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Matrix-Formulated Noncommutative Gauge Theories of Gravity

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In the present article first we review the construction of three- and four-dimensional fuzzy spaces and subsequently we present the construction of fuzzy gravity theories on these noncommutative spaces following the gauge theoretic formulation of gravity.

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1. Introduction

It is widely estimated that physics close to Planck scale might be significantly different than it is in the other scales and what we call "quantum gravity" might be quite different from the quantum field theories of ordinary scales. In addition, another widespread estimation is that singularities and divergences are only technical artifacts which indicate the limitations of a physical theory. However, most theories do contain singularities as an essential element and these are generally considered to contain important information on possible extensions. Among elementary particle physicists, for example, the opinion that infinities in renormalizable field theories not only are not problematic since they can be treated by the renormalization procedure but instead can be considered as signals of a nearby new-physics threshold is formed. A possible way to resolve field-theory singularities, that is infinities related to the field-theoretical description of particle physics is to introduce yet another scale, this time related to the possible unification scale of the non-gravitational interactions. Accordingly, Grand Unified Theories (GUTs) with N = 1 supersymmetry have been constructed which, in turn, can be made finite even to all loops, including the soft supersymmetry breaking sector [1–5] and have predicted, among others, successfully the top [1, 6] and Higgs masses [7].

The above considerations can be applied to singularities in the Einstein theory of gravity as well, extrapolating that these singularities are not of physical significance but only signal the existence of a new structure of space-time beyond a certain scale. This new structure might be offered by noncommutative geometry. The ultimate aim then is the construction of a generalization of the General theory of Relativity (GR) which is assumed to become essentially noncommutative in regions where the commutative limit would be singular. The physical idea we have in mind is that the description of space-time using a set of commuting coordinates is only valid at energy scales smaller than some fundamental one. At higher scales it is impossible to localize a point and a new geometry should be used. As the description by the commuting coordinates breaks down, they must be replaced by elements of a noncommutative algebra. According to the general idea outlined above, a singularity in the metric is due to the extrapolation of the use of commuting coordinates beyond their natural domain of definition into the region where they are physically inappropriate. In a specific example the Kasner manifold has been replaced by a noncommutative algebra, whose Jacobi identities force a modification of the time dependence of the metric [8, 9]. Similar examples have been presented later by other authors [10, 11].

It is well-known that at ordinary scales, which is the arena of particle physics examined mostly in colliders, the Standard Model (SM) of Elementary Particle Physics, which consists of the Strong, Weak and Electromagnetic interactions, has been established. The SM has been successfully formulated using gauge theories, while at much smaller distances the Grand Unified Gauge Theories (GUTs) provide a very attractive unification scheme of the three interactions. The gravitational interaction is not part of this picture, admitting a geometric formulation. However, there exists a gauge-theoretic approach to gravity besides the geometric one [12–23]. This approach started with the pioneer work of Utiyama [12] and was refined by other authors [13–23]; eventually maybe the best description is to consider it as a gauge theory of the de Sitter $SO(1,4)$ group, spontaneously broken by a scalar field to the Lorentz $SO(1,3)$ group [14].

In the noncommutative framework and taking into account the gauge-theoretic description of gravity, the well-established formulation of gauge theories on noncommutative spaces has led to the construction of models of noncommutative gravity [24–34]. In these treatments the authors use the constant noncommutativity (Moyal-Weyl), the formulation of the star-product and the Seiberg-Witten map [35].

In addition to the above treatments, noncommutative gravitational models can be constructed using the noncommutative realization of matrix geometries [36–48], while it should also be noted that there exist alternative approaches [49–51]. Both the latter directions will not be considered here. Our orientation is towards the matrix-realized models. Specifically, we focus on a particular class of noncommutative spaces which are called covariant [52–58], which have the very important property for our purposes that is the preservance of Lorentz covariance [39, 59–61]. In particular we focus to a very interesting class of models which can be constructed on the so-called fuzzy spaces, which is a subclass of noncommutative spaces which preserve the isometries of their commutative analogues. The most typical example of such a space is the fuzzy two-sphere [54], the isometry of which is SO(3) and at the commutative limit the ordinary two-sphere is recovered.

However, a generalization to a higher-dimensional sphere is not straightforward. In particular, in the case of a four-dimensional sphere, the same procedure leads to a number of independent functions which is not a square of an integer. Therefore, one cannot construct a map from functions to matrices. One can restate this difficulty algebraically. Algebras of a fuzzy four-sphere have been constructed in [59] and the difference from the fuzzy two-sphere case is that the commutators of the coordinates do not close in the fuzzy four-sphere case. In [62] (see also [63, 64]), we started a programme realizing gravity as noncommutative gauge theory in three dimensions. In our next contributions in the subject, we worked on the more realistic four-dimensional case [65–71]. Both the constructions and the details involved will be presented in the following.

2. Gauge theories in noncommutative spaces

Before presenting fuzzy gravity, first we need to recall the way gauge theories formulate in noncommutative spaces, according to [72].

The infinitesimal gauge transformation of a scalar field, $\phi(X)$, where X are the coordinates of the noncommutative space, parametrized by $\varepsilon(X)$, will be:

$$\delta\phi(X) = \varepsilon(X)\phi(X). \tag{1}$$

Contrary to the scalar field, the coordinates themselves transform trivially. Because of the latter, the transformation of their product is:

$$\delta(X_\mu\phi(X)) = X_\mu\epsilon(X)\phi(X). \tag{2}$$

Taking into account the noncommutativity of the coordinates, it can be easily shown that the above does not consist a covariant transformation. The resolution of this issue comes with the introduction of the covariant coordinate, which is defined directly through the covariant transformation:

$$\delta\left(\mathcal{X}_\mu\phi(X)\right) \equiv \varepsilon(X)\mathcal{X}_\mu\phi(X). \tag{3}$$

Thanks to the covariant coordinate, the analogy with the ordinary gauge theories is being preserved. Now, from the above covariant transformation, the transformation of the covariant coordinate can be given by the following commutation relation:

$$\delta \mathcal{X}_\mu = \left[\varepsilon(X), \mathcal{X}_\mu\right], \tag{4}$$

which is the covariant transformation of the covariant coordinate itself, that is true by definition. In order to express the covariant coordinate in a more familiar way, it is suggested to introduce a quantity $\mathcal{A}_\mu(X)$, with its transformation rule:

$$\delta \mathcal{A}_\mu(X) = -\left[X_\mu, \varepsilon(X)\right] + \left[\varepsilon(X), \mathcal{A}_\mu(X)\right]. \tag{5}$$

Now, the covariant coordinate can be expressed as $\mathcal{X}_\mu = X_\mu + \mathcal{A}_\mu(X)$, from which it is now clear that $\mathcal{A}_\mu$ plays effectively the role of the gauge connection and, as such, it will be accompanied by a corresponding field strength tensor, which one can in turn find. It is expected that, apart from the defining term of the commutator of the covariant coordinates, the field strength tensor will also contain some extra terms, guaranteeing covariance.

Last, it is necessary to properly treat the anticommutators of the various coordinate-dependent quantities (fields and parameters), as the theory is formulated on a noncommutative space. Specifically, let us consider two elements of an arbitrary algebra, $\varepsilon(X) = \varepsilon^a(X)T_a$ and $\phi(X) = \phi^a(X)T_a$, where T_a are its generators. Their commutator can be expressed as:

$$[\varepsilon, \phi] = \frac{1}{2}\left\{\varepsilon^a, \phi^b\right\}[T_a, T_b] + \frac{1}{2}\left[\varepsilon^a, \phi^b\right]\{T_a, T_b\}. \tag{6}$$

In the ordinary case, where the space is commutative, the last term vanishes as the components ε^a and ϕ^b are ordinary functions of coordinates which naturally commute with each other. On the contrary, when the space is noncommutative, the last term is not vanishing and, thus, the anticommutator of the generators remains in the expression. In the general case, an anticommutator like that will give products that do not belong to the original algebra of the theory, causing the problem that the algebra will not be closing. One possible solution of this problem would be to expand the original algebra of the theory by including all the possible operators that would be produced by the anticommutators as generators. However, the anticommutators of the new generators would also not be closing, leading us to expand again and again the algebra and eventually resulting with an infinite-dimensional algebra. This treatment, although useful in other cases (e.g., in [73], [29] and [30]), is not practical for our cause. Another one -the one that we chose- is to consider the products of the anticommutators to be representation-dependent. In this case, on a specific representation, the anticommutators of the generators will produce finite new operators, and thus, by including them to the algebra, one ends up with an extended algebra, which nevertheless will be of finite dimension.

3. Fuzzy gravity in three dimensions

In the following section, the gauge-theoretic construction of the three-dimensional matrix model of noncommutative gravity will be presented. Of course, in order to build a noncommutative gauge theory, a noncommutative space will be needed to accommodate it. Consequently, we shall firstly specify the appropriate three-dimensional, covariant, noncommutative space that will act as the background for the three-dimensional fuzzy gravity model. Subsequently, the aforementioned three-dimensional gravity model shall be presented, built as a noncommutative gauge theory on the above space.

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The $\mathbb{R}^3_\lambda$ *space*

In order to get to the desired background space, we shall begin from the most well-known covariant, noncommutative space - that is the fuzzy sphere [54, 74]. The fuzzy sphere is defined through its coordinates which are expressed in terms of the three rescaled angular momentum operators $X_i = \lambda J_i$. These consist the Lie algebra generators of a unitary, irreducible representation of SU(2), and satisfy the following relations:

$$[X_i, X_j] = i\lambda\epsilon_{ijk}X_k \ , \quad \sum_{i=1}^{3} X_i X_i = \lambda^2 j(j+1) := r^2 \ , \tag{7}$$

where $i, j, k = 1, \ldots 3$, $\lambda \in \mathbb{R}$ and $2j \in \mathbb{N}$. The second relation of the above comes from the Casimir element. Relaxing this Casimir condition, allowing in turn the coordinates X_i to live in unitary, though reducible representations of SU(2), while at the same time keeping λ fixed, one obtains the three-dimensional noncommutative space known as $\mathbb{R}^3_\lambda$ [75]. This space can be expressed as a direct sum of fuzzy spheres, for every possible radius determined by $2j \in \mathbb{N}$ [75–78]

$$\mathbb{R}^3_\lambda = \sum_{2j\in\mathbb{N}} S^2_{\lambda,j} . \tag{8}$$

Thus, $\mathbb{R}^3_\lambda$ can be thought as a discrete foliation of the three-dimensional Euclidean space by several fuzzy spheres, each of a different radius, with each of them consisting a ‘leaf’ of the foliation [79].

Gauge theory of three-dimensional gravity on $\mathbb{R}^3_\lambda$

Then, the description of a noncommutative version of the three-dimensional gravity will be reviewed, formulated on the space that was presented above.

It has been shown that gravity, in three dimensions, can be successfully formulated as a Chern-Simons gauge theory of the ISO(1,2) group[1], with regard to both the transformations of its gauge fields, as well as its dynamics [23, 62]. This is achieved by introducing the dreibein[2] and the spin connection as gauge fields corresponding to translations and the Lorentz transformations respectively, as well as promoting the common derivative to the appropriate covariant one.

The steps towards the formulation of this noncommutative gauge theory of three-dimensional gravity, are the same that would be followed in order to formulate the corresponding commutative gauge theory [23, 62], albeit in this case, the tools of noncommutative gauge theories (which were mentioned in Section 2) shall be used. This, in turn, means that the covariant derivative shall now contain information about the noncommutative counterparts of the dreibein and spin connection[3].

The first thing that has to be determined, in the process of formulating this three-dimensional fuzzy gravity model on the noncommutative space $\mathbb{R}^3_\lambda$, is the appropriate -starting- gauge group. The relevant group that describes the symmetry of the above fuzzy space, in the Euclidean case, is the SO(4)[4]. This shall lead to a non-abelian noncommutative gauge theory, which in turn will

[1] In the case that a cosmological constant were present, the corresponding algebra would be the de Sitter SO(1,3) or the anti-de Sitter SO(2,2), depending on the sign of the constant.

[2] The dreibein is the three-dimensional case of the vielbein

[3] Similar approaches can be found in [43, 44, 46]

[4] In the Lorentzian case, the relevant group would have been the SO(1,3).

cause the unwelcome feature of the non-closure of the anticommutators of the generators that was described in Section 2. Following the procedure that was explained in that section, gaining motivation by the approach that was followed in the Moyal-Weyl case in [28], the algebra shall be extended appropriately, so that the products of the anticommutators are also included in it.

Having said the above, we begin by considering the spin group of the group of symmetry. In this case, it is the Spin(4) group, which is isomorphic to SU(2)×SU(2). Then, after choosing a specific representation, the elements that are yielded from the anticommutators of the algebra generators in that representation are determined. Subsequently, the above elements are manually included in the algebra as generators, which causes the extension of the SU(2)×SU(2) symmetry to the U(2)×U(2). The latter will be considered as the gauge group of the theory[5]. Since each U(2) is comprised of four generators, given by the Pauli matrices as well as the identity, the U(2)×U(2) gauge group will consist of the following 4×4 matrices

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \; J_0^L = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \; J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}, \; J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \tag{9}$$

where $a = 1, 2, 3$. Still, the identification of the noncommutative dreibein and spin connection in the the expansion of the gauge field should be treated with caution. In order to interpret the above gauge fields correctly, the following linear combinations of the above matrices are considered as generators instead:

$$P_a = \frac{1}{2}\left(J_a^L - J_a^R\right) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \; M_a = \frac{1}{2}\left(J_a^L + J_a^R\right) = \frac{1}{2}\begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \tag{10}$$

as well as

$$\mathbb{I} = J_0^L + J_0^R, \; \gamma_5 = J_0^L - J_0^R. \tag{11}$$

Knowing the commutation and anticommutation relations of the Pauli matrices, the corresponding relations of the above generators are found:

$$\begin{aligned}
&[P_a, P_b] = i\epsilon_{abc}M_c, \quad [P_a, M_b] = i\epsilon_{abc}P_c, \quad [M_a, M_b] = i\epsilon_{abc}M_c, \\
&\{P_a, P_b\} = \frac{1}{2}\delta_{ab}\mathbb{I}, \quad \{P_a, M_b\} = \frac{1}{2}\delta_{ab}\gamma_5, \quad \{M_a, M_b\} = \frac{1}{2}\delta_{ab}\mathbb{I}, \\
&[\gamma_5, P_a] = [\gamma_5, M_a] = 0, \quad \{\gamma_5, P_a\} = 2M_a, \quad \{\gamma_5, M_a\} = 2P_a.
\end{aligned} \tag{12}$$

As was presented in the previous paragraph, regarding the $\mathbb{R}^3_\lambda$ space, the noncommutative coordinates, X_a, will be identified with the three operators which define that fuzzy space. Thus, in the same way that described in Section 2, the covariant coordinates will include information about the deformation of space, through the gauge connection $\mathcal{A}_\mu$, since

$$\mathcal{X}_\mu = \delta_\mu{}^a X_a + \mathcal{A}_\mu, \tag{13}$$

where $\mathcal{A}_\mu$ can be expanded on the generators of the algebra as $\mathcal{A}_\mu = \mathcal{A}_\mu^I(X) \otimes T^I$, T^I being the generators with $I = 1, \ldots, 8$, and $\mathcal{A}_\mu^I$ the U(2)×U(2)-valued gauge fields. It should be noted that

[5]Accordingly, in the Lorentzian case the initial SL(2; $\mathbb{C}$) symmetry would be enlarged to GL(2; $\mathbb{C}$)

the component gauge fields are no longer functions of coordinates in a classical manifold, but are now operator-valued (since the coordinates got promoted), while the generators are represented by 4×4 matrices. This explains the tensor products between the component fields and the generators. According to the above, the explicit expansion of the gauge connection over the generators will be

$$\mathcal{A}_\mu(X) = e_\mu{}^a(X) \otimes P_a + \omega_\mu{}^a(X) \otimes M_a + A_\mu(X) \otimes i\mathbb{I} + \tilde{A}_\mu(X) \otimes \gamma_5, \tag{14}$$

consequently leading to the explicit expansion of the covariant coordinate

$$\mathcal{X}_\mu = X_\mu \otimes i\mathbb{I} + e_\mu{}^a(X) \otimes P_a + \omega_\mu{}^a(X) \otimes M_a + A_\mu(X) \otimes i\mathbb{I} + \tilde{A}_\mu(X) \otimes \gamma_5. \tag{15}$$

Similarly, since the gauge parameter $\varepsilon(X)$ is also an element of the algebra, it will also be expanded on its generators as

$$\varepsilon(X) = \xi^a(X) \otimes P_a + \lambda^a(X) \otimes M_a + \varepsilon_0(X) \otimes i\mathbb{I} + \tilde{\varepsilon}_0(X) \otimes \gamma_5. \tag{16}$$

Next, we proceed with the calculation of the transformations of the gauge fields. Having the above explicit expansions, we can use the relations (5) and (6), to reach the explicit transformation relations for each of the gauge fields:

$$\begin{aligned}
\delta e_\mu{}^a =& - i \left[X_\mu + A_\mu, \xi^a\right] + \frac{i}{2}\left\{\xi_b, \omega_{\mu c}\right\}\epsilon^{abc} + \frac{i}{2}\left\{\lambda_b, e_{\mu c}\right\}\epsilon^{abc} \\
&+ i\left[\varepsilon_0, e_\mu{}^a\right] + \left[\lambda^a, \tilde{A}_\mu\right] + \left[\tilde{\varepsilon}_0, \omega_\mu{}^a\right], \\
\delta \omega_\mu{}^a =& - i \left[X_\mu + A_\mu, \lambda^a\right] + \frac{i}{2}\left\{\xi_b, e_{\mu c}\right\}\epsilon^{abc} + \frac{i}{2}\left\{\lambda_b, \omega_{\mu c}\right\}\epsilon^{abc} \\
&+ i\left[\varepsilon_0, \omega_\mu{}^a\right] + \left[\xi^a, \tilde{A}_\mu\right] + \left[\tilde{\varepsilon}_0, e_\mu{}^a\right], \\
\delta A_\mu =& - i\left[X_\mu + A_\mu, \varepsilon_0\right] - \frac{i}{4}\left[\xi^a, e_{\mu a}\right] - \frac{i}{4}\left[\lambda^a, \omega_{\mu a}\right] - i\left[\tilde{\varepsilon}_0, \tilde{A}_\mu\right], \\
\delta \tilde{A}_\mu =& - i\left[X_\mu + A_\mu, \tilde{\varepsilon}_0\right] + \frac{1}{4}\left[\xi^a, \omega_{\mu a}\right] + \frac{1}{4}\left[\lambda^a, e_{\mu a}\right] + i\left[\varepsilon_0, \tilde{A}_\mu\right].
\end{aligned} \tag{17}$$

At this point, we shall take a moment to comment on the behaviour of the above transformation relations, when the Abelian and the commutative limit are considered.

First, let us consider the case in which an Abelian gauge group was chosen, or in other words, the gauge group that was used were an Abelian U(1) group. Naturally, this would lead to an Abelian gauge theory on the chosen fuzzy space, which effectively amounts to setting $e_\mu{}^a, \omega_\mu{}^a, \tilde{A}_\mu$, as well as their corresponding parameters $\xi^a, \lambda^a, \tilde{\varepsilon}_0$ equal to zero, leaving A_μ as the only non-vanishing gauge field and ε_0 as the only non-vanishing gauge parameter. Consequently, the only non-trivial transformation out of the above would be:

$$\delta A_\mu = -i[X_\mu, \varepsilon_0] + i[\varepsilon_0, A_\mu], \tag{18}$$

which is the anticipated transformation of a noncommutative Maxwell gauge field. Therefore, it is manifested that the Maxwell sector is always present in the theory, intrinsically related to the noncommutative nature of the background space, independently of whether the dreibein is trivial or not, with the covariant coordinate being $X_\mu + A_\mu$.

On the other hand, when the commutative limit is considered, the interplay between gravity-related and Yang-Mills fields ceases to exist, consequently making the gauge fields that were introduced due to the noncommutativity, A_μ and $\tilde{A}_\mu$, vanish. In turn, the inner derivation reduces to the commutative one, that is $[X_\mu, f] \to -i\partial_\mu f$, therefore leading to the following transformation rules of the dreibein and spin connection:

$$\delta e_\mu{}^a = -\partial_\mu \xi^a - \epsilon^{abc}(-i\xi_b\omega_{\mu c} - i\lambda_b e_{\mu c}) \tag{19}$$

$$\delta \omega_\mu{}^a = -\partial_\mu \lambda^a - \epsilon^{abc}(-i\lambda_b\omega_{\mu c} - i\xi_b e_{\mu c}) \tag{20}$$

Upon closer inspection, it is observed that the above relations closely resemble the corresponding ones in the commutative case as they are given in [62]:

$$\delta e_\mu{}^a|_{com} = \partial_\mu \xi^a - \epsilon^{abc}(\xi_b\omega_{\mu c} + \lambda_b e_{\mu c}), \tag{21}$$

$$\delta \omega_\mu{}^a|_{com} = \partial_\mu \lambda^a - \epsilon^{abc}(\lambda_b\omega_{\mu c} + \Lambda\xi_b e_{\mu c}). \tag{22}$$

More specifically, after performing the following rescalings on the generators:

$$P_a \to -\frac{i}{\sqrt{\Lambda}}P_a, \; M_a \to iM_a,$$

as well as on the gauge fields and parameters:

$$e_\mu{}^a \to i\sqrt{\Lambda}e_\mu{}^a, \; \xi^a \to -i\sqrt{\Lambda}e_\mu{}^a, \; \omega_\mu{}^a \to -i\omega_\mu{}^a, \; \lambda_\mu{}^a \to i\lambda_\mu{}^a,$$

the transformations of the noncommutative dreibein and spin connection (19) and (20) at the commutative limit, exactly coincide with their commutative counterparts (21) and (22). Thus, it becomes apparent that in the commutative limit, the transformations of the gauge fields of the three-dimensional gravity, presented in [23], are recovered.

Resuming the formulation of the fuzzy gravity model, the curvature tensor of the theory has to be obtained. This is accomplished by using the usual formula of calculating the commutator of the covariant derivatives, which in our case are the covariant coordinates. It should be noted that since the right hand side of the commutator of the coordinates is linear with respect to the coordinates - as shown in the first equation of (7) - an additional linear term should be included in the definition of the curvature as indicated below:

$$\mathcal{R}_{\mu\nu}(X) = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\lambda\epsilon_{\mu\nu\rho}\mathcal{X}^\rho. \tag{23}$$

The curvature tensor $\mathcal{R}_{\mu\nu}$ is, too, an element of the U(2)×U(2) and as such, it can be expanded on the algebra's generators:

$$\mathcal{R}_{\mu\nu}(X) = T_{\mu\nu}{}^a(X)\otimes P_a + R_{\mu\nu}{}^a(X)\otimes M_a + F_{\mu\nu}(X)\otimes i\mathbb{I} + \tilde{F}_{\mu\nu}(X)\otimes\gamma_5. \tag{24}$$

Following a similar procedure as when calculating the transformation laws of the gauge fields, using the definition of the curvature (23), together with the expansions of the curvature tensor and the covariant coordinate (24) and (15) respectively, the component curvature tensors are calculated:

$$
\begin{aligned}
T_{\mu\nu}{}^{a} =& i\left[X_\mu + A_\mu, e_\nu{}^a\right] - i\left[X_\nu + A_\nu, e_\mu{}^a\right] + \frac{i}{2}\left\{e_{\mu b}, \omega_{\nu c}\right\}\epsilon^{abc} + \frac{i}{2}\left\{\omega_{\mu b}, e_{\nu c}\right\}\epsilon^{abc} \\
&+ \left[\omega_\mu{}^a, \tilde{A}_\nu\right] - \left[\omega_\nu{}^a, \tilde{A}_\mu\right] - i\lambda\epsilon_{\mu\nu\rho}e^{\rho a}, \\
R_{\mu\nu}{}^{a} =& i\left[X_\mu + A_\mu, \omega_\nu{}^a\right] - i\left[X_\nu + A_\nu, \omega_\mu{}^a\right] + \frac{i}{2}\left\{\omega_{\mu b}, \omega_{\nu c}\right\}\epsilon^{abc} + \frac{i}{2}\left\{e_{\mu b}, e_{\nu c}\right\}\epsilon^{abc} \\
&+ \left[e_\mu{}^a, \tilde{A}_\nu\right] - \left[e_\nu{}^a, \tilde{A}_\mu\right] - i\lambda\epsilon_{\mu\nu\rho}\omega^{\rho a}, \\
F_{\mu\nu} =& i\left[X_\mu + A_\mu, X_\nu + A_\nu\right] - \frac{i}{4}\left[e_\mu{}^a, e_{\nu a}\right] - \frac{i}{4}\left[\omega_\mu{}^a, \omega_{\nu a}\right] - i\left[\tilde{A}_\mu, \tilde{A}_\nu\right] \\
&- i\lambda\epsilon_{\mu\nu\rho}\left(X^\rho + A^\rho\right), \\
\tilde{F}_{\mu\nu} =& i\left[X_\mu + A_\mu, \tilde{A}_\nu\right] - i\left[X_\nu + A_\nu, \tilde{A}_\mu\right] + \frac{1}{4}\left[e_\mu{}^a, \omega_{\nu a}\right] + \frac{1}{4}\left[\omega_\mu{}^a, e_{\nu a}\right] \\
&+ -i\lambda\epsilon_{\mu\nu\rho}\tilde{A}^\rho.
\end{aligned}
\tag{25}
$$

Once again, when the commutative limit as well as the rescalings that were mentioned before are considered, the corresponding tensors of the commutative case (as presented in [62]) are recovered:

$$T_{\mu\nu}{}^{a}|_{com} = 2\partial_{[\mu}e_{\nu]}{}^a + 2\epsilon^{abc}\omega_{[\mu b}e_{\nu]c}\,, \tag{26}$$

$$R_{\mu\nu}{}^{a}|_{com} = 2\partial_{[\mu}\omega_{\nu]}{}^a + \epsilon^{abc}(\omega_{\mu b}\omega_{\nu c} + \Lambda e_{\mu c}e_{\nu c}), \tag{27}$$

exactly as expected.

Action of three-dimensional fuzzy gravity

Reaching the conclusion of the three-dimensional case, an action for the aforementioned theory should be found. Once again inspiration is gained by the commutative gauge-theoretic approach of three-dimensional gravity, leading to the well-motivated choice of an action of Chern-Simons type. For the Euclidean case, which we have discussed so far, the suggested action is:

$$S_0 = \frac{1}{g^2}\,\mathrm{Tr}\left(\frac{i}{3}\epsilon^{\mu\nu\rho}X_\mu X_\nu X_\rho - m^2 X_\mu X^\mu\right), \tag{28}$$

which, following its variation, leads to the field equation:

$$\left[X_\mu, X_\nu\right] + 2im^2\epsilon_{\mu\nu\rho}X^\rho = 0. \tag{29}$$

The above field equation admits the space $\mathbb{R}^3_\lambda$ that we have used as a solution, for $2m^2 = -\lambda$.

Next, the gauge fields need to be introduced in the aforementioned action. To that end, there are two possible paths one could follow. The first path would be to consider fluctuations of the above equation of motion by replacing the coordinates with their covariant counterparts. The other, less straightforward path, would be to replace the coordinates in the action with the covariant coordinates and *then* complete its variation, in order to obtain the field equations in terms of the gauge fields. One way or another, the action will eventually be written in terms of the gauge fields itself and because of that an additional trace, **tr**, over the gauge indices should be involved in its expression. Consequently, the proposed action is:

$$S = \frac{1}{g^2}\,\mathrm{Tr}\,\mathrm{tr}\left(\frac{i}{3}\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{X}_\nu\mathcal{X}_\rho + \frac{\lambda}{2}\mathcal{X}_\mu\mathcal{X}^\mu\right), \tag{30}$$

where the first trace is over the matrices X and the second over the generators of the gauge group. The above action can be rewritten as:

$$\begin{aligned} S &= \frac{1}{6g^2}\operatorname{Tr}\operatorname{tr}(i\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{R}_{\nu\rho}) + \frac{\lambda}{6g^2}\operatorname{Tr}\operatorname{tr}(\mathcal{X}_\mu\mathcal{X}^\mu) \\ &= \mathcal{S} + S_\lambda, \end{aligned} \tag{31}$$

where all the λ-related terms have been isolated in S_λ, which vanishes for $\lambda \to 0$.

Now, following the calculation of the traces over the gauge indices of the relations (12) of the generators of the algebra, it is found that the only non-vanishing ones are the following:

$$tr(P_a P_b) = \delta_{ab}, \quad tr(M_a M_b) = \delta_{ab}. \tag{32}$$

Consequently, the first term $\mathcal{S}$ of the above action turns out to be equal to:

$$\mathcal{S} = \frac{i}{6g^2}\operatorname{Tr}\epsilon^{\mu\nu\rho}(e_{\mu a}T_{\nu\rho}{}^a + \omega_{\mu a}R_{\nu\rho}{}^a - 4(X_\mu + A_\mu)F_{\nu\rho} + 4\tilde{A}_\mu\tilde{F}_{\nu\rho}). \tag{33}$$

This action is similar to the one presented in Ref.[23]; when the commutative limit is considered and the rescalings that were mentioned before are applied, the first two terms of the above action are identical to the one presented in [23]. Nevertheless, in this case, an additional sector is unavoidably obtained. This sector is evidently associated with the additional gauge fields, which cannot decouple in the noncommutative case.

Concluding, variation of the action (31) with respect to the covariant coordinate yields the following field equations:

$$T_{\mu\nu}{}^a = 0, \quad R_{\mu\nu}{}^a = 0, \quad F_{\mu\nu} = 0, \quad \tilde{F}_{\mu\nu} = 0. \tag{34}$$

At this point, it is noted that the same equations of motion are obtained, following the variation of (31) with respect to the gauge fields, after using the algebra trace and replacing the tensors with their expansions on the generators of the algebra (25).

4. Fuzzy gravity in four dimensions

In this section, we are going to present the construction of the four-dimensional matrix model of gravity as a noncommutative gauge theory, after establishing the corresponding noncommutative space on which the model will be constructed.

A Fuzzy Version of the Four-Sphere

The noncommutative space on which we are going to construct our model is the fuzzy four-sphere, S_F^4. The fuzzy four-sphere is the four-dimensional analogue of the fuzzy sphere, S_F^2, which is the discrete matrix approximation of the common sphere. Drawing lessons from the ordinary gauge-theoretic approaches of gravitational theories mentioned earlier, it would be plausible to examine first the $SO(5)$ group, as it consists the corresponding isometry group and as such, a subset of its generators could be identified as the coordinate operators. Nevertheless, this group is not suitable for performing the above identification because the subalgebra is not closing. Because of the latter, covariance is not being preserved and thus a larger group is needed [58]. The larger group, that is needed to be used instead, has to possess the property of incorporating all generators and the noncommutativity -with some appropriate identification- in it. In this case, the preservation of the covariance will be manifested as the coordinate operators will transform as vectors under rotational transformations. Eventually, taking into consideration the above arguments, the resulting, minimally extended, group is the $SO(6)$ [65], [68], the generators of which obey the following algebra:

$$[J_{AB}, J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{BC}J_{AD} - \delta_{AD}J_{BC}). \tag{35}$$

Reading the above generators in an $SO(4)$ notation, the following definitions take place:

$$J_{\mu\nu} = \frac{1}{\hbar}\Theta_{\mu\nu}, \qquad J_{\mu 5} = \frac{1}{\lambda}X_5, \qquad J_{\mu 6} = \frac{\lambda}{2\hbar}P_\mu, \qquad J_{56} = \frac{1}{2}h, \tag{36}$$

where $\mu, \nu = 1, ..., 4$, λ is a parameter of non-trivial dimensions and h is an operator inheriting the information of the radius constraint of the fuzzy four-sphere. The X_μ, P_μ and $\Theta_{\mu\nu}$ are identified as coordinates, momenta and noncommutativity tensor respectively. In terms of the above identifications, the above algebra leads to the following commutation relations:

$$[X_\mu, X_\nu] = i\frac{\lambda^2}{\hbar}\Theta_{\mu\nu}, \qquad [P_\mu, P_\nu] = 4i\frac{\hbar}{\lambda^2}\Theta_{\mu\nu}, \tag{37}$$

$$[X_\mu, P_\nu] = i\hbar\delta_{\mu\nu}h, \qquad [X_\mu, h] = i\frac{\lambda^2}{\hbar}P_\mu, \tag{38}$$

$$[P_\mu, h] = 4i\frac{\hbar}{\lambda^2}X_\mu. \tag{39}$$

From the above first two relations, it is clear that the coordinates and momenta both close to an $SO(4)$ subalgebra of the total $SO(6)$. The rest of the commutation relations, which correspond to the spacetime transformations, are:

$$[\Theta_{\mu\nu}, \Theta_{\rho\sigma}] = i\hbar(\delta_{\mu\rho}\Theta_{\nu\sigma} + \delta_{\nu\sigma}\Theta_{\mu\rho} - \delta_{\nu\rho}\Theta_{\mu\sigma} - \delta_{\mu\sigma}\Theta_{\nu\rho}, \tag{40}$$

$$[X_\mu, \Theta_{\nu\rho}] = i\hbar(\delta_{\mu\rho}X_\nu - \delta_{\mu\nu}X_\rho), \tag{41}$$

$$[P_\mu, \Theta_{\nu\rho}] = i\hbar(\delta_{\mu\rho}P_\nu - \delta_{\mu\nu}P_\rho) \tag{42}$$

$$[h, \Theta_{\mu\nu}] = 0. \tag{43}$$

From the first one, the establishment of the $SO(4)$ subalgebra of rotations is understood, while from the second and third ones the vector-like transformation of the coordinates and the momenta, under rotational transformations, is described.

Last, by inspection of the relation between the coordinates and momenta, first one of eq.(38), the underlying quantum structure of the noncommutative space is manifested. This is a crucial observation, that a Heisenberg-like relation is part of this picture, which, along with the fact that the participating generators of the algebra are represented by matrices of finite dimensions, allows the noncommutative space to admit the interpretation of a finite quantum system, hence motivating us to use it as the space on which the four-dimensional gravity model is constructed.

Gauge Group and Representation

As discussed in the previous subsection, the noncommutative framework includes the anti-commutators of the generators, as they do not vanish like in the continuous case. The solution that we choose, as explained above, is the extension of the initial symmetry and the fixing of a specific representation, in order that the new operators to be included as generators and produce a finite-dimensional algebra that is closing. In our case, the extension of $SO(5)$ leads to the $SO(6) \times U(1)$ gauge group with its generators being represented by 4×4 matrices:

$$M_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \quad K_a = \frac{1}{2}\Gamma_a, \quad P_a = -\frac{i}{2}\Gamma_a\Gamma_5, \quad D = -\frac{1}{2}\Gamma_5, \quad \mathbb{I}_4, \tag{44}$$

where the Γ matrices are the 4×4 gamma matrices in the Euclidean signature and form the anticommutation relation $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbb{I}_4$, where $a, b = 1, \ldots, 4$ and $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$. The full algebra of the generators of the extended gauge group, along with their anticommutators is:

$$\begin{aligned}
&[K_a, K_b] = iM_{ab}, \quad [P_a, P_b] = iM_{ab},\\
&[P_a, D] = iK_a, \quad [K_a, P_b] = i\delta_{ab}D, \quad [K_a, D] = -iP_a,\\
&[K_a, M_{bc}] = i\left(\delta_{ac}K_b - \delta_{ab}K_c\right),\\
&[P_a, M_{bc}] = i\left(\delta_{ac}P_b - \delta_{ab}P_c\right),\\
&[M_{ab}, M_{cd}] = i\left(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}\right),\\
&[D, M_{ab}] = 0,\\
&\{M_{ab}, M_{cd}\} = \frac{1}{8}\left(\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}\right)\mathbb{I}_4 - \frac{\sqrt{2}}{4}\epsilon_{abcd}D,\\
&\{M_{ab}, K_c\} = \sqrt{2}\epsilon_{abcd}P_d, \quad \{M_{ab}, P_c\} = -\frac{\sqrt{2}}{4}\epsilon_{abcd}K_d,\\
&\{K_a, K_b\} = \frac{1}{2}\delta_{ab}\mathbb{I}_4, \quad \{P_a, P_b\} = \frac{1}{8}\delta_{ab}\mathbb{I}_4, \quad \{K_a, D\} = \{P_a, D\} = 0,\\
&\{P_a, K_b\} = \{M_{ab}, D\} = -\frac{\sqrt{2}}{2}\epsilon_{abcd}M_{cd}\,.
\end{aligned} \tag{45}$$

Action and Equations of Motion

The action we begin with is the following:

$$\mathcal{S} = \mathrm{Tr}\left(\left[X_\mu, X_\nu\right] - \kappa^2\Theta_{\mu\nu}\right)\left(\left[X_\rho, X_\sigma\right] - \kappa^2\Theta_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}. \tag{46}$$

In order to extract the field equations, we make variations of this action with respect to the X and Θ fields. The equations produced respectively are the following:

$$\epsilon^{\mu\nu\rho\sigma}\left[X_\nu, \left[X_\rho, X_\sigma\right] - \kappa^2\Theta_{\rho\sigma}\right] = 0\,, \;\; \epsilon^{\mu\nu\rho\sigma}\left(\left[X_\rho, X_\sigma\right] - \kappa^2\Theta_{\rho\sigma}\right) = 0\,. \tag{47}$$

We can already observe through the second equation that, when $\kappa^2 = \frac{i\lambda^2}{\hbar}$, the noncommutativity of the space is being recovered and consequently the first one is trivially satisfied.

Now, we move on with examining a dynamical version of the above action. In order to check the analogy with the commutative case, we need to express the action in terms of the curvature field strength tensor. To accomplish it, we introduce the gauge fields treating them as fluctuations of the X and Θ fields:

$$\begin{aligned}\mathcal{S} = \mathrm{Trtr}\, \epsilon^{\mu\nu\rho\sigma} &\left(\left[X_\mu + A_\mu, X_\nu + A_\nu\right] - \kappa^2 \left(\Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}\right)\right) \\ &\cdot \left(\left[X_\rho + A_\rho, X_\sigma + A_\sigma\right] - \kappa^2 \left(\Theta_{\rho\sigma} + \mathcal{B}_{\rho\sigma}\right)\right),\end{aligned} \tag{48}$$

where a trace over the gauge algebra has been included as well.

Now, for later convenience, we define:

- $\mathcal{X}_\mu = X_\mu + A_\mu$, as the covariant coordinate of the noncommutative gauge theory, where A_μ is the gauge connection,
- $\hat{\Theta}_{\mu\nu} = \Theta_{\mu\nu} + \mathcal{B}_{\mu\nu}$, as the covariant noncommutative tensor, where $\mathcal{B}_{\mu\nu}$ is a 2-form field,
- $\mathcal{R}_{\mu\nu} = \left[\mathcal{X}_\mu, \mathcal{X}_\nu\right] - \kappa^2 \hat{\Theta}_{\mu\nu}$, as the field strength tensor of the theory.

Finally we make the replacement $\kappa^2 = \frac{i\lambda^2}{\hbar}$ and get the following expression of the action:

$$\mathcal{S} = \mathrm{Trtr}\left(\left[\mathcal{X}_\mu, \mathcal{X}_\nu\right] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\mu\nu}\right)\left(\left[\mathcal{X}_\rho, \mathcal{X}_\sigma\right] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma} := \mathrm{Trtr}\, \mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,. \tag{49}$$

The above expression is a noncommutative analogue of the four-dimensional Chern-Simons action, in terms of its structure. Performing variations with respect to $\mathcal{X}$ and $\mathcal{B}$ fields, we are lead to the field equations:

$$\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\rho\sigma} = 0\,,\ \ \epsilon^{\mu\nu\rho\sigma}\left[\mathcal{X}_\nu, \mathcal{R}_{\rho\sigma}\right] = 0\,. \tag{50}$$

The first equation implies the vanishing of the curvature tensor. The second one can be interpreted as the noncommutative counterpart of the Bianchi identity.

Before we proceed with the spontaneous symmetry breaking of the action we need to write down some expressions and specifically the field strength tensor's decomposition on the generators,

$$\mathcal{R}_{\mu\nu}(X) = \tilde{R}_{\mu\nu}{}^a \otimes P_a + R_{\mu\nu}{}^{ab} \otimes M_{ab} + R_{\mu\nu}{}^a \otimes K_a + \tilde{R}_{\mu\nu} \otimes D + R_{\mu\nu} \otimes \mathbb{I}_4 \tag{51}$$

and the explicit expressions of its component tensors:

$$\tilde{R}_{\mu\nu}{}^{a} = \left[X_\mu + a_\mu, e_\nu{}^a\right] - \left[X_\nu + a_\nu, e_\mu{}^a\right] - \frac{i}{2}\left\{b_\mu{}^a, \tilde{a}_\nu\right\} + \frac{i}{2}\left\{b_\nu{}^a, \tilde{a}_\mu\right\}$$
$$- \frac{\sqrt{2}}{2}\left(\left[b_\mu{}^b, \omega_\nu{}^{cd}\right] - \left[b_\nu{}^b, \omega_\mu{}^{cd}\right]\right)\epsilon_{abcd} - i\left\{\omega_\mu{}^{ab}, e_{\nu b}\right\} + i\left\{\omega_\nu{}^{ab}, e_{\mu b}\right\} - \frac{i\lambda^2}{\hbar}\tilde{B}_{\mu\nu}{}^a\,,$$
$$R_{\mu\nu}{}^{ab} = \left[X_\mu + a_\mu, \omega_\nu{}^{ab}\right] - \left[X_\nu + a_\nu, \omega_\mu{}^{ab}\right] + \frac{i}{2}\left\{b_\mu{}^a, b_\nu{}^b\right\} + \frac{\sqrt{2}}{4}\left(\left[b_\mu{}^c, e_\nu{}^d\right] - \left[b_\nu{}^c, e_\mu{}^d\right]\right)\epsilon_{abcd}$$
$$- \frac{\sqrt{2}}{4}\left(\left[\tilde{a}_\mu, \omega_\nu{}^{cd}\right] - \left[\tilde{a}_\nu, \omega_\mu{}^{cd}\right]\right)\epsilon_{abcd} + 2i\left\{\omega_\mu{}^{ac}, \omega_\nu{}^b\right\} + \frac{i}{2}\left\{e_\mu{}^a, e_\nu{}^b\right\} - \frac{i\lambda^2}{\hbar}B_{\mu\nu}{}^{ab}\,,$$
$$R_{\mu\nu}{}^{a} = \left[X_\mu + a_\mu, b_\nu{}^a\right] - \left[X_\nu + a_\nu, b_\mu{}^a\right] + i\left\{b_{\mu b}, \omega_\mu{}^{ab}\right\} - i\left\{b_{\nu b}, \omega_\mu{}^{ab}\right\}$$
$$- \frac{i}{2}\left\{\tilde{a}_\mu, e_\nu{}^a\right\} + \frac{i}{2}\left\{\tilde{a}_\nu, e_\mu{}^a\right\} + \frac{\sqrt{2}}{8}\epsilon_{abcd}\left(\left[e_\mu{}^b, \omega_\nu{}^{cd}\right] - \left[e_\nu{}^b, \omega_\mu{}^{cd}\right]\right) - \frac{i\lambda^2}{\hbar}B_{\mu\nu}{}^a\,,$$
$$\tilde{R}_{\mu\nu} = \left[X_\mu + a_\mu, \tilde{a}_\nu\right] - \left[X_\nu + a_\nu, \tilde{a}_\mu\right] + \frac{i}{2}\left\{b_{\mu a}, e_\nu{}^a\right\} - \frac{i}{2}\left\{b_{\nu a}, e_\mu{}^a\right\}$$
$$- \frac{\sqrt{2}}{8}\epsilon_{abcd}\left[\omega_\mu{}^{ab}, m\omega_\nu{}^{cd}\right] - \frac{i\lambda^2}{\hbar}\tilde{B}_{\mu\nu}\,,$$
$$R_{\mu\nu} = \left[X_\mu, a_\nu\right] - \left[X_\nu, a_\mu\right] + \left[a_\mu, a_\nu\right] + \frac{1}{4}\left[b_\mu{}^a, b_{\nu a}\right] + \frac{1}{4}\left[\tilde{a}_\mu, \tilde{a}_\nu\right] + \frac{1}{8}\left[\omega_\mu{}^{ab}, \omega_{\nu ab}\right]$$
$$+ \frac{1}{16}\left[e_{\mu a}, e_\nu{}^a\right] - \frac{i\lambda^2}{\hbar}B_{\mu\nu}\,.$$

Spontaneous Symmetry Breaking of the Noncommutative Action

The action (49) breaks spontaneously after being modified by the introduction of a scalar field, Φ, along with a dimensionful parameter, λ. After this modification, the action becomes:

$$\mathcal{S} = \mathrm{Trtr}_G\, \lambda\Phi(X)\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + \eta\left(\Phi(X)^2 - \lambda^{-2}\mathbf{I}_N \otimes \mathbf{I}_4\right), \tag{52}$$

where η is a Lagrange multiplier. In the on-shell case, the following condition holds:

$$\Phi^2(X) = \lambda^{-2}\mathbb{I}_N \otimes \mathbb{I}_4. \tag{53}$$

In this case the above action, (52), coincides with the dynamic action earlier introduced, (49). Variation of the latter with respect to the Lagrange multiplier results to the above constraint equation as a field equation.

Considering that the scalar field, Φ, consists only of the symmetric part of the decomposition on the generators, it can be expressed as:

$$\Phi(X) = \tilde{\phi}^a(X) \otimes P_a + \phi^a(X) \otimes K_a + \phi(X) \otimes \mathbb{I}_4 + \tilde{\phi}(X) \otimes D\,.$$

In order for the breaking to occur, we make the following gauge fixing of the scalar field Φ, picking a gauge in the direction of the generator D:

$$\Phi(X) = \tilde{\phi}(X) \otimes D|_{\tilde{\phi}=-2\lambda^{-1}} = -2\lambda^{-1}\mathbf{I}_N \otimes D.$$

Substituting to the modified action, (52), and calculating the trace over the algebra, we get the broken symmetry bearing action:

$$\mathcal{S}_{\mathrm{br}} = \mathrm{Tr}\left(\frac{\sqrt{2}}{4}\epsilon_{abcd}R_{\mu\nu}{}^{ab}R_{\rho\sigma}{}^{cd} - 4R_{\mu\nu}\tilde{R}_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}. \tag{54}$$

Due to the fact that we considered the scalar field consisting of only the symmetric part of its decomposition in terms of the generators of $SO(6)$ and, thus, not being charged under $U(1)$, the resulting gauge symmetry of the broken action is $SO(4) \times U(1)$. In other words, seven of the initial sixteen generators remain unbroken. These are: a) the translations' generators, P_a, which lead to the torsionless condition, $\tilde{R}_{\mu\nu}{}^a = 0$ that results to relation between ω, e and $\tilde{a}$, b) the K_a generators, which lead to $R_{\mu\nu}{}^a = 0$ that implies a proportionality relation between e, b gauge fields, and c) the D generator which requires the gauge fixing of $\tilde{a}_\mu = 0$ [80]. In conclusion, the remaining symmetry of the spontaneously broken action is $SO(4) \times U(1)$ and the only remaining independent fields are the e and a gauge fields. Finally, the resulting expression of the component tensor $R_{\mu\nu}{}^{ab}$, after the replacements $\tilde{a}_\mu = 0$ and $b_\mu{}^a = \frac{i}{2} e_\mu{}^a$, is

$$\begin{aligned} R_{\mu\nu}{}^{ab} = &\left[X_\mu + a_\mu, \omega_\nu{}^{ab}\right] - \left[X_\nu + a_\nu, \omega_\mu{}^{ab}\right] + i\left\{\omega_\mu{}^{ac}, \omega_\nu{}^b{}_c\right\} - i\left\{\omega_\mu{}^{bc}, \omega_{\nu c}{}^a\right\} \\ &+ \frac{3i}{8}\left\{e_\mu{}^a, e_\nu{}^b\right\} - \frac{i\lambda^2}{\hbar} B_{\mu\nu}{}^{ab}. \end{aligned}$$

The Commutative Limit

Naturally, in order for the theory to be valid, we need to examine the behaviour of the constructed gravity model in the commutative limit. In other words, in the low-energy regime where noncommutativity becomes negligible, the predictions of the model must coincide with the ones of GR. Although such an assumption is not completely realistic, as noncommutativity-related effects would still have an effect even at low-energy scales, it is important for the consistency of the theory its results to agree with the ones of GR when one considers noncommutativity to be vanishing. Because of that, we are going to examine now the theory at the level of the vanishing of all its noncommutative-related features. In order to do that, firstly we choose the noncommutative space to have Lorentzian signature, which implies that we work in the fuzzy dS^4 space. Following, we make some considerations:

a The 2-form field $\mathcal{B}_{\mu\nu}$ and the a_μ must decouple, because the first one is related with the preservation of covariance of the noncommutative space and the second one is related with the extension of the group for the anticommutators of the generators to be closing;

b When the noncommutativity vanishes, the commutators of functions vanish as well, $[f(x), g(x)] \to 0$, and their anticommutators become double products, $\{f(x), g(x)\} \to 2f(x)g(x)$;

c The inner derivation coincides with the simple derivative: $\left[X_\mu, f\right] \to \partial_\mu f$ and the traces with integrations, $\frac{\sqrt{2}}{4}\,\mathrm{Tr} \to \int d^4x$;

d In the fixed gauge of the spontaneous symmetry breaking, the D-related component tensor $\tilde{R}_{\mu\nu}$ of the field strength tensor reduces to:

$$\tilde{R}_{\mu\nu} = -\frac{\sqrt{2}}{8}\epsilon_{abcd}\left[\omega_\mu{}^{ab}, \omega_\nu{}^{cd}\right] - \frac{i\lambda^2}{\hbar}\tilde{B}_{\mu\nu}.$$

Thus, the second term of the spontaneously broken action, (54), vanishes as it contains the commutator of the spin connection. Furthermore, because of [a], a_μ will not be included in the first term of the action.

e Finally, we make the following reparametrizations:

$$e_\mu{}^a \to ime_\mu{}^a, \quad P_a \to -\frac{i}{m}P_a, \quad \tilde{R}_{\mu\nu}{}^a \to imT_{\mu\nu}{}^a$$
$$\omega_\mu{}^{ab} \to -\frac{i}{2}\omega_\mu{}^{ab}, \quad M_{ab} \to 2iM_{ab}, \quad R_{\mu\nu}{}^{ab} \to -\frac{i}{2}R_{\mu\nu}{}^{ab},$$

where m is an arbitrary, complex constant with dimension $[L]^{-1}$ introduced for $e_\mu{}^a$ to remain dimensionless in the commutative limit. Its introduction is crucial so that $e_\mu{}^a$ can admit the interpretation of the actual vielbein field.

Adopting all of the above, we take the following expression of the torsion tensor, $\tilde{R}_{\mu\nu}{}^a$:

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^{ab} e_{\nu b} + \omega_\nu{}^{ab} e_{\mu b} = 0,$$

which totally coincides with the torsionless condition of the first-order formulation of GR, as shown in [66]. Due to that, the relation between e and ω fields will also coincide to the one of the first-order formulation of GR.

Proceeding with the curvature 2-form, $R_{\mu\nu}{}^{ab}$, we result to the following form:

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\mu{}^{bc} \omega_\nu{}^a{}_c + \frac{3}{2}m^2 e_\mu{}^a e_\nu{}^b = R^{(0)\,ab}_{\mu\nu} + \frac{3}{2}m^2 e_\mu{}^a e_\nu{}^b\,.$$

Again, as shown in [66], the above expression coincides with the one of the first-order formulation of GR, but, with the exception that it also contains an extra term that involves only the vielbein fields.

Last, as commented earlier, because of the vanishing of its second term, the action (54), will now only consist of its first one. In the commutative limit, the action is only Lorentz-invariant, as the initial symmetry has been spontaneously broken. The final expression of the action is going to be of a form initially proposed by MacDowell-Mansouri, that finally leads to the Palatini action - the gauge-theoretic equivalent of the Einstein-Hilbert action.

5. Conclusions

A possible way to resolve the singularities of general relativity was proposed based on the assumption that the description of space-time using commuting coordinates is not valid above a certain fundamental scale. Beyond that scale it is assumed that the space-time has noncommutative structure leading in turn to a resolution of the singularity. Similar aims in Particle Physics Unification schemes, namely attempts to remove the divergences of the field theoretic, led first to supersymmetric field theories as the natural playground for solving the problem of quadratic divergences. Then, requiring complete absence of divergences in GUTs led to all-loop Finite Theories making use of the idea of reduction of couplings. A remarkable consequence of such theories was the prediction, among others, of the top and Higgs masses before their discoveries. It would be very exciting if the four-dimensional noncommutativity, a matrix version of which was discussed in the present article, could lead to similar results in the gravitational-cosmological predictions.

Being more realistic it is first worth noting that the dimensional reduction of higher-dimensional gauge theories over fuzzy manifolds -used as extra dimensions- led to renormalizable four-dimensional theories [81–84]. Then it is not totally inconceivable to think that the four-dimensional fuzzy Euclidean matrix model of gravity discussed here could have improved UV properties, as compared to ordinary gravity; it has finite degrees of freedom, as the extra dimensional theories with fuzzy extra dimensions that were found to be renormalizable. We plan to further explore this possibility and examine to what extent the, hopefully positive, results can be realised in spaces with Minkowskian signature too.

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