PROCEEDINGS OF SCIENCE

Metric algebroid in Double Field Theory

Ursula Carow-Watamura, Kohei Miura and Satoshi Watamura*

Particle Theory and Cosmology Group,
Department of Physics, Graduate School of Science,
Tohoku University
Aoba-ku, Sendai 980-8578, Japan
E-mail: ursula@tuhep.phys.tohoku.ac.jp, miura@tuhep.phys.tohoku.ac.jp, watamura@tuhep.phys.tohoku.ac.jp

The Double Field Theory (DFT) is formulated on a certain class of a metric algebroid, where the structure functions satisfy the pre-Bianchi identity. The derived bracket formulation of the metric algebroid by the Dirac generating operator is applied. The action is formulated by a projected generalized Lichnerowicz formula which is a generalization of the analogous formulation in gravity and supergravity. The flux of the dilaton in this setting is introduced via an ambiguity in the Dirac generating operator, while the dilaton is introduced via an ambiguity of the divergence on the metric algebroid. This talk is based on [1].

Corfu Summer Institute 2021 "School and Workshops on Elementary Particle Physics and Gravity"
29 August - 9 October 2021
Corfu, Greece

*Speaker

PoS(CORFU2021)281

1. Introduction

Double Field Theory (DFT) was introduced to obtain a T-duality covariant formulation of the string effective theory [2–5]. Its relation to the Courant algebroid has been noted already in the very early stage. See [6, 7] for a review and references therein. This relation is very natural since the generalized geometry [8, 9] offers a description of supergravity which formulates diffeomorphism and B-field transformation in a unified way using a Courant algebroid [10, 11]. In generalized geomerty, the unification of the gauge transformation is performed by considering an extended fiber, i.e., $TM \oplus T^*M$. In the DFT, on the other hand, the base manifold is extended to the doubled space as $\mathbb{M} = M \times \tilde{M}$, therefore $T\mathbb{M}$ corresponds to the fiber $TM \oplus T^*M$. The origin of the second space is the winding modes, which is natural from the string view point. However, of course, since supergravity is defined on M and not on $\mathbb{M}$, the space of DFT must be somehow reduced which is done by the so-called section condition or closure constraint. The gauge symmetry of the theory is realized up to the section condition, and this makes the covariance of the theory unclear. Furthermore, the section condition is frame dependent and thus it is troublesome when one analyzes the generalized T-duality like Poisson-Lie T-duality [12, 13] using DFT.

We apply the metric algebroid [14] as a basic structure of the DFT and construct the invariant action without referring to the section condition. The metric algebroid is a generalization of the Courant algebroid, where the Jacobi identity is relaxed. Using the differential graded manifold (QP-manifold) formulation for the Courant algebroid [15–17], we consider the graded symplectic manifold $T[2]T[1]M$, then the operations in the Courant algebroid are defined by the derived bracket using the holomorphic function (Hamiltonian) Θ and the graded Poisson bracket $\{-,-\}$. Then, we can understand the algebraic structure as

$$\begin{array}{lll} \{\Theta,\Theta\} = 0 & \cdots & \text{Courant algebroid} \\ \{\Theta,\Theta\} \neq 0 & \cdots & \text{Metric algebroid} \\ \{\Theta,\Theta\} \approx 0 & \cdots & \text{DFT} \end{array} \tag{1}$$

The graded symplectic manifold is a convenient tool to analyze the algebroid structure e.g. the Bianchi identities of DFT. One drawback is that there is no natural way to accommodate the dilaton in the theory. The geometric aspects with algebroid structure in DFT were also investigated in [18, 19].

It is known that there is another way to realize the derived bracket formulation of the Courant algebroid by using the Dirac operator and a Clifford module [20]. Then, this Dirac operator, which is called Dirac generating operator (DGO), and the graded commutator on the Clifford module play the same roles as the holomorphic function and the graded Poisson bracket in the graded manifold formulation, respectively. We have a similar classification for the algebroids, and the Bianchi identities can be formulated using this DGO.

The merit of using the DGO is that there is an ambiguity in its definition and it is this ambiguity which gives place for the dilaton. The use of the Dirac operator is also very natural for the physicists since, anyway, in supergravity we have fermions. In the graded manifold formulation, there is a natural method to define an action for a topological theory. However, for supergravity or DFT, such a construction is not known. On the other hand, once one introduces a Dirac operator, we have a Lichnerowicz formula to obtain a scalar curvature which gives an Einstein-Hilbert action.

A formulation of supergravity using the Lichnerowicz formula is also known [21]. Our second result is that we formulate the DFT action by a generalization of Lichnerowicz formula to the metric algebroid. We see that the Lichnerowicz formula is not automatic in the metric algebroid approach. Some conditions are needed to make it possible and these correspond to the pre-Bianchi identities which define the class of metric algebroid for DFT. So, the effect of using a formulation with the Lichnerowicz formula is twofold: First, its existence selects a class of metric algebroid on which the DFT should be formulated. Second, it gives the action of DFT.

2. Metric algebroid

The metric algebroid [14] $\mathcal{A} = (E, [-,-], \langle -,- \rangle, \rho)$ is a vector bundle $E \to M$, with a bracket $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, an inner product $\langle -,- \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$, a bundle map (anchor) $\rho : E \to TM$, and a differential $\partial : C^\infty(M) \to \Gamma(E)$ s.t. $\langle \partial f, a \rangle = \rho(a) f$, satisfying

$$(a) \qquad \rho(a)\langle b, c\rangle = \langle [a,b], c\rangle + \langle b, [a,c]\rangle \ , \tag{2}$$

$$(b) \qquad [a,a] = \frac{1}{2}\partial\langle a, a\rangle \ , \tag{3}$$

where $a, b, c \in \Gamma(E)$ are sections. Compared to the Courant algebroid, the Jacobi identity of the bracket is dropped. From the above axioms, the following relation can be derived:

$$(c) \qquad [a,b] = -[b,a] + \partial\langle a, b\rangle \tag{4}$$

$$(d) \qquad [a, fb] = (\rho(a)f)b + f[a,b] \ , \tag{5}$$

$$(e) \qquad [fa, b] = -(\rho(b)f)a + (\partial f)\langle a, b\rangle + f[a,b] \ , \tag{6}$$

For example, (d) can be proven by evaluating $\rho(e)\langle fa, b\rangle = \rho(e)(f\langle a, b\rangle)$ in two ways as

$$\langle [e, fa], b\rangle + \langle fa, [e,b]\rangle = (\rho(e)f)\langle a, b\rangle + f\rho(e)\langle a, b\rangle \ , \tag{7}$$

and using axiom (a) on the l.h.s.

The deviation from the Courant algebroid is characterized by the following maps $\mathfrak{L} : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and $\mathfrak{L}' : \Gamma(E) \times \Gamma(E) \to \Gamma(TM)$:

$$\mathfrak{L}(a,b,c) = [a,[b,c]] - [[a,b],c] - [b,[a,c]] \ , \tag{8}$$

$$\mathfrak{L}'(a,b) = \rho([a,b]) - [\rho(a), \rho(b)]_{TM} \ , \tag{9}$$

where $[-,-]_{TM}$ denotes the standard Lie bracket on TM. The map $\mathfrak{L}$ in (8) is a Jacobiator in Leibniz like form. We added here $\mathfrak{L}'$ which does not vanish in general. If we restrict the algebroid $\mathcal{A}$ by the conditions $\mathfrak{L} = 0$, $\mathfrak{L}' = 0$ then the metric algebroid reduces to the Courant algebroid.

$$\begin{array}{lllll} \mathfrak{L} = 0 & \to & \mathfrak{L}' = 0 & \to & \text{Courant algebroid} \\ \mathfrak{L} \neq 0 & & \mathfrak{L}' = 0 & \to & \text{pre-Courant algebroid} \\ \mathfrak{L} \neq 0 & & \mathfrak{L}' \neq 0 & \to & \text{metric algebroid} \end{array} \tag{10}$$

Thus, CA $\subset$ pre-CA $\subset$ MA, and the DFT structure is somewhere between the pre-CA and MA.

3. DFT condition and pre-Bianchi identity

To formulate the DFT action using the algebroid structure, we take here the following assumption on the metric algebroid which we call DFT condition.

1. $dim(E) = dim(T\mathbb{M})$ and the anchor is invertible.

2. We identify the inner product on E and on $T\mathbb{M}$ as $\langle\rho(a), \rho(b)\rangle_{T\mathbb{M}} = \langle a, b\rangle$.

This means that we consider the minimum case, i.e. the dimension of E is $2D$, where D is the dimension of M and of $\tilde{M}$, and E has an $O(D, D)$ structure.

To describe DFT we first introduce a local basis E_A on the bundle E, s.t.

$$\langle E_A, E_B\rangle = \eta_{AB} \quad , \tag{11}$$

where η_{AB} is a symmetric constant $O(D, D)$ metric. We introduce η^{AB} by $\eta_{AB}\eta^{BC} = \delta_A^C$ and the raising and lowering of indices by η. Using a local coordinate, the anchor map is defined by a vielbein as

$$\rho(E_A) = E_A{}^M \partial_M . \tag{12}$$

Note that the metric on the base manifold $\eta_{MN} = \langle\partial_M, \partial_N\rangle$ is not necessarily a constant in general.

In this basis we can write the differential operator as

$$\partial f = \sum_A (\rho(E_A) f) E^A \quad . \tag{13}$$

We then define a structure function $F_{AB}{}^C \in C^\infty(M)$ of the bracket by

$$[E_A, E_B] = F_{AB}{}^C E_C \quad . \tag{14}$$

Using this basis, we can show that

$$F_{ABC} = \langle [E_A, E_B], E_C\rangle = F_{AB}{}^D \eta_{DC} \tag{15}$$

is totally antisymmetric.

We also introduce the structure functions corresponding to maps $\mathfrak{L}$, $\mathfrak{L}'$ by using the action on the local frame E_A:

$$\mathfrak{L}(E_A, E_B, E_C) = \phi_{ABC}{}^D E_D , \tag{16}$$

$$\mathfrak{L}'(E_A, E_B) = \phi'_{AB}{}^C \rho(E_C) , \tag{17}$$

where the structure functions ϕ and ϕ' are represented by the above maps as

$$\phi_{ABCD} = \langle\mathfrak{L}(E_A, E_B, E_C), E_D\rangle \quad , \quad \phi'_{ABC} = \langle\mathfrak{L}'(E_A, E_B), \rho(E_C)\rangle . \tag{18}$$

A metric algebroid gives a wider class than necessary for DFT even with the above DFT conditions. We want to restrict this class by conditions on the structure functions. It is easy to see that the structure function ϕ_{ABCD} is totally antisymmetric and represented by the structure function F_{ABC} as:

$$\begin{aligned}\phi_{ABCD} &= \langle [E_A,[E_B,E_C]] - [[E_A,E_B],E_C] - [E_B,[E_A,E_C]], E_D\rangle \\ &= \frac{1}{4!}\Big(4\rho(E_{[A})F_{BCD]} - 3F_{[AB}{}^{A'}F_{CD]A'}\Big) .\end{aligned} \tag{19}$$

However, $\phi_{ABCD} = 0$ is not the right restriction, since it is not a tensor and it depends on the choice of the local basis. We see that ϕ' also depends on the choice of the local basis E_A, but we can prove that $\tilde{\phi}_{ABCD}$

$$\tilde{\phi}_{ABCD} = \phi_{ABCD} + \phi'_{ABC'}\phi'_{CD}{}^{C'} + \phi'_{ADC'}\phi'_{BC}{}^{C'} - \phi'_{ACC'}\phi'_{BD}{}^{C'} . \tag{20}$$

is a covariant totally antisymmetric tensor. Thus, the condition $\tilde{\phi} = 0$ gives the relation on the structure functions which is independent of the choice of the local basis E_A:

$$\begin{aligned}\tilde{\phi}_{ABCD} &= \phi_{ABCD} + \frac{1}{8}\phi'_{[AB}{}^{E}\phi'_{CD]E} \\ &= \frac{1}{6}\rho(E_{[A})F_{BCD]} - \frac{1}{8}F_{[AB}{}^{E}F_{CD]E} + \frac{1}{8}\phi'_{[AB}{}^{E}\phi'_{CD]E} = 0 .\end{aligned} \tag{21}$$

We call this condition the pre-Bianchi identity [22]. The pre-Bianchi identity defines a class of metric algebroid which includes the standard DFT.

There is another condition on the structure functions obtained from the Jacobi identity of the Lie bracket $[-,-]_{T\mathbb{M}}$ on $T\mathbb{M}$. From (9) it is clear that the structure function ϕ'_{ABC} can be written as

$$\phi'_{ABC} = F_{ABC} - F'_{ABC}. \tag{22}$$

where F'_{ABC} is the so-called geometric flux defined by $[\rho(E_A),\rho(E_B)]_{T\mathbb{M}} = F'_{AB}{}^{C}\rho(E_C)$. The Jacobi identity of the Lie bracket gives the condition on F'_{ABC} as

$$\mathcal{J}_{ABCD} = \rho(E_{[A})F'_{BC]D} + F'_{[BC}{}^{C'}F'_{A]C'D} = 0 \tag{23}$$

which shows that F'_{ABC} is a Lie algebroid structure function on $T\mathbb{M}$. This gives another condition on the structure functions F_{ABC} and ϕ' as

$$\begin{aligned}\mathcal{J}_{ABC}{}^{C} &= \rho(E_C)F_{AB}{}^{C} - \rho(E_C)\phi'_{AB}{}^{C} - \rho(E_{[A})\phi'_{B]C}{}^{C} \\ &\quad - F_{AB}{}^{C'}\phi'_{CC'}{}^{C} + \phi'_{AB}{}^{C'}\phi'_{CC'}{}^{C} = 0 .\end{aligned} \tag{24}$$

This condition is related to the pre-Bianchi identity including the dilaton as we shall see below.

4. Geometry on metric algebroid

The geometry on the metric algebroid can be introduced as in the generalized geometry. On the metric algebroid E we define an E-connection compatible with the inner product $\langle -,-\rangle$. The E-connection ∇^E is a map $\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ satisfying for $a,b,c\in\Gamma(E)$ and $f\in C^\infty(M)$

$$\nabla^E_a fb = (\rho(a)f)b + f\nabla^E_a b , \tag{25}$$

$$\nabla^E_{fa} b = f\nabla^E_a b . \tag{26}$$

PoS(CORFU2021)281

and the compatibility with the inner product is

$$\rho(a)\langle b,c\rangle = \langle \nabla^E_a b, c\rangle + \langle b, \nabla^E_a c\rangle \ . \tag{27}$$

Using the basis E_A, the connection ∇^E is defined by a gauge field W_{ABC} as

$$\nabla^E_{E_A} E_B = W_{AB}{}^C E_C \ . \tag{28}$$

In the following, we also use the abbreviation $\nabla^E_A = \nabla^E_{E_A}$ as long as it does not cause confusion. Compatibility with the fiber metric yields that W_{ABC} is antisymmetric in the last two indices.

We can also define E-torsion [20, 23]. The same torsion was also introduced in DFT context in [24],

$$T_{ABC} = \frac{1}{2} W_{[ABC]} - F_{ABC} \ , \tag{29}$$

which is independent of the choice of the local basis [25]. We impose the torsionless condition and thus the totally antisymmetric part of the spin connection is defined by a metric algebroid structure function.

For a generalized curvature on the metric algebroid, the independence of the choice of the basis requires a modification to the generalized curvature introduced in DFT as

$$\mathcal{R}_{ABCD} = R^\nabla_{ABCD} + R^\nabla_{CDAB} + \phi'_{ABE}\phi'_{CD}{}^E \ , \tag{30}$$

where

$$R^\nabla_{ABCD} := \rho(E_{[A})W_{B]CD} - W_{[A|C}{}^{E'} W_{|B]E'D} - F_{AB}{}^E W_{ECD} + \frac{1}{2} W_{EAB} W^E{}_{CD} \ . \tag{31}$$

The first two terms in (30) are the curvature given in [24], and the last term is a modification to recover the tensorial property.

Using these quantities, the pre-Bianchi identity $\tilde{\phi}(a,b,c,d) = 0$ can be represented in curvature and torsion as

$$3\mathcal{R}_{[ABCD]} = 4\nabla_{[A} T_{BCD]} + 3\sum_{A'} T_{[AB|A'} T_{CD]}{}^{A'} \ . \tag{32}$$

4.1 Divergence

We define a divergence on the metric algebroid following the definition on a Courant algebroid [20, 25, 26] as a map $div : \Gamma(E) \to C^\infty(\mathbb{M})$, for $a \in \Gamma(E)$ and $f \in C^\infty(\mathbb{M})$ s.t.

$$div(fa) = \rho(a)f + f\,div(a) \ . \tag{33}$$

For a given E-connection, we can define a divergence div_∇ by using a local basis as

$$div_\nabla a = \langle \nabla^E_A a, E^A\rangle \ , \tag{34}$$

where div_∇ satisfies the relation (33) as

$$div_\nabla fa = \langle \nabla^E_A fa, E^A\rangle = \langle (\rho(E_A)f)a, E^A\rangle + \langle f\nabla^E_A a, E^A\rangle = \rho(a)f + f\,div_\nabla a \ . \tag{35}$$

However, the divergence defined by the property (33) is not unique and has an ambiguity by $U \in \Gamma(E)$. A general divergence for a given connection is

$$div^U_\nabla(a) = div_\nabla(a) - \langle U, a\rangle \ . \tag{36}$$

The Laplacian of the given E-connection ∇^E is defined by

$$\Delta a = div^U_{\nabla^E}(\nabla^E a) \ . \tag{37}$$

Since the divergence has an ambiguity, the Laplacian has also an ambiguity of $U \in \Gamma(E)$.

The reason to use the metric algebroid is to control the closure of the gauge transformation generated by the generalized Lie derivative of DFT. In order to include the dilaton, we have to consider the gauge transformation of a field with weight. And the closure of the weight term gives an extra condition which is not related to the metric algebroid but to the Dirac generating operator defined below. We define the generalized Lie derivative with weight p in the metric algebroid by

$$\delta_X\psi = \mathcal{L}_X\psi + p(div\, X)\psi \tag{38}$$

The violation of the closure of the gauge transformation can be written as

$$([\delta_X, \delta_Y] - \delta_{[X,Y]})\psi = ([\mathcal{L}_X, \mathcal{L}_Y] - \mathcal{L}_{[X,Y]})\psi + p(\rho(X)divY - \rho(Y)divX - div[X,Y])\psi \tag{39}$$

The first term is controlled by the above map $\mathfrak{L}$. To discuss the closure of the weight term we define for a given connection

$$\mathfrak{B}i_\nabla(X,Y) = div_\nabla[X,Y] - \rho(X)div_\nabla Y + \rho(Y)div_\nabla X. \tag{40}$$

As a closure condition, we can not simply take $\mathfrak{B}i_\nabla$ zero, since it is not a tensor and such a condition depends on the choice of the basis. We can show that the difference $\mathfrak{B}i_\nabla(X,Y) - div_\nabla\mathfrak{L}'(X,Y)$ is a tensor. Furthermore, for a connection $\nabla^{\phi'}$ where the spin connection is defined as $W_{ABC} = \phi'_{BCA}$, we find that the following holds

$$\mathfrak{B}i_{\nabla^{\phi'}}(X,Y) - div_{\nabla^{\phi'}}\mathfrak{L}'(X,Y) = 0 \ , \tag{41}$$

due to the Jacobi identity of the Lie bracket on $T\mathbb{M}$.

For a given connection ∇ we know that the difference of the divergences div_∇ and $div_{\phi'}$ is a $C^\infty(M)$-linear function, and thus, there exists a generalized vector $e_\nabla \in T\mathbb{M}$ satisfying

$$div_\nabla X - div_{\nabla^{\phi'}}X = \langle e_\nabla, X\rangle \ . \tag{42}$$

From the above considerations we conclude

$$\mathfrak{B}i_\nabla(X,Y) - div_\nabla\mathfrak{L}'(X,Y) = \langle[e_\nabla, X], Y\rangle - \langle e_\nabla, \mathfrak{L}'(X,Y)\rangle \tag{43}$$

Since the difference of the two terms on the l.h.s. of (43) is a tensor, taking a basis we obtain a relation:

$$\rho(E_C)F'_{AB}{}^C + F'_{AB}{}^C W_{DC}{}^D - \rho(E_{[A})W_{CB]}{}^C = -\rho(E_{[A})e_{\nabla B]} + e^C_\nabla(F_{CAB} - \phi'_{ABC}) \tag{44}$$

This can be rewritten as

$$\rho(E_C)F'_{AB}{}^C + \rho(E_{[A})(e_{\nabla B]} - W_{CB]}{}^C) - F'_{ABC}(e^C_\nabla - W_D{}^{CD}) = 0 \tag{45}$$

Subtracting the Jacobi identity (24) from (45), we obtain an equation for $U_B = e_{\nabla B} + \phi'_{BC}{}^C - W_{CB}{}^C$:

$$\rho(E_{[A})U_{B]} - U^C(F_{ABC} - \phi'_{ABC})) = 0 \tag{46}$$

We identify the dilaton flux F_A in DFT in terms of the above object as

$$F_A = -W_{CA}{}^C + e_{\nabla A} = U_A - \phi'_{AC}{}^C \tag{47}$$

where U_A satisfies (46), as in [1]. Then, we obtain from (45) a DFT Bianchi identity of the dilaton flux in the metric algebroid structure as

$$\rho(E_C)(F_{AB}{}^C - \phi'_{AB}{}^C) - (F_{AB}{}^{C'} - \phi'_{AB}{}^{C'})F_{C'} + \rho(E_{[A})(F_{B]}) = 0 \tag{48}$$

As we have shown in [1], the equation for U_A has a solution $U_A = 2\rho(E_A)d$, then

$$F_A = 2\rho(E_A)d - \phi'_{AC}{}^C , \tag{49}$$

and therefore, we identify the field d with the dilaton.

5. Dirac generating operator

In the formulation by using graded symplectic manifold, bracket and anchor in Courant algebroid are defined by the derived bracket [27]. The same can be done by using the Dirac operator [20, 25, 28] on a Clifford bundle $Cl(E)$. We consider a linear map $\gamma : \Gamma(E) \to \Gamma(Cl(E))$ and denote the image of the base E_A by $\gamma_A = \gamma(E_A)$ which satisfies a Clifford algebra:

$$\{\gamma_A, \gamma_B\} = \gamma_A\gamma_B + \gamma_B\gamma_A = 2\eta_{AB} , \tag{50}$$

then $\gamma(a) = \gamma(a_A E^A) = a_A\gamma^A$. By extending the Clifford product $\Gamma(Cl(E)) \times \Gamma(Cl(E)) \ni (a, b) \to ab \in \Gamma(Cl(E))$ in the standard way, we get a degree n element $X \in Cl(E)$ as

$$X = X_{A_1A_2\cdots A_n}\gamma^{A_1A_2\cdots A_n} , \tag{51}$$

where $\gamma^{A_1\cdots A_n}$ is an antisymmetric product of γ^A, and we define a graded bracket of $X, Y \in Cl(E)$ as

$$\{X, Y\} = XY - (-1)^{|X||Y|}YX = -(-1)^{|X||Y|}\{Y, X\} . \tag{52}$$

Note that we do not write the map γ explicitly for the simplicity.

We also consider the induced connection on $Cl(E)$ for a given E-connection as

$$\{\nabla^{Cl}_a, b\} = \nabla^E_a b , \tag{53}$$

and Leibniz rule with respect to the Clifford product. Then we introduce a spin bundle $\mathbb{S}$ where a spinor $\chi \in \Gamma(\mathbb{S})$ is a module over the Clifford bundle.

The Dirac operator is defined on the spinor $\not{D} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ and we can find a Dirac generating operator (DGO) which generates the operations by the derived bracket:

$$\partial f = 2\{\not{D}, f\} \, , \tag{54}$$

$$[a, b] = \{\{\not{D}, a\}, b\} \, , \tag{55}$$

$$\rho(a) f = \{\{\not{D}, a\}, f\} \, . \tag{56}$$

They satisfy the axiom of the metric algebroid. We can also get the expressions of $\mathfrak{L}$ and $\mathfrak{L}'$ by this DGO as

$$\mathfrak{L}(a, b, c) = -\{\{\{\not{D}^2, a\}, b\}, c\} \, , \tag{57}$$

$$\mathfrak{L}'(a, b) f = \{\{\not{D}^2, a\}, b\}, f\} \, . \tag{58}$$

The derived bracket construction works completely analogous as in the differential graded manifold approach and we can derive all the relations of the metric algebroid.

The advantage of the formulation using the DGO is that we can define the action by generalizing the Lichnerowicz formula. The original Lichnerowicz formula on a manifold is given by a Levi-Civita connection ∇ and corresponding Dirac operator $\not{\nabla}$ as

$$\not{\nabla}^2 - \nabla^2 = -\frac{1}{4} R \, , \tag{59}$$

where R is a scalar curvature. One can also include H-flux, and a corresponding formula for the supergravity has also been given. See [21] and reference therein.

To obtain the Lichnerowicz formula for DFT, we take a torsion free E-connection $\nabla_{E_A} E_B = W_{AB}{}^C E_C$. Then, a corresponding connection on the spinor is

$$\nabla^{\mathbb{S}}_A = \partial_A - \frac{1}{4} W_{ABC} \gamma^{BC} \, , \tag{60}$$

where $\partial_A = \rho(E_A)$. The DGO can be given by this connection as $\not{D} = \frac{1}{2}\gamma^a \nabla^{\mathbb{S}}_A$ which can be written as

$$\not{D} = \frac{1}{2}(\gamma^A \partial_A - \frac{1}{12} F_{ABC} \gamma^{ABC} - \frac{1}{2} F_A \gamma^A) \, , \tag{61}$$

where we denoted the totally antisymmetric part of the connection as $\frac{1}{2} W_{[ABC]} = F_{ABC}$ and the trace part as $W^B{}_{BA} = F_A$. Using this DGO, we get a metric algebroid with the structure function F_{ABC}. The flux F_A which appeared here is an ambiguity of the DGO, i.e., it is not determined and is free parameter from the metric algebroid viewpoint.

As in the original Lichnerowicz formula, $\not{D}^2$ is not a scalar function and we have to subtract a derivative term. We subtract a Laplacian constructed by a specific connection given by the structure function ϕ'_{ABC}. This is possible since the structure function ϕ'_{ABC} has the same transformation under a local $O(D, D)$ rotation as the W_{CAB}. Thus, we have a connection on the Clifford module $\Gamma(\mathbb{S})$ s.t.

$$\nabla^{\phi'}_A = \partial_A - \frac{1}{4} \phi'_{BCA} \gamma^{BC} \, . \tag{62}$$

The corresponding Laplace operator is then given by

$$\Delta^{\phi'} = div^U_{\nabla^{\phi'}} \nabla^{\phi'} = \eta^{AB} \nabla^{\phi'}_A \nabla^{\phi'}_B - (\phi'_B{}^{AB} + U^A) \nabla^{\phi'}_A \, , \tag{63}$$

where U_A is a vector representing the ambiguity in the divergence. The generalized Lichnerowicz formula is given by the difference of the square of the Dirac generating operator and the Laplacian $\Delta^{\phi'}$

$$
\begin{aligned}
4\not{\mathcal{D}}^2 - \Delta^{\phi'} = & -\frac{1}{24}F_{ABC}F^{ABC} - \frac{1}{2}(\rho(E^A)F_A) + (-F^A + {\phi'_E}^{AE} + U^A)\partial_A + \frac{1}{4}F_AF^A + \frac{1}{8}\phi'_{BCA}\phi'^{BCA} \\
& +\frac{1}{4}\left(-{\mathcal{J}_{BCD}}^D + (\rho(E_{[B})(-F_{C]} + {\phi'^D}_{C]D}) - (-F^A + {\phi'_D}^{AD})F_{ABC} - U_A{\phi'_{BC}}^A\right)\gamma^{BC} \\
& -\frac{1}{2}\tilde{\phi}_{BCB'C'}\gamma^{BCB'C'}
\end{aligned} \tag{64}
$$

We see that the second order derivative term is canceled by the Laplacian. However, it is not a function and there are still non-scalar terms proportional to ∂_A,γ^{AB} and γ^{ABCD}. We require that these non-scalar terms should vanish then we obtain the conditions

$$
\partial_A \quad : \quad F_A = {\phi'_{BA}}^B + U_A \tag{65}
$$

$$
\gamma^{BC} \quad : \quad \rho(E_{[A})U_{B]} - {F'_{AB}}^C U_C + {\mathfrak{J}_{ABC}}^C = 0 \tag{66}
$$

$$
\gamma^{ABCD} \quad : \quad \tilde{\phi}_{ABCD} = 0 \tag{67}
$$

The terms proportional to γ^{ABCD} are represented as $\tilde{\phi}_{ABCD}$ and the condition (67) coincides with the pre-Bianchi identity. The condition from the terms proportional to ∂_A gives a relation between the ambiguity of the DGO and the ambiguity of the divergence as (65). Using (65), the condition from the terms proportional to γ^{AB} becomes (66), giving a condition on U_A which is solved by $U_A = 2\partial_A d$ with a function d. In this way we get the ambiguity F_A of the DGO where ${\mathcal{J}_{BCD}}^D$ is the tensor defined in the identity (24). Thus, with the pre-Bianchi identity we get a generalized Lichnerowicz formula

$$
4\not{\mathcal{D}}^2 - \Delta^{\phi'} = \frac{1}{8}\mathbf{R}\,. \tag{68}
$$

5.1 Projected Lichnerowicz formula

The scalar curvature given in the generalized Lichnerowicz formula is not the scalar in the action of DFT. This is natural since we did not introduce a Riemann structure yet. As in the generalized geometry, we introduce the Riemann structure by splitting the bundle into positive and negative sub-bundle as

$$
E = V^+ \oplus V^- \quad \text{, where } \; V^\pm = \{a \in E | \langle a, a\rangle = \pm|\langle a, a\rangle|\}\,. \tag{69}
$$

By this splitting, the $O(D,D)$ symmetry reduces to $O(D-1,1) \otimes O(1,D-1)$. We consider the corresponding basis and distinguish correspondingly the indices as

$$
E_a, E_{\bar{a}} \in V^+ \oplus V^-\,. \tag{70}
$$

A compatible connection is introduced by

$$
\langle \nabla^E_{E_A} E_b, E_{\bar{c}}\rangle = 0 \;\text{ and }\; \langle \nabla^E_{E_A} E_{\bar{b}}, E_c\rangle = 0. \tag{71}
$$

Thus, the non-zero components are W_{Aab} and $W_{A\bar{a}\bar{b}}$ and the compatible spin connection is given by

$$
{\nabla\mathbb{S}^+}_A = \partial_A - \frac{1}{4}W_{Aab}\gamma^{ab}\,. \tag{72}
$$

PoS(CORFU2021)281

To obtain a projected Lichnerowicz formula, we consider a Dirac operator

$$\begin{aligned} \mathcal{D}\!\!\!/^{+} &= \frac{1}{2}\gamma^a\nabla_a^{\mathbb{S}^+} \\ &= \frac{1}{2}\gamma^a\partial_a - \frac{1}{24}F_{abc}\gamma^{abc} - \frac{1}{4}F_a\gamma^a\ . \end{aligned} \tag{73}$$

and also a connection by ϕ'_{ABC} as

$$\nabla_A^{\phi'^+} = \partial_A - \frac{1}{4}\phi'_{bcA}\gamma^{bc}\ . \tag{74}$$

Then, the projected Lichnerowicz formula is given by

$$L^+ = 4\mathcal{D}\!\!\!/^{+2} + div_\nabla\nabla_-^{\mathbb{S}^+} - \Delta^{\phi'^+}\ . \tag{75}$$

As in the generalized Lichnerowicz formula case, the terms proportional to ∂_A give a relation $F_A = \phi'_{BA}{}^B + U_A$ and with this relation, we obtain

$$L^+ = \mathcal{R}^{DFT} - \frac{1}{4}\Big(\mathcal{J}_{abC}{}^C + \rho(E_{[a})(U_{b]}) - U^C(F_{Cab} - \phi'_{abC})\Big)\gamma^{ab} - \frac{1}{2}\tilde{\phi}_{abcd}\gamma^{abcd}\ , \tag{76}$$

where

$$\mathcal{R}^{DFT} = -\frac{1}{24}F_{abc}F^{abc} - \frac{1}{8}F^{\bar{a}bc}F_{\bar{a}bc} - \frac{1}{2}\rho(E_a)F^a + \frac{1}{4}F_aF^a - \frac{1}{8}\phi'_{abC}\phi'^{abC}\ . \tag{77}$$

We see that with the pre-Bianchi identity $\tilde{\phi}_{ABCD} = 0$, and taking the U_A as a solution of the condition from γ^{ab} we get the Lichnerowicz formula for DFT.

6. Discussion

We presented the projected generalized Lichnerowicz formula which gives the scalar curvature of the DFT action. To define this action we have to introduce the integration measure. The consistent measure is $\mu = d^{2D}X\sqrt{\det\eta_{MN}}\,e^{-2d}$. The DGO gives a way for this. There is a natural generalized Lie derivative generated by the DGO of the spinor $\chi \in \Gamma(\mathbb{S})$ by $a \in \Gamma(E)$ as

$$\mathcal{L}_a\chi = \{\mathcal{D}\!\!\!/, a\}\chi\ . \tag{78}$$

One can show that this is a generalized Lie derivative with weight $\frac{1}{2}$. Then, we split the spinor as $\chi = \chi_0 \otimes \mu^{\frac{1}{2}} \in \Gamma(\mathbb{S}) = \Gamma(\mathbb{S}') \otimes \Gamma(\Lambda^{\frac{1}{2}})$ with a factor of the weight $\frac{1}{2}$ bundle $\Lambda^{\frac{1}{2}}$. We define an inner product $\Gamma(\mathbb{S}) \times \Gamma(\mathbb{S}) \to \Gamma(\Lambda)$ which defines a section $\mu \in \Gamma(\Lambda)$ which we identify with the integration measure [26]. We introduced a dilaton d as a function defined by the section $\mu = e^{-2d}\mu_0$ with a base μ_0 of $\Gamma(\Lambda)$. The gauge transformation of the dilaton is defined from the Lie derivative of μ as

$$\mathcal{L}_a(e^{-2d}\mu_0) = (-2\delta_a d)e^{-2d}\mu_0 \tag{79}$$

In this formulation, using the metric algebroid, there is no need to refer to the section condition in order to obtain the action. Just we need to restrict the class of metric algebroid by imposing the pre-Bianchi identity on the structure functions.

References

[1] U. Carow-Watamura, K. Miura, S. Watamura, and T. Yano, "Metric algebroid and dirac generating operator in double field theory,". https://doi.org/10.1007%2Fjhep10%282020%29192.

[2] W. Siegel, "Two-vierbein formalism for string-inspired axionic gravity," *Physical Review D* **47** no. 12, (1993) 5453–5459.

[3] W. Siegel, "Superspace duality in low-energy superstrings," *Physical Review D* **48** no. 6, (1993) 2826–2837. http://dx.doi.org/10.1103/PhysRevD.48.2826.

[4] C. Hull and B. Zwiebach, "The gauge algebra of double field theory and Courant brackets," *JHEP* **09** (2009) 090, arXiv:0908.1792. https://arxiv.org/pdf/0908.1792.pdf.

[5] O. Hohm, C. Hull, and B. Zwiebach, "Generalized metric formulation of double field theory," *Journal of High Energy Physics* **08** no. 8, (2010) 008, arXiv:1006.4823. http://dx.doi.org/10.1007/JHEP08(2010)008.

[6] B. Zwiebach, "Doubled Field Theory, T-Duality and Courant-Brackets," *Lecture Notes in Physics* **851** (2012) 265–291. http://dx.doi.org/10.1007/978-3-642-25947-0_7.

[7] G. Aldazabal, D. Marqués, and C. Núñez, "Double Field Theory: A Pedagogical Review," *Classical and Quantum Gravity* **30** no. 16, (2013) 163001, arXiv:1305.1907. http://dx.doi.org/10.1088/0264-9381/30/16/163001.

[8] N. Hitchin, "Brackets, forms and invariant functionals," *ASIAN J. MATH* **10** no. 3, (2006) 541–560, arXiv:arXiv:math/0508618 [math-DG].

[9] M. Gualtieri, "Generalized complex geometry," arXiv:math/0703298 [mathDG]. https://arxiv.org/pdf/math/0703298.pdf.

[10] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as generalised geometry I: type II theories," *Journal of High Energy Physics* **11** no. 11, (2011) 091, arXiv:1107.1733. http://dx.doi.org/10.1007/JHEP11(2011)091.

[11] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Generalised Geometry and type II Supergravity," *Fortschr. Phys.* **60** no. 9-10, (2012) 982–986, arXiv:1202.3170. https://arxiv.org/pdf/1202.3170.pdf.

[12] C. Klimčík and P. Ševera, "Dual non-Abelian duality and the Drinfeld double," *Physics Letters B* **351** no. 4, (1995) 455 – 462, hep-th/9502122. http://www.sciencedirect.com/science/article/pii/037026939500451P.

[13] C. Klimčík, "Poisson-Lie T-duality," *Nuclear Physics B - Proceedings Supplements* **46** no. 1, (1996) 116 – 121, hep-th/9509095. http://www.sciencedirect.com/science/article/pii/0920563296000138.

[14] I. Vaisman, "On the geometry of double field theory," *Journal of Mathematical Physics* **53** no. 3, (Mar, 2012) 033509, arXiv:1203.0836. http://dx.doi.org/10.1063/1.3694739.

[15] D. Roytenberg, "Courant algebroids, derived brackets and even symplectic supermanifolds," *Dissertation* (1999) , arXiv:math/9910078.

[16] A. S. Cattaneo and G. Felder, "On the AKSZ Formulation of the Poisson Sigma Model," *Letters in Mathematical Physics* **56** no. 2, (2001) 163–179. http://dx.doi.org/10.1023/A:1010963926853.

[17] N. Ikeda, "Lectures on AKSZ Sigma Models for Physicists," *Noncommutative Geometry and Physics* **4** (2017) 79, arXiv:1204.3714. http://dx.doi.org/10.1142/9789813144613_0003.

[18] A. Chatzistavrakidis, L. Jonke, F. S. Khoo, and R. J. Szabo, "The Algebroid Structure of Double Field Theory," arXiv:arXiv:1903.01765 [hep-th].

[19] H. Mori, S. Sasaki, and K. Shiozawa, "Doubled Aspects of Vaisman Algebroid and Gauge Symmetry in Double Field Theory," arXiv:1901.04777. https://arxiv.org/pdf/1901.04777.pdf.

[20] A. Alekseev and P. Xu, "Derived brackets and Courant algebroids," *unpublished* **available at http://www.math.psu.edu/ping/anton-final.pdf** (2001) .

[21] A. Coimbra and R. Minasian, "M-theoretic lichnerowicz formula and supersymmetry," *Journal of High Energy Physics* **2019** no. 10, (Oct, 2019) , arXiv:1705.04330. http://dx.doi.org/10.1007/jhep10(2019)036.

[22] U. Carow-Watamura, N. Ikeda, T. Kaneko, and S. Watamura, "DFT in supermanifold formulation and group manifold as background geometry," *Journal of High Energy Physics* **04** (2019) 002, arXiv:1812.03464. http://dx.doi.org/10.1007/JHEP04(2019)002.

[23] M. Gualtieri, "Branes on Poisson Varieties," *in "The Many Facets of Geometry"* (Jul, 2010) 368–394, arXiv:0710.2719.http://dx.doi.org/10.1093/acprof:oso/9780199534920.003.0018.

[24] O. Hohm and B. Zwiebach, "Towards an invariant geometry of double field theory," *Journal of Mathematical Physics* **54** no. 3, (Mar, 2013) 032303, arXiv:1212.1736. http://dx.doi.org/10.1063/1.4795513.

[25] M. Garcia-Fernandez, "Ricci flow, Killing spinors, and T-duality in generalized geometry," *Advances in Mathematics* **350** (2019) 1059 – 1108, arXiv:1611.08926. https://arxiv.org/pdf/1611.08926.pdf.

[26] P. Ševera and F. Valach, "Courant algebroids, Poisson-Lie T-duality, and type II supergravities," *Communications in Mathematical Physics* **375** no. 1, (10, 2020) 307–344, arXiv:1810.07763. http://dx.doi.org/10.1007/s00220-020-03736-x.

[27] Y. Kosmann-Schwarzbach, "Derived brackets," *Lett.Math.Phys.* **69** (2004) 61–87, math/0312524. https://arxiv.org/pdf/math/0312524.pdf.

[28] P. Ševera and F. Valach, "Ricci flow, Courant algebroids, and renormalization of Poisson–Lie T-duality," *Letters in Mathematical Physics* **107** no. 10, (May, 2017) 1823–1835, arXiv:1610.09004. http://dx.doi.org/10.1007/s11005-017-0968-5.