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Quantum principal bundles and noncommutative differential calculus

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We give a sheaf theoretic approach to the notion of quantum principal bundle over projective bases and its first order differential calculus.

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1. Introduction

Quantum groups are born to encode quantum symmetries. We treat physical geometric objects like the Minkowski space as *homogeneous spaces*, with natural actions coming from their physical symmetries (see [18] and refs. therein). With quantization, however, we lose the geometrical point of view and we need the algebra of function, or better its quantization, to retrieve all the properties we are interested in.

The notion of principal bundle over an affine base is replaced by Hopf-Galois extension of the algebra of functions on the base and triviality is generalized by algebraic conditions, like the *cleft property* [2, 3, 30]. The case of an affine base was pursued successfully, via the notion of Hopf-Galois extension, by several authors, see [6–8, 19, 20] and refs. therein. However, this approach is not suitable when the base is not affine, because in this case we need an affine open cover to properly describe our geometrical object. The algebra of functions must be replaced by the structural sheaf and care must be exterted in gluing and localizing. The key idea is to associate a graded quantum ring to a projective base encoding its embedding into a projective space. This is obtained by a quantum line bundle, via the key notion of *quantum section* (see [11]), with a gluing procedure similar to the classical Proj one in [1, 28, 33, 34] (see also [12, 13, 15–17, 26] for the description of quantum projective varieties and flags). Once the quantum principal bundle concept is established in this more general setting, we can proceed and define a first order differential calculus (FODC) in sheaf theoretic terms. An alternative approach to quantum differential calculi on quantum flags for quantum algebraic groups can be also found in [9, 21–23, 31], but with different techniques.

The present work is organized as follows: we first describe a sheaf theoretic construction of quantum principal bundles and then how to define a differential calculus using Ore extension, summarizing the work [3] and announcing the main results in [4], yet to be published.

We discuss in detail the example of the special linear group $\mathrm{SL}_q(2, \mathbb{C})$ as a principal bundle on the projective line to elucidate our construction.

2. Quantum principal bundles via sheaves

In this section we give a sheaf theoretic point of view on the concept of principal bundle. We start with the classical definition (see [24]).

Definition 2.1. Let E and M be topological spaces, P a topological group and $\wp : E \longrightarrow M$ a continuous function. We say that $(E, M, \wp, P)$ is a *P-principal bundle* (or *principal bundle* for short) with total space E and base M, if the following conditions hold

1. $\wp$ is surjective.
2. P acts freely from the right on E.
3. P acts transitively on the fiber $\wp^{-1}(m)$ of each point $m \in M$.
4. E is locally trivial over M, i.e. there is an open covering $M = \cup U_i$ and homeomorphisms $\sigma_i : \wp^{-1}(U_i) \longrightarrow U_i \times P$ that are P-equivariant i.e., $\sigma_i(up) = \sigma_i(u)p$, $u \in U_i$, $p \in P$.

Let us look at a simple, yet elucidating example.

Example 2.2. Consider the special linear group $\mathrm{SL}_2(\mathbb{C})$ as a principal bundle over the complex projective line:

$$\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})/P = \mathbf{P}^1(\mathbb{C})$$

where

$$P = \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}$$

Notice that the base $M = \mathbf{P}^1(\mathbb{C})$ has no global non constant holomorphic functions and, furthermore, that we are considering the special, yet interesting, case in which the total space E has a group structure.

The first three properties of our previous definition are clear. The local triviality can be easily checked on the two open sets in $\mathrm{SL}_2(\mathbb{C})$:

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \det = \alpha\delta - \beta\gamma = 1,\ \alpha \neq 0 \right\} \quad \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \det = \alpha\delta - \beta\gamma = 1,\ \gamma \neq 0 \right\} \tag{1}$$

The fact that on projective bases like $\mathbf{P}^1(\mathbb{C})$ there are no non constant holomorphic functions requires to go beyond the ring theoretic description of the properties of principal bundles, when advancing to the quantum setting.

We start with a sheaf theoretic definition of quantum ringed space, following Manin [29] and Pflaum [32], that takes into account the non affine setting and then we proceed to give a sheaf theoretic definition of principal bundle.

Definition 2.3. We say that (M, O_M) is a *quantum ringed space*, if M is a classical topological space and O_M is a sheaf of algebras over M.

Let us look at our previous example.

Example 2.4. Consider $M = \mathrm{SL}_2(\mathbb{C})/P = \mathbf{P}^1(\mathbb{C})$. We can define an affine open cover of $\mathbf{P}^1(\mathbb{C}) = U \cup V$:

$$U = \{[x_0, x_1] \in \mathbf{P}^1(\mathbb{C}) \mid x_0 \neq 0\}, \quad V = \{[x_0, x_1] \in \mathbf{P}^1(\mathbb{C}) \mid x_1 \neq 0\}$$

and the ring of functions on the open sets:

$$\mathcal{O}_M(U) := \mathbb{C}[u], \qquad \mathcal{O}_M(V) := \mathbb{C}[v]$$

where $u = x_1/x_0$, $v = x_0/x_1$. As one can readily check, there is compatibility on intersection (change of chart):

$$u \mapsto v := 1/u$$

so we have $\mathcal{O}_M(U \cap V) = \mathbb{C}[u, u^{-1}]$. The assignment:

$$U \cup V \mapsto \mathbb{C}, \quad U \mapsto \mathcal{O}_M(U), \quad V \mapsto \mathcal{O}_M(V), \quad U \cap V \mapsto \mathcal{O}_M(U \cap V)$$

defines a sheaf on M. Notice that we give the sheaf *not* on the full Zariski topology of M. This is necessary in the quantum setting in order to obtain non trivial noncommutative localizations, i.e. the quantized algebras of functions on open sets, in meaningful examples. We shall see that the rough topology we use is however sufficient to describe the principal bundle $\mathrm{SL}_2(\mathbb{C}) \to \mathbf{P}^1(\mathbb{C})$.

We are ready for the key definition of this section, motivated by the classical Example 2.2. Let $(M, \mathcal{O}_M)$ be a quantum ringed space and H a Hopf algebra.

Definition 2.5. We say that a sheaf $\mathcal{F}$ on M is an H-*quantum principal bundle* over the quantum ringed space $(M, \mathcal{O}_M)$ if:

- $\mathcal{F}$ is a sheaf of right H-comodule algebras.
- There exists an open covering $\{U_i\}$ of M such that:
  1. $\mathcal{F}(U_i)^{\mathrm{co}H} = \mathcal{O}_M(U_i)$,
  2. $\mathcal{F}$ is *locally cleft*, i.e., $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \otimes H$, as left $\mathcal{F}(U_i)^{\mathrm{co}H}$-modules and right H-comodules.

Notice that for a P-principal bundle $E \longrightarrow M = E/P$, according to Def. 2.1, the H-comodule structure encodes the right action of the group P on E, while the locally cleft condition encodes effectively the local triviality in the classical sense (see also [20], [2] for more details on this subtle question).

We are now going to reinterpret Example 2.2 of a classical principal bundle in the light of Def. 2.5.

Example 2.6. Consider, as before:

$$E = \mathrm{SL}_2(\mathbb{C}) \longrightarrow M = \mathrm{SL}_2(\mathbb{C})/P \simeq \mathbf{P}^1(\mathbb{C})$$

where

$$P = \left\{ \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \, \det = \alpha\delta - \beta\gamma = 1 \right\}$$

We define the sheaf $\mathcal{F}$ on $M = U \cup V$

$$\begin{array}{ccc} U & \mapsto & \mathcal{F}(U) := \mathbb{C}[\alpha,\beta,\gamma,\delta][\alpha^{-1}]/(\det -1) \\ V & \mapsto & \mathcal{F}(V) := \mathbb{C}[\alpha,\beta,\gamma,\delta][\gamma^{-1}]/(\det -1) \\ U\cap V & \mapsto & \mathcal{F}(U\cap V) := \mathbb{C}[\alpha,\beta,\gamma,\delta][\alpha^{-1},\gamma^{-1}]/(\det -1) \end{array}$$

and $\mathcal{F}(U \cup V) = \mathbb{C}[\mathrm{SL}_2] := \mathbb{C}[\alpha,\beta,\gamma,\delta]/(\det -1)$.

Notice that the opens U and V in M lift to open sets defined in (1) in Example 2.2.

As for the right coaction of $H := \mathbb{C}[t, p][t^{-1}]$ on the sheaf $\mathcal{F}$, define:

$$\begin{array}{ccc} \mathbb{C}[\mathrm{SL}_2] & \longrightarrow & \mathbb{C}[\mathrm{SL}_2] \otimes H \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \mapsto & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} \end{array}$$

This coaction extends to $\mathcal{F}$ by the universal property of commutative localizations. Notice that H is a Hopf algebra with coalgebra structure and antipode

$$\Delta\begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} \otimes \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix}, \ \epsilon\begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ S\begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} t^{-1} & -p \\ 0 & t \end{pmatrix}.$$

Hence we have that $\mathcal{F}$ is a (quantum) principal bundle on $\mathbf{P}^1(\mathbb{C})$.

We now give a quantum version of the previous example.

Example 2.7. Let us consider the *quantum special linear group* as defined by Manin in [29]:

$$\mathbb{C}_q[\mathrm{SL}_2] \ = \ \mathbb{C}_q\langle\alpha,\beta,\gamma,\delta\rangle \ / \ I_M + (\alpha\delta - q^{-1}\beta\gamma - 1) \ .$$

I_M is the ideal of the *Manin relations*:

$$\alpha\beta = q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \gamma\delta = q^{-1}\delta\gamma,$$

$$\beta\gamma = \gamma\beta \qquad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma$$

$\mathbb{C}_q[\mathrm{SL}_2]$ is a Hopf algebra:

$$\Delta\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ \epsilon\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q\beta \\ -q^{-1}\gamma & \alpha \end{pmatrix}$$

Define the sheaf on $\mathbf{P}^1(\mathbb{C}) = U \cup V$:

$$\begin{aligned} U &\mapsto \mathcal{F}(U) := \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1}] \\ V &\mapsto \mathcal{F}(V) := \mathbb{C}_q[\mathrm{SL}_2][\gamma^{-1}] \\ U\cap V &\mapsto \mathcal{F}(U\cap V) := \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1},\gamma^{-1}] \\ U\cup V &\mapsto \mathcal{F}(U\cup V) := \mathbb{C}_q[\mathrm{SL}_2] \end{aligned}$$

where here adjoining the element α^{-1} is with respect to the Ore localization (see [3] Sec. 3 and refs. therein for more details).

This sheaf defines a quantum principal bundle on $\mathbf{P}^1(\mathbb{C})$ for $H = \mathbb{C}_q[\mathrm{SL}_2]/(\gamma)$, the quotient Hopf algebra of $\mathbb{C}_q[\mathrm{SL}_2]$ with respect to the Hopf ideal (γ) generated by γ. See also Example 2.2. We have that

$$\begin{array}{ccc} \mathbb{C}_q[\mathrm{SL}_2] & \longrightarrow & \mathbb{C}_q[\mathrm{SL}_2] \otimes H \\ \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \mapsto & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & p \\ 0 & t^{-1} \end{pmatrix} \end{array}$$

is a well-defined coaction. The properties of the Ore localization ensure that the latter extends to the whole sheaf. We leave to the reader the easy checks involved to verify Def. 2.5. In particular $\mathcal{F}^{\mathrm{co}H}(U) = O_M(U)$ is generated by $u = \gamma\alpha^{-1}$ and $\mathcal{F}^{\mathrm{co}H}(V) = O_M(V)$ is generated by $v = \alpha\gamma^{-1}$.

3. The quantum principal bundle $G \to G/P$

Let G be a complex semisimple algebraic group and P a closed algebraic subgroup of G. Let $O_q(G)$ and $O_q(P) := O_q(G)/I_P$ be the Hopf algebra quantizations of the algebraic functions on G and P respectively.

Notice: here q is a parameter, when q is specialized to 1 we obtain the commutative algebras $O(G)$ and $O(P)$ of algebraic functions on G and P respectively.

Let $s \in O_q(G)$ be a *quantum section*, that is, an element $s \in O_q(G)$ such that $\Delta(s) - s \otimes s \in O_q(G) \otimes I_P$. The element s can be seen as the quantum version of the lift to $O(G)$ of the character of P defining the line bundle $\mathcal{L}$ giving the projective embedding of G/P. Hence the name quantum section. This is a key concept, s controls the ring of algebraic functions on G/P and its quantization. The elements s_i appearing in the expression of the comultiplication of s

$$\Delta_G(s) = s^i \otimes s_i$$

give an open cover of G:

$$G = \cup U_i, \qquad U_i := \{g \in G \,|\, s_i(g) \neq 0\}$$

so that $O(U_i) = O(G)[s_i^{-1}]$. The projection of the open cover U_i under $G \longrightarrow G/P$ is an open cover of G/P (see [3] Sec. 3).

Notice that the opens $U_I := \cap_{i \in I} U_i$, $I = (i_1, \dots, i_r)$, $r \in \mathbb{N}$, form a base $\mathcal{B}$ for a topology and that in order to give a sheaf for such topology it is enough to give an assignment for each U_I (see [14] Ch. 1).

We now state the main result in [3], Thm 4.8.

Let $U_I := \cap_{i \in I} U_i$ and $O_q(G)S_I^{-1}$ denote the subsequent localizations with respect to $S_{i_1} \dots S_{i_r}$, $I = (i_1, \dots, i_r)$, $S_i := \{s_i^k,\ k \in \mathbb{N}\}$. Similarly, let $s_I := s_{i_1} \dots s_{i_r}$.

Theorem 3.1. *Let the notation be as above. Assume S_i is Ore and that subsequent Ore localizations of $O_q(G)$ with respect to the S_i's do not depend on the order.*

1. *The assignment*
$$U_I \mapsto O_M(U_I) := \langle s_K s_I^{-1} \rangle \subset O_q(G)S_I^{-1},$$
where $\langle s_K s_I^{-1} \rangle$ *is the subalgebra in* $O_q(G)S_I^{-1}$ *generated by the elements* $s_K s_I^{-1}$*, defines a quantum ringed space on* M*.*

2. *The assignment* $U_I \mapsto \mathcal{F}_G(U_I) := O_q(G)S_I^{-1}$ *defines a sheaf of right* $O_q(P)$*-comodule algebras on the quantum ringed space* M*.*

3. $\mathcal{F}_G(U_I)^{\mathrm{co}\, O_q(P)} = O_M(U_I)$.

If locally trivial, $\mathcal{F}_G$ *is a quantum principal bundle.*

Example 3.2. Let us look again at the Example 2.2. Here $G = \mathrm{SL}_2(\mathbb{C})$ and a quantum section is $\alpha \in O_q(G)$. We have that
$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma$$
so the s_i's are given by α and γ. As one can readily check, the construction in Example 2.7 verifies all the hypothesis of the theorem, including the local triviality. Hence it is a quantum principal bundle.

It is possible to construct in the same fashion also an example of a quantum principal bundle over the n-dimensional projective space. This is done in Sec. 5 in [3].

4. Quantum differential calculus: the affine setting

We start with the affine picture, that is we define a *quantum differential calculus on* H*-comodule algebras*. Let H be a Hopf algebra, A a right H-comodule algebra with coaction $\delta_A\colon A \longrightarrow A \otimes H$.

Definition 4.1. A right H-covariant *first order differential calculus* (*FODC*) on A is an A-bimodule Γ, together with a $\mathbb{C}$-linear map $\mathrm{d}\colon A \longrightarrow \Gamma$ such that:

- $\mathrm{d}(fg) = \mathrm{d}(f)g + f\mathrm{d}g$ (Leibniz Rule)
- $\Gamma = A\mathrm{d}A$ (Surjectivity)
- $\Delta_\Gamma\colon \Gamma \longrightarrow \Gamma \otimes H,\ f\mathrm{d}g \mapsto f_0\mathrm{d}g_0 \otimes f_1 g_1$ is well-defined (right H-covariance)

where, as usual, we denote $\delta_A(f) = f_0 \otimes f_1 \in A \otimes H$ for $f \in A$.

Similarly left H-covariant FODC are defined on left H-comodule algebras and H-bicovariant FODC are defined on H-bicomodule algebras. If H is viewed as an H-comodule algebra with respect to its comultiplication we simply refer to right H-covariant FODC on H as right covariant FODC and similarly for left and bicovariant FODC on H. If we do not require covariance we refer to (Γ, d) as a FODC on an algebra A.

It follows that (Γ, d) is a right H-covariant FODC on (A, δ_A) if and only if (Γ, Δ_Γ) is a right H-covariant A-bimodule and d is right H-colinear, c.f. [35]. For a detailed introduction to covariant FODC we refer to [5, 27].

A morphism $(\Phi, \phi)\colon (\Gamma, \mathrm{d}) \to (\Gamma', \mathrm{d}')$ of right H-covariant FODC on A and A' is a morphism $\phi\colon A \to A'$ of right H-comodule algebras and a morphism $\Phi\colon \Gamma \to \Gamma'$ of right H-covariant A-bimodules such that $\Phi \circ \mathrm{d} = \mathrm{d}' \circ \phi$. If Φ and ϕ are isomorphisms we call (Γ, d) and (Γ', d') equivalent. More in general, we call an algebra morphism $\phi\colon A \to A'$ *differentiable* if there exists an A-bimodule morphism $\mathrm{d}_\phi\colon \Gamma \to \Gamma'$ such that

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ {\scriptstyle \mathrm{d}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}'} \\ \Gamma & \xrightarrow{\mathrm{d}_\phi} & \Gamma' \end{array}$$

commutes, where the A-bimodule action on Γ' is given by ϕ. In the following proposition, taken from [4], it is shown that for injective or surjective algebra morphisms we can induce FODC such that the algebra map is differentiable.

Proposition 4.2 (Induced FODC)**.** *Consider a right H-covariant FODC (Γ, d) on a right H-comodule algebra (A, δ_A).*

i.) Every injective right H-comodule algebra morphism $\iota\colon B \hookrightarrow A$ gives a right H-covariant FODC $(\Gamma_\iota, \mathrm{d}_\iota)$ on B, where $\Gamma_\iota = \iota(B)\mathrm{d}\iota(B) \subseteq \Gamma$ and $\mathrm{d}_\iota = \mathrm{d} \circ \iota\colon B \to \Gamma_\iota$.

ii.) Every surjective right H-comodule algebra morphism $\pi\colon A \twoheadrightarrow B$ gives a right H-covariant FODC $(\Gamma_\pi, \mathrm{d}_\pi)$ on B, where $\Gamma_\pi = \Gamma/\Gamma_I$ with $\Gamma_I = A\mathrm{d}I + I\mathrm{d}A$, $I = \ker\pi$ and $\mathrm{d}_\pi\colon B \to \Gamma_\pi$ is induced by d on the quotient $B \cong A/I$.

iii.) If $0 \to \ker\pi \to A \xrightarrow{\pi} B \to 0$ is a split exact sequence of right H-comodule algebras with section $\iota\colon B \to A$ the induced right H-covariant FODC $(\Gamma_\iota, \mathrm{d}_\iota)$ and $(\Gamma_\pi, \mathrm{d}_\pi)$ are equivalent.

We call $(\Gamma_\iota, \mathrm{d}_\iota)$ the pullback calculus and $(\Gamma_\pi, \mathrm{d}_\pi)$ the quotient calculus.

In the rest of this section we discuss the example of a bicovariant FODC on $\mathbb{C}_q[\mathrm{SL}_2]$. Via Proposition 4.2 we induce a bicovariant calculus on the Hopf algebra quotient $\pi\colon \mathbb{C}_q[\mathrm{SL}_2] \to \mathcal{O}_q(P)$.

Example 4.3. The Hopf algebra $A = \mathbb{C}_q[\mathrm{SL}_2]$ discussed in Example 2.7 admits a 4-dimensional bicovariant FODC (Γ, d), as described in [36], see also [25]. Here $\Gamma = \mathrm{span}_A\{\omega^1, \omega^2, \omega^3, \omega^4\}$ is the free left A-module generated by four 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$. The right A-module action on Γ

is determined by

$$\begin{aligned}
\omega^1\alpha &= q\alpha\omega^1, & \omega^1\beta &= q^{-1}\beta\omega^1,\\
\omega^2\alpha &= \alpha\omega^2, & \omega^2\beta &= -q^{-1}\lambda\alpha\omega^1 + \beta\omega^2,\\
\omega^3\alpha &= -q^{-1}\lambda\beta\omega^1 + \alpha\omega^3, & \omega^3\beta &= \beta\omega^3,\\
\omega^4\alpha &= -\lambda\beta\omega^2 + q^{-1}\alpha\omega^4, & \omega^4\beta &= q^{-1}\lambda^2\beta\omega^1 - \lambda\alpha\omega^3 + q\beta\omega^4,\\
\omega^1\gamma &= q\gamma\omega^1, & \omega^1\delta &= q^{-1}\delta\omega^1,\\
\omega^2\gamma &= \gamma\omega^2, & \omega^2\delta &= \delta\omega^2 - q^{-1}\lambda\gamma\omega^1,\\
\omega^3\gamma &= \gamma\omega^3 - q^{-1}\lambda\delta\omega^1, & \omega^3\delta &= \delta\omega^3,\\
\omega^4\gamma &= -\lambda\delta\omega^2 + q^{-1}\gamma\omega^4, & \omega^4\delta &= q^{-1}\lambda^2\delta\omega^1 - \lambda\gamma\omega^3 + q\delta\omega^4,
\end{aligned} \tag{2}$$

where $\lambda := q^{-1} - q$. The differential $\mathrm{d}\colon A \longrightarrow \Gamma$ is defined by

$$\begin{array}{rcl}
\mathrm{d}\alpha &=& \frac{q-1}{\lambda}\,\alpha\omega^1 + \frac{q^{-1}-1}{\lambda}\,\alpha\omega^4 - \beta\omega^2\\
\mathrm{d}\beta &=& Q\beta\omega^1 + \frac{q-1}{\lambda}\,\beta\omega^4 - \alpha\omega^3\\
\mathrm{d}\gamma &=& \frac{q-1}{\lambda}\,\gamma\omega^1 + \frac{q^{-1}-1}{\lambda}\,\gamma\omega^4 - \delta\omega^2\\
\mathrm{d}\delta &=& Q\delta\omega^1 + \frac{q-1}{\lambda}\,\delta\omega^4 - \gamma\omega^3,
\end{array} \tag{3}$$

$Q := \frac{q^{-1}(\lambda^2+1)-1}{\lambda}$, and is extended via the Leibniz rule. Then $(\Gamma, \Delta_\Gamma, {}_\Gamma\Delta)$ is a bicovariant H-bimodule such that

$$\Delta_\Gamma\left(\sum_i a^i \mathrm{d}b^i\right) = \sum_i a^i_1 \mathrm{d}b^i_1 \otimes a^i_2 b^i_2, \qquad {}_\Gamma\Delta\left(\sum_i a^i \mathrm{d}b^i\right) = \sum_i a^i_1 b^i_1 \otimes a^i_2 \mathrm{d}b^i_2$$

for all $a^i, b^i \in A$, i.e. (Γ, d) is a bicovariant FODC on A. With this definition the 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$ are left coinvariant. The induced quotient calculus (Γ_H, d_H) on $H = O_q(P)$ can be shown to be the 2-dimensional bicovariant FODC with Γ_H being the free left H-module generated by the basis $\{[\omega^3], [\omega^4]\}$ of left coinvariant elements (notice that $[\omega^1] = [\omega^2] = 0$, while $[\omega^3], [\omega^4]$ are linearly independent). The resulting commutation relations are

$$[\omega^3]t = t[\omega^3]\,,\ \ [\omega^3]p = p[\omega^3]\,,\ \ [\omega^4]t = q^{-1}t[\omega^4]\,,\ \ [\omega^4]p = qp[\omega^4] - \lambda t[\omega^3]\,,$$

and the differential reads

$$\mathrm{d}_H t = \tfrac{q^{-1}-1}{\lambda}\, t\,[\omega^4]\,, \quad \mathrm{d}_H p = -t\,[\omega^3] + \tfrac{q-1}{\lambda}\, p\,[\omega^4].$$

Via the Hopf algebra projection $\pi\colon A \longrightarrow H$ we can view (A, δ_A) as a right H-comodule algebra and $(\Gamma, \mathrm{d}, \Delta_H)$ as a right H-covariant FODC, where $\delta_A := (\mathrm{id} \otimes \pi) \circ \Delta\colon A \longrightarrow A \otimes H$ and $\Delta_H := (\mathrm{id} \otimes \pi) \circ \Delta_\Gamma\colon \Gamma \longrightarrow \Gamma \otimes H$.

5. A sheaf approach to noncommutative differential calculi

We now want to take a sheaf theoretic point of view on the construction of the previous section.

Assume we have a quantum ringed space (M, O_M) with open cover $M = \cup U_i$ and $\{U_I\}$ a topological basis. Recall from Definition 2.5 that a quantum principal bundle $\mathcal{F}$ on M is a sheaf of H-comodule algebras which is locally cleft.

Definition 5.1. A right H-covariant FODC on $\mathcal{F}$ is a sheaf Υ of right H-covariant $\mathcal{F}$-bimodules with a morphism of sheaves of right H-comodules

$$\mathrm{d}\colon \mathcal{F} \longrightarrow \Upsilon$$

satisfying *locally*

- $\mathrm{d}_I(fg) = \mathrm{d}_I(f)g + f\mathrm{d}_I g$ for all $f, g \in \mathcal{F}(U_I)$ (Leibniz Rule)
- $\Upsilon(U_I) = \mathcal{F}(U_I)\mathrm{d}_I\mathcal{F}(U_I)$ (Surjectivity)

where $\mathrm{d}_I := \mathrm{d}|_{U_I}\colon \mathcal{F}(U_I) \to \Upsilon(U_I)$.

Notice that for each open $U \subseteq M$ $\mathrm{d}|_U\colon \mathcal{F}(U) \to \Upsilon(U)$ is a right H-comodule map. Let P be an affine Lie group and $H = \mathbb{C}[P]$ the associated Hopf algebra. Clearly the de Rham calculus on a smooth P-manifold fits Definition 5.1. The remaining part of this proceeding is devoted to provide non-trivial noncommutative examples of Definition 5.1.

We start with the general construction of the *Ore calculus* on the quantization $\mathcal{O}_q(G)$ of a complex semisimple algebraic group G. Take a parabolic subgroup P of G and a quantization $\mathcal{O}_q(P)$ thereof. Given a quantum section s on $\mathcal{O}_q(G)$ we consider the corresponding open cover $\{U_i\}$, the induced topological basis $\{U_I\}$ and the sheaves $\mathcal{F}_G, \mathcal{O}_M$ given in Theorem 3.1. We denote the restriction morphisms of $\mathcal{F}_G$ by $r_{JI}\colon \mathcal{F}_G(U_I) \to \mathcal{F}_G(U_J)$ for $U_J \subseteq U_I$.

Lemma 5.2. *Given a right $\mathcal{O}_q(P)$-covariant FODC (Γ, d) on $\mathcal{O}_q(G)$ the assignment*

$$\Upsilon_G\colon U_I \mapsto \Upsilon_G(U_I) := \mathcal{F}_G(U_I) \otimes_{\mathcal{O}_q(G)} \Gamma \otimes_{\mathcal{O}_q(G)} \mathcal{F}_G(U_I)$$

determines a sheaf Υ_G of right H-covariant $\mathcal{F}_G$-bimodules on M. The restriction morphisms $r^{\Upsilon}_{JI}\colon \Upsilon_G(U_I) \to \Upsilon_G(U_J)$, $U_J \subseteq U_I$, are defined by

$$r^{\Upsilon}_{JI}(f \otimes_{\mathcal{O}_q(G)} \omega \otimes_{\mathcal{O}_q(G)} g) := r_{JI}(f) \otimes_{\mathcal{O}_q(G)} \omega \otimes_{\mathcal{O}_q(G)} r_{JI}(g),$$

where $f, g \in \mathcal{F}_G(U_I)$ and $\omega \in \Gamma$.

We can extend $\mathrm{d}\colon \mathcal{O}_q(G) \to \Gamma$ to a differential $\mathrm{d}_I\colon \mathcal{F}_G(U_I) \to \Upsilon_G(U_I)$ on $\mathcal{F}_G(U_I)$ by defining $\mathrm{d}_I|_{\mathcal{O}_q(G)} = \mathrm{d}$ and

$$\mathrm{d}_I(s_{i_k}^{-1}) := -s_{i_k}^{-1} \otimes_{\mathcal{O}_q(G)} \mathrm{d}s_{i_k} \otimes_{\mathcal{O}_q(G)} s_{i_k}^{-1}, \tag{4}$$

where $I = (i_1, \dots, i_r)$ and $1 \leq k \leq r$. In the following we shall omit the tensor product in (4) thus simply writing $\mathrm{d}_I(s_{i_k}^{-1}) = -s_{i_k}^{-1}\mathrm{d}(s_{i_k})s_{i_k}^{-1}$. Then d_I is extended to arbitrary elements of $\mathcal{F}_G(U_I)$ via the Leibniz rule. As a consequence we obtain the following main result of this proceeding, c.f. [4] for more explanations and the proof of this non trivial result.

Theorem 5.3. *Any right $\mathcal{O}_q(P)$-covariant FODC (Γ, d) on $\mathcal{O}_q(G)$ induces a right $\mathcal{O}_q(P)$-covariant FODC $(\Upsilon_G, \mathrm{d}_G)$ on $\mathcal{F}_G$. The subsheaf $(\Upsilon_M, \mathrm{d}_M)$ given by the pullback calculi corresponding to the algebra embeddings $\mathcal{O}_M(U_I) = \mathcal{F}_G^{\mathrm{co}H}(U_I) \subseteq \mathcal{F}_G(U_I)$ is a FODC on $\mathcal{O}_M$.*

We exemplify the machinery provided by Theorem 5.3 with the covariant FODC presented in Example 4.3.

Example 5.4. Consider the 4-dimensional FODC (Γ, d) on $A = \mathbb{C}_q[\mathrm{SL}_2]$ viewed as a right $H = O_q(P)$-covariant FODC, c.f. Example 4.3. The sheaf $\mathcal{F}_G$ is spelled out explicitly in Example 2.7, with opens U and V defined in Example 2.4. According to Theorem 5.3, on the level of differential 1-forms we have the assignment

$$\begin{aligned}
U &\mapsto \Upsilon_G(U) := \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1}] \otimes_A \Gamma \otimes_A \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1}] \\
V &\mapsto \Upsilon_G(V) := \mathbb{C}_q[\mathrm{SL}_2][\gamma^{-1}] \otimes_A \Gamma \otimes_A \mathbb{C}_q[\mathrm{SL}_2][\gamma^{-1}] \\
U \cap V &\mapsto \Upsilon_G(U \cap V) := \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1}, \gamma^{-1}] \otimes_A \Gamma \otimes_A \mathbb{C}_q[\mathrm{SL}_2][\alpha^{-1}, \gamma^{-1}] \\
U \cup V &\mapsto \Upsilon_G(U \cup V) := \Gamma
\end{aligned}$$

with covariant differential on U and V defined as in equation (4)

$$\mathrm{d}_U(\alpha^{-1}) = -\alpha^{-1}\mathrm{d}(\alpha)\alpha^{-1}, \qquad \mathrm{d}_V(\gamma^{-1}) = -\gamma^{-1}\mathrm{d}(\gamma)\gamma^{-1}$$

and similarly for the differential on $U \cap V$.

We now give a more explicit description of this right H-covariant FODC on the sheaf $\mathcal{F}_G$. We show that

i.) $\Upsilon_G(U_I)$ is a free left $\mathcal{F}_G(U_I)$-module, generated by the left coinvariant 1-forms $\omega^1, \omega^2, \omega^3, \omega^4$.

ii.) The base forms $(\Upsilon_M, \mathrm{d}_M)$ are determined by the free modules $\Upsilon_M(U) = \mathrm{span}_{O_M(U)}\{\alpha^{-2}\omega^2\}$ and $\Upsilon_M(V) = \mathrm{span}_{O_M(V)}\{\gamma^{-2}\omega^2\}$, i.e. they are the left span over $O_M(U)$ and $O_M(V)$ of the forms $\alpha^{-2}\omega^2$ and $\gamma^{-2}\omega^2$, respectively. The algebras $O_M(U)$ and $O_M(V)$ are generated by $u = \gamma\alpha^{-1}$ and $v = \alpha\gamma^{-1}$ as discussed at the end of Example 2.7. We thus recover the commutation relations

$$(\mathrm{d}_U u)u = q^2 u \mathrm{d}_U u, \qquad (\mathrm{d}_V v)v = q^{-2} v \mathrm{d}_V v$$

described by Chu, Ho and Zumino in [10]. Furthermore $\Upsilon_M(U \cap V) = \mathrm{span}_{O_M(U\cap V)}\{\alpha^{-2}\omega^2\}$ is a free left $O_M(U \cap V)$-module and $\mathrm{d}_U u = -q^2 u^2 \mathrm{d}_V v$ in $\Upsilon_M(U \cap V)$.

Proof. We show $i.)$ and $ii.)$ on the open U; the results for the other opens follow similarly. Recall form Example 4.3 that we have the free left A-module of 1-forms $\Gamma = \mathrm{span}_A\{\omega^1, \omega^2, \omega^3, \omega^4\}$, then from Lemma 5.2 we have $\mathrm{span}_{\mathcal{F}_G(U)}\{\omega^1, \omega^2, \omega^3, \omega^4\} \subseteq \Upsilon_G(U)$. Point $i.)$ is proven by showing the other inclusion. This reduces to show that $\mathrm{d}_U(\alpha^{-1}) \in \mathrm{span}_{\mathcal{F}_G(U)}\{\omega^1, \omega^2, \omega^3, \omega^4\}$. Using equations (2) and (3) we obtain

$$\alpha \mathrm{d}\alpha = \frac{q-1}{\lambda}\alpha^2\omega^1 + \frac{q^{-1}-1}{\lambda}\alpha^2\omega^4 - \alpha\beta\omega^2 = q^{-1}\frac{q-1}{\lambda}\alpha\omega^1\alpha + q\frac{q^{-1}-1}{\lambda}\alpha\omega^4\alpha - \beta\omega^2\alpha,$$

which implies

$$\mathrm{d}(\alpha)\alpha^{-1} = \alpha^{-1}\alpha\mathrm{d}(\alpha)\alpha^{-1} = q^{-1}\frac{q-1}{\lambda}\omega^1 + q\frac{q^{-1}-1}{\lambda}\omega^4 - \alpha^{-1}\beta\omega^2. \tag{5}$$

This shows that $\mathrm{d}_U\alpha^{-1} = -\alpha^{-1}\mathrm{d}(\alpha)\alpha^{-1} \in \mathrm{span}_{\mathcal{F}_G(U)}\{\omega^1, \omega^2, \omega^3, \omega^4\}$. In order to prove point $ii.)$ we show that $\mathrm{d}_U(u) = -\alpha^{-2}\omega^2$. The commutation relations (2) and (3) give

$$\begin{aligned}
\mathrm{d}(\gamma)\alpha^{-1} =& \frac{q-1}{\lambda}\gamma\omega^1\alpha^{-1} + \frac{q^{-1}-1}{\lambda}\gamma\omega^4\alpha^{-1} - \delta\omega^2\alpha^{-1} \\
=& q^{-1}\frac{q-1}{\lambda}\gamma\alpha^{-1}\omega^1 + \frac{q^{-1}-1}{\lambda}\gamma(q\alpha^{-1}\omega^4 + q\lambda\alpha^{-1}\beta\omega^2\alpha^{-1}) - \delta\omega^2\alpha^{-1}
\end{aligned}$$

and so

$$\begin{aligned}
\mathrm{d}_U(u) =&\mathrm{d}(\gamma)\alpha^{-1} + \gamma \mathrm{d}_U \alpha^{-1} \\
=&\frac{q^{-1}-1}{\lambda}\gamma(q\alpha^{-1}\omega^4 + q\lambda\alpha^{-1}\beta\omega^2\alpha^{-1}) - \delta\omega^2\alpha^{-1} - q\frac{q^{-1}-1}{\lambda}\gamma\alpha^{-1}\omega^4 + \alpha^{-1}\gamma\alpha^{-1}\beta\omega^2 \\
=&q(q^{-1}-1)\gamma\alpha^{-1}\beta\omega^2\alpha^{-1} - \delta\omega^2\alpha^{-1} + \alpha^{-1}\gamma\alpha^{-1}\beta\omega^2 \\
=&q^{-1}\gamma\alpha^{-1}\beta\omega^2\alpha^{-1} - \delta\omega^2\alpha^{-1} \\
=&-\alpha^{-2}\omega^2,
\end{aligned}$$

where we also used (5) and the determinant relation $\delta\alpha^{-1} - q^{-1}\alpha^{-1}\beta\gamma\alpha^{-1} = \alpha^{-2}$. The commutation relation

$$\mathrm{d}_U(u)u = -\alpha^{-2}\omega^2\gamma\alpha^{-1} = -\alpha^{-2}\gamma\alpha^{-1}\omega^2 = -q^2\gamma\alpha^{-1}\alpha^{-2}\omega^2 = q^2 u \mathrm{d}_U u$$

shows that the $\mathcal{F}_G^{\mathrm{co}H}(U) = \mathcal{O}_M(U)$-bimodule $\Upsilon_M(U)$ is the free left $\mathcal{F}_G^{\mathrm{co}H}(U) = \mathcal{O}_M(U)$-module freely generated by $\mathrm{d}_U(u) = -\alpha^{-2}\omega^2$. □

6. Conclusions

We have defined a notion of Quantum Principal Bundle over a non affine base and provided a concrete example, the quantization of the principal bundle $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathbf{P}^1(\mathbb{C})$, the complex projective line. We have also stated a general theorem regarding the flag varieties of classical groups (see [3] Ch. 4, for more details).

We have defined a first order differential calculus on quantum principal bundles over projective bases elucidating our definition with a calculus on the quantization of the bundle $\mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathbf{P}^1(\mathbb{C})$.

Appendix: The gluing procedure

We pedagogically illustrate the gluing of two right H-comodule algebras $\mathcal{F}_G(U_I)$, $\mathcal{F}_G(U_J)$ obtained from Theorem 3.1 on two opens $U_I, U_J \in \mathcal{B}$. We set

$$\mathcal{F}_G(U_I \cup U_J) := \{\vec{f} \in \Pi_{U_K \in \mathcal{B}:\, U_K \subseteq U_I \cup U_J} \mathcal{F}_G(U_K) \mid \forall U_K \subseteq U_L \subseteq U_I \cup U_J :\ f_K = r_{KL}(f_L)\}, \quad (6)$$

where f_K is the component of $\vec{f}$ corresponding to $\mathcal{F}_G(U_K)$ for all $U_K \subseteq U_I \cup U_J$. Note that $\mathcal{F}_G(U_I \cup U_J)$ is not empty since it certainly contains the diagonal in $\mathcal{O}_q(G) \times \ldots \times \mathcal{O}_q(G)$. Since $\mathcal{B}$ is finite we can number all index sets $I_1, \ldots, I_n$ such that $U_{I_k} \subseteq U_I \cup U_J$ for all $0 < k \le n$. Then (6) is a $\mathbb{C}_q$-module with respect to the action $\lambda \cdot (f^1, \ldots, f^n) = (\lambda \cdot f^1, \ldots, \lambda \cdot f^n)$ and an associative algebra with respect to componentwise multiplication $(f^1, \ldots, f^n) \cdot (g^1, \ldots, g^n) = (f^1 g^1, \ldots, f^n g^n)$ and unit $(1, \ldots, 1)$, for all $\lambda \in \mathbb{C}_q$ and $(f^1, \ldots, f^n), (g^1, \ldots, g^n) \in \mathcal{F}_G(U_I \cup U_J)$, where $f^k, g^k \in \mathcal{F}_G(U_{I_k})$ for all $0 < k \le n$. Moreover, we structure (6) as a right $\mathcal{O}_q(P)$-comodule algebra via

$$\Delta_R(f^1, \ldots, f^n) := (f^1_0, 0, \ldots, 0) \otimes f^1_1 + \ldots + (0, \ldots, 0, f^n_0) \otimes f^n_1$$

for all $(f^1,\dots,f^n)\in\mathcal{F}_G(U_I\cup U_J)$. In fact $\Delta_R(1,\dots,1)=(1,0,\dots,0)\otimes 1+\dots+(0,\dots,0,1)\otimes 1=(1,\dots,1)\otimes 1$ and

$$\begin{aligned}\Delta_R(f^1,\dots,f^n)\Delta_R(g^1,\dots,g^n)=&((f_0^1,0,\dots,0)\otimes f_1+\dots+(0,\dots,0,f_0^n)\otimes f_1^n)\\ &\cdot((g_0^1,0,\dots,0)\otimes g_1^1+(0,\dots,0,g_0^n)\otimes g_1^n)\\ =&(f_0^1g_0^1,0,\dots,0)\otimes f_1^1g_1^1+\dots+(0,\dots,0,f_0^ng_0^n)\otimes f_1^ng_1^n\\ =&\Delta_R((f^1,\dots,f^n)\cdot(g^1,\dots,g^n))\end{aligned}$$

for all $(f^1,\dots,f^n),(g^1,\dots,g^n)\in\mathcal{F}_G(U_I\cup U_J)$.

For an open $U_K\subseteq U_I\cup U_J$ we define the restriction morphism $r_{U_K,U_I\cup U_J}:\mathcal{F}_G(U_I\cup U_J)\to\mathcal{F}_G(U_K)$ by $r_{U_K,U_I\cup U_J}(\vec{f}):=f_K$, where f_K is the component of $\vec{f}$ in $\mathcal{F}_G(U_K)$. An arbitrary open is the union $U=U_{J_1}\cup\dots U_{j_m}$ of elements in $\mathcal{B}$. Note that $\mathcal{B}$ is finite, so it suffices to consider finite unions. Then $\mathcal{F}_G(U)$ is defined in analogy to (6). If $U\subseteq U_I\cup U_J$ we define the restriction morphism $r_{U,U_I\cup U_J}:\mathcal{F}_G(U_I\cup U_J)\to\mathcal{F}_G(U)$ by

$$r_{U,U_I\cup U_J}(\vec{f}):=\{\vec{g}\in\Pi_{U_K\in\mathcal{B}:\,U_K\subseteq U}\mathcal{F}_G(U_K)\mid\forall U_K\subseteq U:\ g_K=f_K\}.$$

The previous description generalizes from $U_I\cup U_J$ to arbitrary opens $U_{I_1}\cup\dots\cup U_{I_\ell}$ in M.

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