{n-1}h_{a_1\ldots a_n}\frac{dx^{a_1}}{dt}\ldots\frac{dx^{a_1}}{dt}\right] \tag{8}$$

where, we introduced for convenience the usual *velocity Lorentz factor*

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \tag{9}$$

with $v^i \doteq dx^i/dt$ and $v^2 = \delta_{ij}v^iv^j$.

We can now make some assumptions about the lifetime dilated of fundamental particulars accelerated in LHC or FCC accelerators. In this case, the three-dimensional velocity norm $v^2$ is approximately a constant, which allows for a simplification of the above expression. Next, $\Delta t$ will be the time measured in the laboratory reference frame in which the particle is accelerated, while $\Delta\tau$ is the proper time experienced by the particle, respectively measured by a comoving observer to the particle.

### 2.1 The Finsler version of bicrossproduc basis of $\kappa$-Poincaré

The first order correction that arises from the GR Finsler quadratic function is a polynomial of degree $n = 3$. We consider as our working case the so called $\kappa$-Poincaré dispersion relation on the bicrossproduct basis [13]

$$m^2 = p_0^2 - \delta^{ij}p_ip_j - \ell p_0\delta^{ij}p_ip_j\,. \tag{10}$$

For this construction, the symbols $h_{a_1a_2a_3}$ read $h_{a_1a_2a_3} = -\frac{1}{3}(\delta^0_{a1}\delta_{ij}\delta^i_{a2}\delta^j_{a3} + \delta^0_{a2}\delta_{ij}\delta^i_{a1}\delta^j_{a3} + \delta^0_{a3}\delta_{ij}\delta^i_{a1}\delta^j_{a2})$.

It is expected that the quantum gravitational corrections occur close to the Planck energy scale, which indicates that the parameter $\epsilon$ is in the order of the inverse of the energy scale. However, in the context of $\kappa$-Poincaré algebra, we generally denote $\epsilon$ as the deformation parameter $\kappa^{-1}$, which leads us to make the following definition $\ell = \epsilon = \kappa^{-1}$, where $\kappa$ is expected to be of the order the Planck energy $E_p \approx 1.2 \times 10^{19}$GeV. The lifetime of a fundamental particle propagating in a Finsler spacetime induced by the bicrossproduct basis of $\kappa$-Poincaré dispersion relation can then be expressed as being (we define $\Delta t \doteq t_B - t_A$)

$$\Delta\tau = \frac{\Delta t}{\gamma}\left[1 + \frac{\ell}{2}m\gamma(\gamma^2 - 1)\right]. \tag{11}$$

This result gives us the proper time that a particle experiences and is related to the time spent in the laboratory, relative to which the particle is accelerated. Thus, it is possible to relate the measured lifetime of a particle in the laboratory, denoted by $\Delta t$, with the particle's own lifetime $\Delta\tau$, depending on its coordinate velocity $v$ by the factor $\gamma$. For the first order $\ell$, we find for the lifetime of the particle's laboratory frame

$$\Delta t = \gamma\Delta\tau\left[1 - \frac{\ell}{2}m\gamma(\gamma^2 - 1)\right]. \tag{12}$$

As we wish to compare this result with the data from particle accelerators, the lifetime obtained must be expressed in terms of the velocity $\gamma$ factor defined in (9), as well as in terms of the energy $p_0$ and mass $m$ of the particles. For this, we derive the 4-momentum of the particles, which automatically satisfy the MDR:

$$p_0 = m\frac{\partial}{\partial\dot{x}^0}F(x,\dot{x}) = m\gamma - \frac{\ell}{2}m^2(\gamma^2 - 1)(2\gamma^2 - 1), \tag{13}$$

$$p_i = m\frac{\partial}{\partial\dot{x}^i}F(x,\dot{x}) = -v_i\gamma m + \ell m^2 v_i\gamma^4, \tag{14}$$

and solving the first relation for $\gamma$ as a function of $p_0$ yields $\gamma = \frac{p_0}{m} + \frac{\ell}{2}m\left(1 - 3\frac{{p_0}^2}{m^2} + 2\frac{{p_0}^4}{m^4}\right)$. We get the lifetime as a function of $p_0$ by substituting this result in (12)

$$\Delta t = \frac{p_0}{m}\Delta\tau\left[1 + \frac{\ell}{2}\left(\frac{m^2}{p_0} - 2p_0 + \frac{{p_0}^3}{m^2}\right)\right] \doteq \gamma_{DSR}\Delta\tau, \tag{15}$$

where $\Delta\tau$ corresponds to the lifetime a fundamental particle in its rest frame. An important point is the definition of the modified Lorentz factor, which is derived from the geometric clock defined by the Finsler function (4). This result has an impact when analysing the effects of the Finsler deformed relativistic approach on the phenomenology of time dilation of accelerated particles.

Another important point is to reaffirm the type of correction initially proposed in (1) in which considering Eq.(12) and that the particles are accelerated to close to the speed of light ($\gamma \gg 1$) and realizing that the energy of the particle can be read directly from the Finsler function (and is related to the velocity Lorentz factor) as $p_0 = m\partial F/\partial x^0 \approx m\gamma - \ell m^2\gamma^4$. This implies that the dilated lifetime of special relativity $\Delta t_{SR} = \Delta\tau p_0/m$ is modified by the quantum gravity parameter $\ell$ as

$$\Delta t \approx \Delta t_{SR}\left[1 + \frac{\ell}{2}\frac{{p_0}^3}{m^2}\right]. \tag{16}$$

.

Exemplifying this result, if we consider as hypothetical input the optimal energy scale results of the LHC ($p_0 \sim 6.5$ TeV) and muons ($m \simeq 106$ MeV) as our test particles, we find that we would be able to constrain the $\ell$ parameter with *Planck scale sensitivity* if a measurement of the dilated muon lifetime were made at the LHC with a relative uncertainty $\sim 10^{-6}$, which lies within the precision of this measurement for low energy muons [14]:

$$\frac{1}{\ell} \gtrsim E_{\text{Planck}} = 1.22 \times 10^{19}\,\text{GeV}. \tag{17}$$

We should stress that this approach opens up the possibility for phenomenological investigations based on different kinds of particles, like pions, whose mass is $\sim 30\%$ larger than the muon's, which could facilitate the detection of their decays in accelerators (without much loss in the necessary precision to reach the Planck scale). This also allows one to investigate the decay of particles in atmospheric showers initiated by cosmic rays. Besides that, different deformations could arise from alternative approaches to quantum gravity that could give rise to other kinds of contributions, as further discussed in [9].

However, if one aims to extend this approach to the study of cosmic rays, for instance, it becomes of paramount importance to properly describe the decay of particle in modified relativity, and to get there, it is necessary to obtain the expression of the deformed Lorentz transformations in terms between frames that move relative to the other with arbitrary finite velocity $v$.

## 3. Finite boost transformation between arbitrary momenta

To continue with our investigation, let us consider two inertial frames, $S$ and $\tilde{S}$, which move with relative velocity $v$ and, for simplicity, we assume spacetime in $1+1$ dimensions. We also assume that each observer assigns momenta $p_\mu$ and $\tilde{p}_\mu$ to a particle. For this configuration, we observe that in first order in $\ell$, the most general deformed transformations connecting these $p_\mu \to \tilde{p}_\mu$ momenta should be as follows

$$\tilde{p}_0 = \gamma(p_0 - v p_1) + \ell\left[A p_0 p_1 + B p_1^2 - \frac{1}{2} p_0^2 (\gamma^2 - 1)(2\gamma^2 - 1)\right], \tag{18}$$

$$\tilde{p}_1 = \gamma(p_1 - v p_0) + \ell(p_0^2 v \gamma^4 + F p_0 p_1 + G p_1^2), \tag{19}$$

where $A$, $B$, $F$, $G$ are general functions of $v$. For dimensional reasons, the terms that multiply the perturbation parameter $\ell$ must be quadratic in momenta. Applying the MDR (10) invariance, to then derive a deformed symmetry transformation, we find the following necessary conditions for these functions:

$$B = -\frac{A}{v} - \frac{(1-v^2)^{3/2} - v^2 - 1}{2(1-v^2)}, \tag{20}$$

$$F = -\frac{A}{v}, \tag{21}$$

$$G = A - \frac{v[2v^2 - (1-v^2)^{3/2}]}{2(1-v^2)^2}, \tag{22}$$

Setting $A = 0$, for simplicity, we find the following set of DSR transformations

$$[\Lambda(v,p)]_\mu = \tilde{p}_\mu = \begin{cases} \tilde{p}_0 = \gamma(p_0 - vp_1) + \frac{\ell}{2}\left[p_1^2\gamma(2\gamma^3-\gamma-1) - p_0^2(\gamma^2-1)(2\gamma^2-1)\right], \\ \tilde{p}_1 = \gamma(p_1 - vp_0) + \ell v[p_0^2\gamma^4 - \frac{p_1^2}{2}\gamma(2\gamma^3-2\gamma-1)]. \end{cases} \tag{23}$$

It is important to note, that in this notation, we are referring to $[\Lambda(v,p)]_\mu$ as the transformed $\mu-$component of momenta "$p$" using the boost parameter "$v$". We notice that theses equations are dominated by the term $\gamma^4$ when $\gamma \gg 1$. The emergence of the term $\gamma^4$ becomes the window for the possibility of detecting new effects, working as an amplifier to Planck scale effects. This form of the finite transformation has not been yet considered in the $\kappa$-Poincaré literature. On the other hand, the infinitesimal (in $v$) version of this transformation coincides with the usual ones of $\kappa$-Poincaré [10, 15, 16]. They are given by (we also replace $v \to -v$):

$$\tilde{p}_0 \approx p_0 + vp_1, \tag{24}$$

$$\tilde{p}_1 \approx p_1 + vp_0 - \ell v\left(p_0^2 + \frac{p_1^2}{2}\right). \tag{25}$$

## 4. Modified composition law

Continuing in the solving this puzzle, another important piece for the elaboration of deformed relativistic kinematics is the formulation of a modified energy/momentum conservation law. In this way, we guarantee that inertial observers agree on the existence or prohibition of interactions between elementary particles.

Taking dimensionality into account, the most general form of the composition law in first-order perturbation is[2]

$$(p\oplus q)_0 = p_0 + q_0 + \ell(\alpha p_0 q_0 + \beta p_1 q_1 + \omega p_0 q_1 + \eta p_1 q_0), \tag{26a}$$

$$(p\oplus q)_1 = p_1 + q_1 + \ell(\delta p_1 q_0 + \epsilon p_0 q_1 + \lambda p_1 q_1 + \mu p_0 q_0), \tag{26b}$$

where $(\alpha,\ \beta,\ \omega,\ \eta,\ \delta,\ \epsilon,\ \lambda,\ \mu)$ are dimensionless parameters yet to be determined. In order for us to have a deformed relativistic compatibility, the action of the Lorentz transformation (23) in the compound momenta must fulfill a relation of the form:

$$\Lambda(v, p\oplus q) = \Lambda(v_q, p) \oplus \Lambda(v_p, q), \tag{27}$$

generally, the boost parameters $v_p$ and $v_q$, which appear on the right side of this relationship, may be dependent on the moment $p$ and $q$ respectively. An observation that was initially proposed in [17] indicates that these elements, known as "back-reaction", are a necessity to assure the relativistic nature of the law of composition of the $\kappa$-Poincaré algebra on the bicrossproduct basis.

Here, the most general "back-reaction" parameters for first order deformations that we can use in the boosted composition law (27) are

$$v_q = v + \ell(Hq_0 + Jq_1), \tag{28a}$$

$$v_p = v + \ell(Mp_0 + Rp_1). \tag{28b}$$

[2]This law satisfies $p\oplus 0 = p$ and $0\oplus p = p$.

The presence of the "back-reaction" in both entries of the composition law has already been considered in [10], for instance. Here we implement it completely in this article and analyze its phenomenological consequences. The imposition of the relativistic condition on each component of (27) from Eqs. (23) and (26) give the following set of conditions between the composition law parameters and those of the back-reaction:

$$\alpha = 0 = \lambda\,, \tag{29a}$$

$$\beta = -\frac{2+\gamma^3(J+R-2)+4\gamma^4-\gamma^2[4+J+R-v\gamma(H+M)]}{2(\gamma-1)}\,, \tag{29b}$$

$$\delta = -\gamma\frac{\{1+\gamma^2(R-1)+2\gamma^3-\gamma(2+R-v\gamma M)\}}{2(\gamma-1)}\,, \tag{29c}$$

$$\epsilon = \delta(R\to J\,;\,M\to H)\,, \tag{29d}$$

$$\omega = -\gamma\frac{\{M\gamma(\gamma-1)+v\gamma[2\gamma^2+\gamma(R-1)-1]\}}{2(\gamma-1)}\,, \tag{29e}$$

$$\eta = \omega(R\to J\,;\,M\to H)\,, \tag{29f}$$

$$\mu = \frac{\{-2+\gamma^2(2+J+R)+4\gamma^3+\gamma[J+R-4+v\gamma(H+M)]\}}{2v}\,. \tag{29g}$$

Note that the above expressions are not supposed to be seen as if the composition law parameters depended on $v$; instead, these expressions describe inverse functions between the back-reaction parameters and $v$ and the composition law parameters, i.e., $(H, J, M, R)$ are functions of $(v, \alpha, \beta, \delta, \ldots)$. Obviously, some ambiguities will emerge and some conditions will have to be fixed as we analyze in the following.

### 4.1 Parity-invariant composition law

As can be seen from (10), the 3 + 1-dimensional case is invariant under parity transformations ($k_0 \to k_0$, $\vec{k} \to -\vec{k}$, where $k$ describes momenta $p$, $q$ and $p \oplus q$). For this reason, we implement this symmetry also in the 1 + 1-dimensional case under consideration, so that the results could be translated to the general one. In order to have this property, we require $\omega = \eta = \mu = 0$ (the term $\lambda$ is null due to (29a)). This gives the following set of conditions on the back-reaction parameters found from Eqs. (29a)–(29g)

$$J = -\frac{Hv\gamma}{1+\gamma}+\frac{1+\gamma-2\gamma^2}{\gamma}\,, \quad M = -\frac{Rv\gamma}{\gamma-1}-v(1+2\gamma)\,. \tag{30}$$

This leads to two cases to analyze that arise from this restriction. They are the cases of undeformed momentum and undeformed energy composition.

#### 4.1.1 Undeformed spatial momentum conservation

The first case, whose importance is significant because it allows applications in the study of the decay of one particle into two others, when the analysis is performed in the frame of the rest of the parent particle. This procedure is done when we choose $\delta = 0 = \epsilon$ in (26) to $\omega = \eta = \mu = 0$.

It implies for the back-reaction parameters (30)

$$R = J = \frac{1}{\gamma}-\gamma\,, \quad H = M = -v\gamma\,, \tag{31}$$

which also leads to $\beta = 1$. This way, the composition law reads

$$\begin{cases} (p \oplus q)_0 = p_0 + q_0 + \ell p_1 q_1 \,, \\ (p \oplus q)_1 = p_1 + q_1 \,, \end{cases} \tag{32}$$

which is compatible with deformed Lorentz transformations $\Lambda(v, p)$ given by Eq. (23) for the back-reacting parameter

$$v_k = v + \ell \left[ \left( \frac{1}{\gamma} - \gamma \right) k_1 - v\gamma k_0 \right] , \tag{33}$$

where $k$ refers to momenta $p$ or $q$. In this case, the back-reaction acts equally on the first and second argument of the composition law. One can also check this result by a straightforward calculation considering the infinitesimal transformations (24) and (25) and infinitesimal back-reacting parameter $v_k = v(1 - \ell k_0)$.

#### 4.1.2 Undeformed energy conservation

The next case consists of preserving the conservation of energy, as it will be the last one in the recovery of the known addition of momenta that is defined by the coproduct structure of the bicrossproduct basis of $\kappa$- Poincaré algebra. In fact, this condition is fulfilled by requiring $\beta = 0$ in (26). This provides the following conditions:

$$H = \frac{v[\gamma^2(1 + R) - 1]}{\gamma(\gamma - 1)} , \tag{34}$$

which implies in the following composition law

$$\begin{cases} (p \oplus q)_0 = p_0 + q_0 \,, \\ (p \oplus q)_1 = p_1 + q_1 + \ell(\delta p_1 q_0 + \epsilon p_0 q_1) \,, \end{cases} \tag{35}$$

where

$$\delta = \frac{\gamma(\gamma^2 + R\gamma - 1)}{\gamma - 1} , \quad \epsilon = \frac{\gamma^3 + R\gamma^2 - 1}{\gamma - 1} . \tag{36}$$

Choosing $R = \gamma^{-1} - \gamma$, we obtain further simplifications for our general case which turn the composition law into the one from the bicrossproduct $\kappa$-Poincaré coproduct structure, [18],

$$\begin{cases} (p \oplus q)_0 = p_0 + q_0 \,, \\ (p \oplus q)_1 = p_1 + q_1 - \ell p_0 q_1 \,. \end{cases} \tag{37}$$

This law is compatible with deformed Lorentz transformations $\Lambda(v, p)$ given by Eq. (23) for the back-reaction parameters

$$v_q = v - \ell \left[ v \frac{(\gamma^2 - 1)}{\gamma} q_0 + \left( \gamma - \frac{1}{\gamma^2} \right) q_1 \right] , \tag{38}$$

$$v_p = v - \ell \left[ v\gamma p_0 + \left( \gamma - \frac{1}{\gamma} \right) p_1 \right] , \tag{39}$$

where we can see different types of reverse reaction parameters. It is possible to verify this result by a direct calculation considering the infinitesimal transformations (24) and (25) and infinitesimal inverse reaction parameters $v_q = v$ and $v_p = v(1 - \ell p_0)$, which coincides with the inverse reaction of the deformed infinitesimal Lorentz transformation extensively studied in the literature [17, 18].

## 5. Signatures for two-body decays

The framework constructed so far can be applied to the study of the equations that govern particle decays in cosmic-ray showers. This is possible because these calculations are performed by comparing the rest frame of the decaying particle with the laboratory frame. Moreover, the composition of the momenta is an essential piece of information to describe the spectrum of the particles produced. In this section, we study the case of a mother particle of mass $M$ decaying into two offspring particles, $m_p$ and $m_q$, with momentum $p$ and $q$ respectively. We continue considering the cases described in the previous section: *undeformed moment* and *undeformed energy conservation*. For other cases, the procedure discussed here can be applied. Consider the deformed conservation law $P_\mu \doteq (p \oplus q)_\mu$. In the following subsection, we refer to frame "$*$" as the one in which the parent particle (the one with momenta $P_\mu$) is at rest.

### 5.1 Conservation of undeformed space momentum in particle decay

From the rest frame condition and the composition law we find $0 = P_1^* = (p_1^* + q_1^*)$, and thus $p_1^* = -q_1^*$. Then, the modified dispersion relation (10 ) and de composition law (32) imply

$$M^2 = (P_0^*)^2 = (p_0^* + q_0^* - \ell(p_1^*)^2)^2 = (p_0^*)^2 + (q_0^*)^2 + 2(p_0^*)(q_0^*) - 2\ell(p_1^*)^2[p_0^* + q_0^*]\,. \tag{40}$$

Further, $p$ and $q$ themselves satisfy the MDR (10), and thus one can replace $p_0$ and $q_0$ as function of $p_1^*$, $q_1^*$, $m_p$ and $m_q$. After this substitution, (40) can be solved for $p^* = |p_1^*| = |q_1^*|$ and we find

$$p^* = \frac{\sqrt{M^4 - 2M^2(m_p^2 + m_q^2) + (m_p^2 - m_q^2)^2}}{2M}\,. \tag{41}$$

The result obtained indicates that the composition law exactly compensates for the effects of the MDR, providing the same expression that would be found in the framework of Special Relativity (SR).

An interesting application that can be done is the analysis of the decay of a particle into a massless and a massive one. What we get is $m_q = 0$, and since this law of composition is commutative we would have found the same result if we set $m_p = 0$. The result we get based on the previous equation is the momentum of the massive and massless particles produced in the rest frame of the parent particle

$$p^* = \frac{M}{2}\left(1 - \frac{m_p^2}{M^2}\right)\,. \tag{42}$$

### 5.2 Conservation of undeformed energy in particle decay

The next step is to analyze the case that coincides with the composition law that follows from the structure of the coproduct of the bicrossproduct base $\kappa$-Poincaré, given by Eq. (37). Again, we define $P_\mu = (p \oplus q)_\mu$, and consider the rest frame "$*$" of the parent particle with momenta $P_\mu$. Now, from $P_1^* = 0$, we find a non-trivial relation between the momenta of the descendant particles 1 and 2. In fact, assuming on-shell particles dispersion relation (10) and the first order in $\ell$ approach, we deduce

$$p_1^* = -q_1^* + \ell(q_1^*)(p_0^*) \approx -q_1^* + \ell(q_1^*)\sqrt{m_p^2 + (p_1^*)^2} \approx -q_1^* + \ell(q_1)^*\sqrt{m_p^2 + (q_1^*)^2}\,. \tag{43}$$

Using the MDR (10) for the momenta $p$ and $q$ and $P^*_0$, i.e. the equation $(P^*_0)^2 = M^2 = (p^*_0)^2 + (q^*_0)^2 + 2(p^*_0)(q^*_0)$ we can write the energies $p^*_0$, $q^*_0$ as a function of the spatial momenta $p^*_1$, $q^*_1$ and the masses $m_p$ and $m_q$. In addition, with Eq. (43) we can finally express $p^*_1$, $q^*_1$ as functions of the masses alone

$$|p_1|^* = \frac{\sqrt{M^4 - 2M^2(m_p^2 + m_q^2) + (m_p^2 - m_q^2)^2}}{2M}\left[1 - \frac{\ell(M^2 + m_p^2 - m_q^2)}{2M}\right]. \tag{44a}$$

$$|q_1|^* = \frac{\sqrt{M^4 - 2M^2(m_p^2 + m_q^2) + (m_p^2 - m_q^2)^2}}{2M}. \tag{44b}$$

We find corrections only for the first momentum in this composition law. In fact, since the composition law is non-commutative, we can have two distinct cases depending on the order in which the particles' momenta "enter" the deformed sum. To illustrate this issue, let us once again consider the case of the decay into a massive and a massless particle. If the massless particle is the first one in the composition law $p \oplus q$, i.e., if $m_p = 0$, we find the following relations:

$$m_p = 0 \Rightarrow \begin{cases} |p_1|^* = \frac{M}{2}\left(1 - \frac{m_q^2}{M^2}\right)\left[1 - \frac{\ell}{2M}(M^2 - m_q^2)\right], \\ |q_1|^* = \frac{M}{2}\left(1 - \frac{m_q^2}{M^2}\right). \end{cases} \tag{45}$$

On the other hand, if the massless one is the second particle, $m_q = 0$, we find

$$m_q = 0 \Rightarrow \begin{cases} |p_1|^* = \frac{M}{2}\left(1 - \frac{m_p^2}{M^2}\right)\left[1 - \frac{\ell}{2M}(M^2 + m_p^2)\right], \\ |q_1|^* = \frac{M}{2}\left(1 - \frac{m_p^2}{M^2}\right). \end{cases} \tag{46}$$

So, for instance, consider a pion decaying into a muon and a neutrino $\pi^\pm \to \mu^\pm + \nu_\mu(\bar{\nu}_\mu)$, (for simplicity we shall refer these processes simply as $\pi \to \mu + \nu$). If we have the composition law of the form $p_\pi = p_\mu \oplus p_\nu$, where $p_\pi$, $p_\mu$ and $p_\nu$ refer to the energy/momentum of each of these particles, then the spatial momenta of the muon and the neutrino are given by the first and second expressions of (46), respectively. This means that only the relation between the momentum of the muon, its mass, and the pion mass gets corrected in this frame. On the other hand, if we express the conservation law of this decay as $p_\pi = p_\nu \oplus p_\mu$, then the spatial momenta of the neutrino and the muon are given by the first and second expressions of (45), respectively. This means that only the relation between the momentum of the neutrino, the mass of the muon, and the pion mass gets corrected in this frame. Such situation does not happen in the previous subsection (undeformed momentum conservation) due to the commutativity of that composition law.

## 6. Conclusion

As seen in the previous article [9], a finite deformed Lorentz symmetry connecting the rest frame and the lab frames emerged from Finsler geometry, containing an amplification factor that can lead to observations of the Planck scale sensitivity. Here, we continue this two-step analysis:

- First we generalized the results inspired by $\kappa$-Poincaré from [9] by constructing finite deformed Lorentz transformations that connect the momenta of particles in two different frames (in $1+1$D) that move relative to each other. Together with the construction of the general law of composition of the moment that is compatible with this finite transformation of the first order in the deformation scale, in which it was necessary to introduce the back-reaction acting on both "inputs" of the composition law, such a condition guarantees that all inertial observers agree on the nature of the vertices of interactions between fundamental particles.

- The second stage was to apply the entire framework (modified dispersion relation, deformed Lorentz transformation compatible with Finsler and composition law with back-reaction) to consider the decay of a massive parent particle into two descendants. We derived kinematic equations that can be used to deduce the corrections in equations that shall be considered in the near future.

This paper is part of a larger project that aims to implement Planck-scale-deformed relativistic kinematics in the phenomenological equations of particle decays in cosmic-ray showers. The case of $\kappa$-Poincaré was considered as a first approach to this problem, in which we are learning important lessons to be implemented in the future. The steps followed here can also be generalized to allow the exploration of higher orders of perturbation in the Planck scale from the analysis of very-high-energy particles, which complement and expand the studies started in [19].

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