

Signature change of the emergent space-time in the IKKT matrix model

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The IKKT matrix model (or the type IIB matrix model) is known as a promising candidate for a nonperturbative formulation of superstring theory in ten dimensions. As a most attractive feature, the model admits the emergence of (3+1)-dimensional space-time associated with the spontaneous breaking of the (9+1)-dimensional Lorentz symmetry. Numerical confirmation of such a phenomenon has been attempted for more than two decades. Recently it has been found that the sign problem, the main obstacle in simulating this model, can be overcome by the complex Langevin method. It has been shown that the Lorentzian version of the model is smoothly connected with the Euclidean version, in which the $SO(10)$ symmetry is found to be spontaneously broken to $SO(3)$. Here we propose to add a Lorentz invariant “mass” term to the original model and discuss a scenario that (3+1)-dimensional expanding space-time with Lorentzian signature appears at late times. Some numerical results supporting this scenario are presented.

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1. Introduction

In view of the success of lattice gauge theory in understanding the nonperturbative dynamics of QCD such as quark confinement and the spontaneous chiral symmetry breaking, it is quite natural to consider that some kind of nonperturbative formulation of superstring theory should play a crucial role in understanding the expected dynamics such as the compactification of six extra dimensions. Indeed in 1996, the IKKT matrix model [1] (or the type IIB matrix model) was proposed as such a formulation. Since then, numerical studies have been performed by many people in order to understand the dynamical properties of the model.

The model has ten bosonic $N \times N$ Hermitian matrices, whose eigenvalue distribution describes the ten-dimensional space-time in the large- N limit. It is therefore possible that the eigenvalue distribution collapses to a lower-dimensional manifold, which implies that the “compactification” can occur dynamically in this model. When this happens, the (9+1)D Lorentz symmetry of the model has to be spontaneously broken.

There are quite a few pieces of evidence that the IKKT model provides a nonperturbative formulation of superstring theory. In particular, direct connections to perturbative formulation of superstring theory can be seen by considering the type IIB superstring theory in 10D. First, the action of the model can be regarded as a kind of matrix regularization of the worldsheet action of type IIB superstring theory in the Schild gauge [1]. Unlike the worldsheet formulation of superstring theory, however, the matrix model is expected to be a “second quantization” of superstrings because multiple worldsheets appear naturally in the matrix model as block-diagonal configurations, where each block represents a single worldsheet embedded into the 10-dimensional target space. Second, under a few modest assumptions, one can derive the string field Hamiltonian for type IIB superstring theory from Schwinger-Dyson equations for the Wilson loop operators, which play the role of creation and annihilation operators of strings [2]. If this is true, the IKKT matrix model can reproduce perturbative expansions in type IIB superstring theory to all orders.

In these connections to type IIB superstring theory, one identifies the eigenvalues of the matrices A_μ as the target space coordinates. This identification is suggested also by the supersymmetry algebra of the model, in which the translation that appears from the anti-commutator of supersymmetry generators is identified with the shift symmetry $A_\mu \mapsto A_\mu + \alpha_\mu \mathbf{1}$ of the model, where $\alpha_\mu \in \mathbf{R}$. It is also important to note that the model has extended $\mathcal{N} = 2$ supersymmetry in ten dimensions, which suggests that the model should include gravity since it is known in field theory that $\mathcal{N} = 1$ supersymmetry is the maximal one that can be achieved in ten dimensions without including gravity.

For many years, the IKKT matrix model was studied in its $SO(10)$ symmetric Euclidean version, which is related to the Lorentzian version by deforming the integration contour of the 10 bosonic matrices. This contour deformation amounts to multiplying some phase factors to the temporal and spatial matrices, which is allowed since there is no singularity that one has to go through. We can confirm explicitly that the Lorentzian version is indeed equivalent to the Euclidean version by measuring correlation functions, which are identical up to some phase factors [3, 4]. Since the emergent space-time in the Euclidean version has Euclidean signature, the equivalence between the Lorentzian and Euclidean versions clearly poses a big challenge in obtaining a Lorentzian space-time in the IKKT model. (See Refs. [5–7] for other recent developments on this model.)

Here we overcome this situation by adding a Lorentz invariant “mass” term to the original

model [8]. The motivation for this term comes from our observation [9] that such a modified model have classical solutions representing space-time with Lorentzian signature, which exhibits expanding behavior with any number of expanding directions. If the mass is large enough, the path integral is expected to be dominated by one of such classical solutions. The equivalence to the Euclidean model is avoided since the corresponding $SO(10)$ invariant mass term will have a phase factor $e^{\frac{3}{4}\pi i}$ with a negative real part, which makes the integral divergent. The mass is sent to zero after taking the large- N limit so that the supersymmetry is restored. In such a limit, we expect to obtain an inequivalent model, in which (3+1)D expanding space-time with Lorentzian signature appears at late times.¹

Unfortunately, it is extremely hard to perform Monte Carlo studies of these matrix models due to the so-called sign problem caused by the complex weight in the partition function. In the Euclidean IKKT model, it comes from the Pfaffian that is obtained by integrating out fermionic matrices, while in the Lorentzian IKKT model, it comes from the phase factor e^{iS_b} with the bosonic action S_b . If we treat the phase of the complex weight by reweighting, huge cancellation among configurations with different phases occurs, which makes the calculation impractical. Recently the complex Langevin method (CLM) [18, 19] has been attracting much attention as a promising approach to this problem [20–24]. Here we use the same method in addressing the issues discussed above using a technique [16] which enables us to extract the time evolution [10] from matrices generated by the CLM.

The rest of this article is organized as follows. In section 2 we discuss the problem in the original IKKT matrix model that one cannot obtain space-time with Lorentzian signature. In section 3 we introduce the Lorentz invariant mass term to solve the problem of the original model. In section 4 we discuss how to extract the time evolution from matrix configurations. In section 5 we discuss how we apply the CLM to this model. In section 6 we present our numerical results for the bosonic model with the mass term, which show the emergence of (1+1)D expanding space-time. Section 7 is devoted to a summary and discussions. In particular, we speculate on the emergence of (3+1)D expanding space-time when the fermionic matrices are added.

2. A problem in the original IKKT matrix model

The action of the IKKT matrix model is given by [1]

$$S = S_b + S_f, \quad (1)$$

$$S_b = -\frac{1}{4g^2} \text{Tr}([A_\mu, A_\nu][A^\mu, A^\nu]), \quad (2)$$

$$S_f = -\frac{1}{2g^2} \text{Tr}(\Psi_\alpha (C\Gamma^\mu)_{\alpha\beta} [A_\mu, \Psi_\beta]), \quad (3)$$

where the bosonic variables A_μ ($\mu = 0, \dots, 9$) and the fermionic variables Ψ_α ($\alpha = 1, \dots, 16$) are $N \times N$ Hermitian matrices. Γ^μ are 10D gamma-matrices after the Weyl projection and C

¹The Monte Carlo simulations in Refs. [10–14] avoided the sign problem by an approximation, which turned out to be unjustifiable [15]. Moreover, the observed 3D expanding space turned out to be represented by Pauli matrices [15]. The present work is based on the CLM to overcome the sign problem [16], which revealed a new phase with the emergent space-time being continuous instead of having the Pauli-matrix structure [17].

is the charge conjugation matrix. The ‘‘coupling constant’’ g is merely a scale parameter in this model since it can be absorbed by rescaling A_μ and Ψ_α appropriately. In what follows, we set $g^2 = \frac{1}{N}$ without loss of generality. The indices μ and ν are contracted using the Lorentzian metric $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$.

The partition function of the Lorentzian IKKT matrix model can be written as [10]

$$Z = \int dA e^{iS_b} \text{Pf} \mathcal{M}(A) , \quad (4)$$

where $\text{Pf} \mathcal{M}(A)$ is obtained by integrating out the fermionic matrices, and it is a polynomial in A that is known to take real values. The ‘‘ i ’’ in front of the bosonic action is motivated from the fact that the string worldsheet metric should also have Lorentzian signature. Note also that the bosonic action (2) can be written as

$$S_b = \frac{1}{4} N \text{Tr} (F_{\mu\nu} F^{\mu\nu}) = \frac{1}{4} N \left\{ -2 \text{Tr} (F_{0i})^2 + \text{Tr} (F_{ij})^2 \right\} , \quad (5)$$

where we have introduced the Hermitian matrices $F_{\mu\nu} = i[A_\mu, A_\nu]$. Since the two terms in the last expression have opposite signs, S_b is not positive semi-definite, and it is not bounded from below.

Clearly the integral that appears in (4) is not absolutely convergent. In order to cure this problem, we use Cauchy’s theorem and deform the integration contour for A_μ in (4) as

$$\begin{aligned} A_0 &= e^{-3s\pi i/8} \tilde{A}_0 , \\ A_i &= e^{s\pi i/8} \tilde{A}_i , \end{aligned} \quad (6)$$

where \tilde{A}_μ are Hermitian. This amounts to making the bosonic action

$$S_b = N \left\{ -\frac{1}{2} e^{-s\pi i/2} \text{Tr} (\tilde{F}_{0i})^2 + \frac{1}{4} e^{s\pi i/2} \text{Tr} (\tilde{F}_{ij})^2 \right\} \quad (7)$$

$$= N e^{s\pi i/2} \left\{ \frac{1}{2} \text{Tr} [e^{-s\pi i/2} \tilde{A}_0, \tilde{A}_i]^2 + \frac{1}{4} \text{Tr} (\tilde{F}_{ij})^2 \right\} , \quad (8)$$

where we have defined $\tilde{F}_{\mu\nu} = i[\tilde{A}_\mu, \tilde{A}_\nu]$. The overall phase factor $e^{s\pi i/2}$ in (8) can be identified as the Wick rotation of the worldsheet coordinates, whereas the phase factor $e^{-s\pi i/2}$ in front of \tilde{A}_0 can be identified as the Wick rotation of the target space coordinates. Similarly, the Pfaffian $\text{Pf} \mathcal{M}(A)$ in (4) is replaced by $\text{Pf} \mathcal{M}(e^{-s\pi i/2} \tilde{A}_0, \tilde{A}_i)$ up to some irrelevant constant phase factor. In particular, at $s = 1$, one obtains the Euclidean IKKT model, for which the bosonic part e^{iS_b} in (4) becomes real positive for (8) and the fermionic part $\text{Pf} \mathcal{M}(-i\tilde{A}_0, \tilde{A}_i)$ becomes complex. From (7), on the other hand, one finds $\text{Im} S_b > 0$ for generic configurations² if $0 < s < 2$, which suggests that the model (4) becomes well defined in that region. Therefore, one can define the Lorentzian model by taking the $s \rightarrow 0$ limit³.

²There is a subtlety due to the flat direction $S_b = 0$ corresponding to the configurations that satisfy $[\tilde{A}_\mu, \tilde{A}_\nu] = 0$. Despite this subtlety, the finiteness of the partition function (4) is confirmed for $s = 1$, which corresponds to the Euclidean IKKT model [25, 26]. We consider that the proof in Ref. [26] can be extended to $0 < s < 2$.

³The author would like to thank Yuhma Asano for pointing this out to him in 2018.

The model defined in this way is actually equivalent to the Euclidean IKKT model. For instance, we obtain [3, 4]

$$\left\langle \frac{1}{N} \text{Tr} (A_0)^2 \right\rangle_{\text{L}} = e^{-\frac{3\pi}{4}i} \left\langle \frac{1}{N} \text{Tr} (\tilde{A}_0)^2 \right\rangle_{\text{E}}, \quad (9)$$

$$\left\langle \frac{1}{N} \text{Tr} (A_i)^2 \right\rangle_{\text{L}} = e^{\frac{\pi}{4}i} \left\langle \frac{1}{N} \text{Tr} (\tilde{A}_i)^2 \right\rangle_{\text{E}}, \quad (10)$$

where the suffixes “L” and “E” imply that the expectation values are defined in the Lorentzian model and the Euclidean model, respectively. One can prove that the expectation values $\langle \frac{1}{N} \text{Tr} (\tilde{A}_0)^2 \rangle_{\text{E}}$ and $\langle \frac{1}{N} \text{Tr} (\tilde{A}_i)^2 \rangle_{\text{E}}$ are real positive using the fact that the Pfaffian becomes complex conjugate under the parity transformation $\tilde{A}_\mu \mapsto -\tilde{A}_\mu$ for some μ . This means that $\langle \frac{1}{N} \text{Tr} (A_0)^2 \rangle_{\text{L}}$ and $\langle \frac{1}{N} \text{Tr} (A_i)^2 \rangle_{\text{L}}$ have the phase factors $e^{-\frac{3\pi}{4}i}$ and $e^{\frac{\pi}{4}i}$, respectively. Thus the space-time that appears dynamically in the Lorentzian model has Euclidean signature.

The Euclidean IKKT model has been studied recently by the complex Langevin method [27], and the rotational SO(10) symmetry is found to be spontaneously broken down to SO(3) as suggested earlier by the Gaussian expansion method [28]. While this is certainly an interesting dynamical property of the Euclidean IKKT model, its relevance to our real world is unclear.

3. Adding a Lorentz invariant “mass” term

In order to overcome this situation, we propose to add a Lorentz invariant “mass” term to the original model [8]. Namely, we add to the bosonic action (5), a quadratic term

$$S_{\text{m}} = -\frac{1}{2} N \gamma \text{Tr} (A_\mu A^\mu) = \frac{1}{2} N \gamma \{ \text{Tr} (A_0)^2 - \text{Tr} (A_i)^2 \}, \quad (11)$$

where $\gamma > 0$ is a parameter which is sent to zero after taking the large- N limit.

The motivation for this mass-deformed model defined above comes from our observation [9] that the classical equation of motion⁴

$$[A^\nu, [A_\nu, A_\mu]] - \gamma A_\mu = 0 \quad (12)$$

derived from it has infinitely many solutions with Hermitian A_μ , which represent space-time with Lorentzian signature. Moreover, these solutions exhibit expanding behavior generically with any number of expanding directions. If the coefficient γ of the mass term is large enough, the path integral is expected to be dominated by one of such classical solutions.⁵ Our numerical results presented below suggest that the expanding behavior continues longer as we decrease γ although there is a transition to the “Euclidean phase” at some critical $\gamma_c(N)$ for finite N . It is conceivable that $\gamma_c(N) \rightarrow 0$ in the large- N limit since the expanding phase is entropically favored in that limit compared to the Euclidean phase, in which the extent of space-time remains finite. Thus we expect to obtain a space-time with Lorentzian signature and expanding behavior if we take the $\gamma \rightarrow 0$ limit after the large- N limit.

⁴The mass term was introduced originally to represent the effects of the infrared cutoff used in the simulation [10].

⁵This can be understood by rescaling $A_\mu = \sqrt{\gamma} \tilde{A}_\mu$, which makes the total action proportional to γ^2 . Hence at large γ , the path integral is dominated by some saddle-point configuration satisfying $\frac{\partial S}{\partial A_\mu} = 0$, which is nothing but (12).

The reason why we can avoid the problem of the original model discussed in the previous section can be understood as follows. Upon rotation of the integration contour (6), one obtains

$$S_m = \frac{1}{2} N \gamma \left\{ e^{-3s\pi i/4} \text{Tr}(\tilde{A}_0)^2 - e^{s\pi i/4} \text{Tr}(\tilde{A}_i)^2 \right\} . \quad (13)$$

Unlike (7), one finds that $\text{Im}S_m < 0$ for $0 < s < \frac{4}{3}$, which implies that the bosonic part e^{iS_m} in the partition function can become arbitrarily large in magnitude for (13). Therefore, one cannot make the model well defined by simply introducing $s > 0$ and taking the $s \rightarrow 0$ limit. In particular, at $s = 1$, one obtains

$$S_m = \frac{1}{2} N \gamma i e^{3\pi i/4} \left\{ \text{Tr}(\tilde{A}_0)^2 + \text{Tr}(\tilde{A}_i)^2 \right\} , \quad (14)$$

which makes the corresponding Euclidean model ill defined due to the phase factor $e^{3\pi i/4}$ with a negative real part in front of the SO(10) invariant mass term.

In order to make the mass-deformed IKKT model well defined for finite γ and finite N , we can think of introducing some convergence factor in (11) as

$$S_m^{(\varepsilon)} = \frac{1}{2} N \gamma \left\{ e^{i\varepsilon} \text{Tr}(A_0)^2 - e^{-i\varepsilon} \text{Tr}(A_i)^2 \right\} . \quad (15)$$

We deform the integration contour as (6) and take the $s \rightarrow 0$ limit before we take the $\varepsilon \rightarrow 0$ limit.

Let us emphasize that the sign of the mass term (11) is crucial. For $\gamma < 0$, we have $\text{Im}S_m > 0$ for $0 < s < \frac{4}{3}$, which enables us to connect the theory to the Euclidean model with the mass term (14), which is well defined. Therefore the situation is qualitatively the same as in the $\gamma = 0$ case, and nothing dramatic happens.

It is also known [9] that the classical equation of motion (12) does not have expanding solutions for $\gamma < 0$. When $\gamma = 0$, the classical equation of motion (12) is satisfied if and only if all the matrices are commutative; i.e., $[A_\mu, A_\nu] = 0$ as is proved in Appendix A of ref. [29].

The mass term can be interpreted as the cosmological constant in the Einstein equation, which is derived from the IKKT matrix model [30]. Incidentally, the mass term is introduced in obtaining interesting classical solutions in Refs. [6, 29, 31–34]. See also Refs.[35–38] for related work, which discuss the signature change in the IKKT type of matrix models from a different viewpoint.

4. How to extract the time evolution

Let us explain how we can extract the time evolution from matrix configurations following ref. [10]. For that, we use the SU(N) symmetry of the model to bring the temporal matrix A_0 into the diagonal form

$$A_0 = \text{diag}(\alpha_1, \dots, \alpha_N) , \quad \text{where } \alpha_1 < \dots < \alpha_N . \quad (16)$$

By ‘‘fixing the gauge’’ in this way, we can rewrite the partition function (4) as

$$Z = \int \prod_{a=1}^N d\alpha_a \Delta(\alpha)^2 \int dA_i e^{i(S_b + S_m)} \text{Pf} \mathcal{M}(A) , \quad (17)$$

$$\Delta(\alpha) \equiv \prod_{a>b}^N (\alpha_a - \alpha_b) , \quad (18)$$

where $\Delta(\alpha)$ is the van der Monde determinant. The factor $\Delta(\alpha)^2$ in (17) appears from the Fadeev-Popov procedure for the gauge fixing, and it acts as a repulsive potential between α_a .

We can extract a time-evolution from matrix configurations of A_μ . A crucial observation is that the spatial matrices A_i have a band-diagonal structure in the $SU(N)$ basis in which A_0 has the diagonal form (16). See Fig. 3 (Right). More precisely, there exists some integer n such that the elements of spatial matrices $(A_i)_{ab}$ for $|a - b| > n$ are much smaller than those for $|a - b| \leq n$. Based on this observation, we may naturally consider $n \times n$ submatrices of A_i ,

$$(\bar{A}_i)_{IJ}(t_\nu) \equiv (A_i)_{\nu+I, \nu+J}, \quad (19)$$

where $I, J = 1, \dots, n$, $\nu = 0, 1, \dots, N - n$, and t_ν is defined by

$$t_\nu = \sum_{\rho=1}^{\nu} |\bar{\alpha}_\rho - \bar{\alpha}_{\rho-1}|, \quad (20)$$

$$\bar{\alpha}_\nu = \frac{1}{n} \sum_{I=1}^n \langle \alpha_{\nu+I} \rangle. \quad (21)$$

We interpret the $\bar{A}_i(t)$ as representing the state of the universe at time t . Note that $\bar{\alpha}_\nu \in \mathbb{C}$ in general, since the weight in (17) is complex. The appearance of real time implies that $\bar{\alpha}_\nu - \bar{\alpha}_{\nu-1} \in \mathbb{R}$.

Using $\bar{A}_i(t)$, we can define, for example, the extent of space at time t as

$$R^2(t) = \left\langle \frac{1}{n} \text{tr} \sum_i (\bar{A}_i(t))^2 \right\rangle, \quad (22)$$

where tr represents a trace over the $n \times n$ submatrix. Since $R^2(t) \in \mathbb{C}$ in general, let us define

$$R^2(t) = e^{2i\theta_s(t)} |R^2(t)|. \quad (23)$$

The appearance of real space implies that $\theta_s(t) = 0$.

We also define the ‘‘moment of inertia tensor’’

$$T_{ij}(t) = \frac{1}{n} \text{tr} \left(\bar{A}_i(t) \bar{A}_j(t) \right), \quad (24)$$

which is a 9×9 real symmetric matrix since $\bar{A}_i(t)$ is Hermitian. The eigenvalues of $T_{ij}(t)$, which we denote by $\lambda_i(t)$ with the order

$$\lambda_1(t) > \lambda_2(t) > \dots > \lambda_9(t), \quad (25)$$

represent the spatial extent in each of the nine directions at time t . Note that the expectation values $\langle \lambda_i(t) \rangle \in \mathbb{C}$ tend to be equal in the large- N limit if the $SO(9)$ symmetry is not spontaneously broken. If some of the eigenvalues $\langle \lambda_i(t) \rangle$ ($i = 1, \dots, d$) have significantly larger modulus than the rest, it implies that d -dimensional space appears dynamically.

5. Applying the complex Langevin method

In this section, we review the CLM and discuss how we apply it to the model (17).

5.1 Brief review of the CLM

Let us consider a system

$$Z = \int dx e^{-S(x)} \quad (26)$$

of N real variables x_k ($k = 1, \dots, N$) as a simple example. Here the action $S(x)$ is a complex-valued function, which causes the sign problem.

In the CLM, the original real variables x_k are complexified as $x_k \rightarrow z_k = x_k + iy_k \in \mathbb{C}$ and one considers a fictitious time evolution of the complexified variables z_k using the complex Langevin equation given, in its discretized form, by

$$z_k^{(\eta)}(t + \epsilon) = z_k^{(\eta)}(t) + \epsilon v_k(z^{(\eta)}(t)) + \sqrt{\epsilon} \eta_k(t), \quad (27)$$

where t is the fictitious time with a stepsize ϵ . The second term $v_k(z)$ on the right-hand side is called the drift term, which is defined by holomorphic extension of the one

$$v_k(x) = -\frac{\partial S(x)}{\partial x_k} \quad (28)$$

for the real variables x_k . The variables $\eta_k(t)$ appearing on the right-hand side of eq. (27) are a real Gaussian noise with the probability distribution $\propto e^{-\frac{1}{4} \sum_t \eta_k(t)^2}$, which makes the time-evolved variables $z_k^{(\eta)}(t)$ stochastic. The expectation values with respect to the noise $\eta_k(t)$ are denoted as $\langle \dots \rangle_\eta$ in what follows.

Let us consider the expectation value of an observable $O(x)$. In the CLM, one computes the expectation value of the holomorphically extended observable $O(z)$ for complexified variables z_k . Then, the correct convergence of the CLM implies the equality

$$\lim_{t \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \langle O(z^{(\eta)}(t)) \rangle_\eta = \frac{1}{Z} \int dx O(x) e^{-S(x)}, \quad (29)$$

where the right-hand side is the expectation value of $O(x)$ in the original theory (26). A proof of eq. (29) was given in refs. [20, 21], where the notion of the time-evolved observable $O(z; t)$ plays a crucial role. In particular, it was pointed out that the integration by parts used in the argument cannot be justified when the probability distribution of z_k that appears during the simulation falls off slowly in the imaginary direction.

While this argument provided theoretical understanding of the cases in which the CLM gives wrong results, the precise condition on the probability distribution was not specified. Furthermore, there is actually a subtlety in defining the time-evolved observable $O(z; t)$. Recently ref. [24] provided a refined argument for justification of the CLM, which showed that the probability for the drift term $v_k(z)$ to become large has to be suppressed strongly enough. More precisely the histogram of the magnitude of the drift term should fall off exponentially or faster. This criterion tells us whether the results obtained by the CLM are reliable or not.

5.2 Applying the CLM to the bosonic IKKT model with the mass term

Let us apply the CLM to the model (17) following ref. [16]. From now on, we omit the Pfaffian and consider the bosonic model for simplicity.

The first step of the CLM is to complexify the real variables. As for the spatial matrices A_i , we simply treat them as general complex matrices instead of Hermitian matrices. As for the temporal matrix A_0 , which is diagonalized as (16), we have to take into account the ordering of the eigenvalues. For that purpose, we make the change of variables as

$$\alpha_1 = 0, \quad \alpha_2 = e^{\tau_1}, \quad \alpha_3 = e^{\tau_1} + e^{\tau_2}, \quad \dots, \quad \alpha_N = \sum_{a=1}^{N-1} e^{\tau_a} \quad (30)$$

so that the ordering is implemented automatically, and then complexify τ_a ($a = 1, \dots, N-1$). Using the shift symmetry $A_0 \mapsto A_0 + \text{const.} \mathbf{1}$ of the original IKKT action, we make a shift $A_0 \rightarrow A_0 - \frac{1}{N} \text{Tr} A_0$ so that A_0 becomes traceless in what follows.

The effective action that appears in the Boltzmann weight $e^{-S_{\text{eff}}}$ reads

$$S_{\text{eff}} = -\frac{1}{4}iN \left\{ 2 \text{Tr} [A_0, A_i]^2 - \text{Tr} [A_i, A_j]^2 \right\} - \frac{1}{2}iN\gamma \left\{ \text{Tr}(A_0)^2 - \text{Tr}(A_i)^2 \right\} \\ - \log \Delta(\alpha) - \sum_{a=1}^{N-1} \tau_a, \quad (31)$$

where the last term comes from the Jacobian associated with the change of variables (30). The complex Langevin equation is given by

$$\frac{d\tau_a}{dt} = -\frac{\partial S_{\text{eff}}}{\partial \tau_a} + \eta_a(t), \\ \frac{d(A_i)_{ab}}{dt} = -\frac{\partial S_{\text{eff}}}{\partial (A_i)_{ba}} + (\eta_i)_{ab}(t), \quad (32)$$

where the $\eta_a(t)$ in the first equation are random real numbers obeying the probability distribution $\exp(-\frac{1}{4} \int dt \sum_a \{\eta_a(t)\}^2)$ and the $\eta_i(t)$ in the second equation are random Hermitian matrices obeying the probability distributions $\exp(-\frac{1}{4} \int dt \sum_i \text{Tr} \{\eta_i(t)\}^2)$.

The expectation values of observables can be calculated by defining them holomorphically for complexified τ_a and A_i and taking an average using the configurations generated by solving the discretized version of (32) for sufficiently long time. In order for this method to work, the probability distribution of the drift terms, namely the first terms on the right-hand side of (32), has to fall off exponentially [24]. We have checked that this criterion is indeed satisfied for all the values of parameters used in this paper.

6. Results for the bosonic IKKT model with the mass term

In this section we present our preliminary results for the bosonic IKKT model (17) with the mass term [8]. We choose the matrix size to be $N = 32$ and set $\varepsilon = 0$ in (15) for simplicity.

When the mass term is absent $\gamma = 0$, we obtain the results equivalent to the Euclidean model as in (9) and (10). The complex Langevin simulation is completely stable. This is understandable since for $\gamma = 0$ the simulation can find the contour deformation by itself ending up in simulating the Euclidean model, which is free from the sign problem.⁶

⁶Note that this is no more the case if we incorporate the Pfaffian in the simulation.

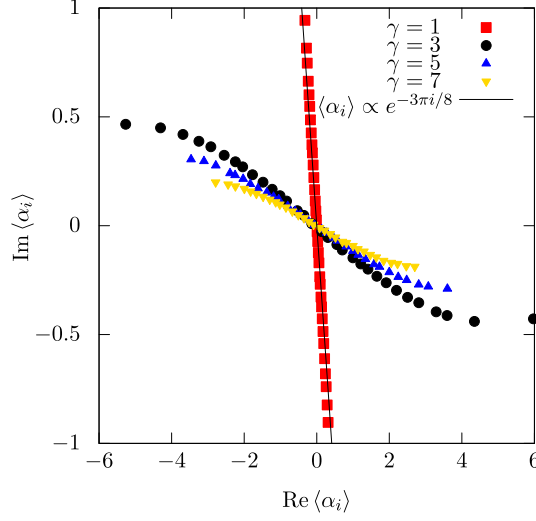


Figure 1: The expectation values $\langle \alpha_i \rangle$ are plotted in the complex plane for $\gamma = 1, 3, 5, 7$. The solid line represents the prediction (9) for $\gamma = 0$ obtained from the equivalence to the Euclidean IKKT model.

When we start our simulation with large γ , however, the simulation turns out to be unstable, which forces us to use some trick to obtain meaningful results. Let us note here that one of the classical solutions represented by Hermitian A_μ is expected to dominate at large γ . Therefore, we insert a procedure

$$A_i \mapsto \frac{1}{1+\eta} (A_i + \eta A_i^\dagger) \quad \text{for } i = 1, \dots, 9 \quad (33)$$

after each Langevin step to stabilize the simulation, which is similar in spirit to the dynamical stabilization proposed in the complex Langevin simulation of finite density QCD [39]. For $\eta = 1$, this amounts to Hermitizing A_i after each Langevin step, whereas $\eta = 0$ corresponds to doing nothing. We tried to decrease η as much as possible, and found that the simulation is stable for $\eta \gtrsim 0.001$ and the results do not depend much on η within the region $0.001 \lesssim \eta \lesssim 0.01$. In what follows, we present our results for $\eta = 0.01$.

After we obtain a thermalized configuration for $\gamma = 7$ in this way, we decrease γ adiabatically and obtain results for smaller γ . At $\gamma \sim 2.5$, there is a drastic change in the results. The results for $\gamma = 1$ are close to those for $\gamma = 0$, and we set $\eta = 0$ since the technique (33) is not only needless but also unjustifiable since the configurations are not close to Hermitian in this Euclidean phase.

In Fig. 1, we plot our results for $\langle \alpha_i \rangle$ in the complex plane for $\gamma = 1, 3, 5, 7$. Note that the aspect ratio is chosen as 1 : 6. For $\gamma = 7$, the distribution of $\langle \alpha_i \rangle$ is close to the real axis, which is consistent with the fact that one of the classical solutions represented by Hermitian A_μ dominates at large γ . As γ becomes smaller, we observe that the distribution moves away from the real axis, but the flat region at both ends extends, suggesting the emergence of real time in that region. The results for $\gamma = 1$ are close to the prediction (9) for $\gamma = 0$ obtained from the equivalence to the Euclidean model, and they are qualitatively different from the results for $\gamma \geq 3$, suggesting a first order phase transition⁷ at some $\gamma = \gamma_c(N)$, which lies around 2.5 for the present matrix size $N = 32$. From our preliminary results for $N = 64$, the (lower) critical point $\gamma_c(N)$ seems to decrease for larger N .

⁷This is also suggested by the existence of hysteresis; starting from a thermalized configuration at $\gamma = 0$ and increasing

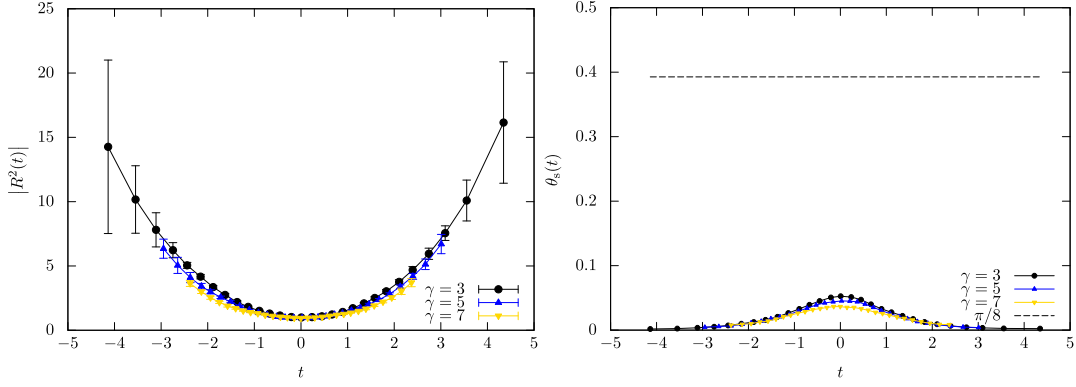


Figure 2: (Left) The extent of space $|R^2(t)|$ is plotted against time t for $\gamma = 3, 5, 7$. (Right) The complex phase $\theta_s(t)$ of the space is plotted against time t for $\gamma = 3, 5, 7$. The dashed line $\theta_s(t) = \frac{\pi}{8}$ represents the prediction (10) for $\gamma = 0$ obtained from the equivalence to the Euclidean model.

In Fig. 2, we plot $|R^2(t)|$ (Left) and $\theta_s(t)$ (Right) defined by (23) for $\gamma \geq 3$. The block size used in defining $\bar{\alpha}_\nu$ in (21) and $\bar{A}_i(t)$ in (19) is chosen to be $n = 4$. From the left panel, we find that the expanding behavior of $|R^2(t)|$ is analogous to that observed for classical solutions [9]. Scaling behavior is observed for different values of γ , and decreasing γ results in extending the time direction and hence the space becomes larger at the end time. From the right panel, we find that the space becomes real at late times, while the phase $\theta_s(t)$ becomes positive near $t \sim 0$. Emergence of the real space-time at late times observed even for $\gamma = 3$ can be understood as a consequence of classicalization since the value of the action increases with the expansion [32].

Figures 1 and 2 exhibit symmetries around $t = 0$, which is due to the symmetry of the model (17) under $A_0 \mapsto -A_0$. The behavior is reminiscent of bouncing cosmology.

Let us also look at the order parameter for the SSB of $SO(9)$ symmetry for $\gamma \geq 3$. It is not straightforward to calculate the expectation values of (25) by the CLM since they cannot be regarded as holomorphic functions of τ_a and A_i . Here we estimate them by defining the ‘‘moment of inertia tensor’’ (24) using only the Hermitian part of $\bar{A}_i(t)$, which is expected to be a good approximation according to our results in Fig. 2 (Right). The results for $\gamma = 3$ are shown in Fig. 3 (Left). We observe that only one direction expands and the other directions remain small. Thus the expanding space is actually one dimensional. We can fit our data for the largest eigenvalue $\langle \lambda_1(t) \rangle$ to an exponential behavior, which shows that our data are consistent with an exponential expansion.

Finally, let us confirm that the spatial matrices $A_i(t)$ has a band diagonal structure, which is important in defining the submatrices (19). For that, we plot

$$\mathcal{A}_{pq} = \frac{1}{9} \sum_{i=1}^9 |(A_i)_{pq}|^2 \quad (34)$$

against p and q for $\gamma = 3$ in Fig. 3 (Right). We find that the off-diagonal elements are quite small. Similar behaviors are observed for $\gamma = 5, 7$, which justifies our choice $n = 4$ of the block size for $\gamma \geq 3$. Such band-diagonal structure is not observed for $\gamma \leq 2$.

γ adiabatically, we find that the system is in the Euclidean phase even for $\gamma \gtrsim 2.5$.

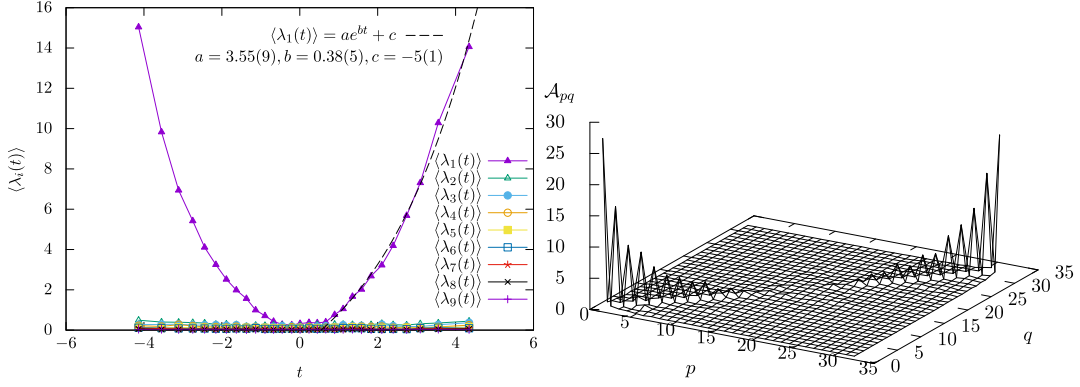


Figure 3: (Left) The expectation values $\langle \lambda_i(t) \rangle$ are plotted against t for $\gamma = 3$. The dashed line represents a fit of $\langle \lambda_1(t) \rangle$ to the behavior $\langle \lambda_1(t) \rangle = ae^{bt} + c$, where $a = 3.55(9)$, $b = 0.38(5)$ and $c = -5(1)$. (Right) The magnitude \mathcal{A}_{pq} of each element of the spatial matrices is plotted against p and q for $\gamma = 3$.

7. Summary and discussions

In this article, we have discussed the signature of the emergent space-time in the IKKT matrix model, which was proposed as a nonperturbative formulation of superstring theory. A naive definition of the model leads to a space-time with Euclidean signature. In order to avoid this, we have proposed to add a Lorentz invariant mass term to the original action. We investigate the bosonic IKKT model with the mass term by the CLM. When the mass parameter γ is large enough, the path integral is dominated by one of the classical solutions with Lorentzian signature and expanding behavior. As γ is decreased, the extent of the emergent time increases and the emergent space at the end time becomes larger. The signature of the space-time is Lorentzian at late times, while it seems to deviate from Lorentzian towards Euclidean at early times. The expansion at late times is consistent with an exponential behavior, and we also observed that only one out of nine spatial directions expands. We speculate that an expanding space-time with Lorentzian signature emerges at late times even in the $\gamma \rightarrow 0$ limit after taking the large- N limit.

The mechanism for the appearance of the (1+1)D space-time may be understood from the bosonic action (5). Since the spatial direction expands exponentially, the $\text{Tr} [A_i, A_j]^2$ term becomes dominant. The fluctuation of this term can be made small by having only one expanding direction.

As a future prospect, it would be important to include the effect of fermionic matrices, which is represented by the Pfaffian in (4). It is known that the Pfaffian vanishes if we set A_μ to zero except for two of them [25, 40]. Therefore the (1+1)D space-time observed in our simulation of the bosonic model is strongly suppressed by the Pfaffian. Considering that the expansion of space is exponential with respect to time, it is conceivable that the Pfaffian favors the emergence of three exponentially extended spatial directions. Such effects are already confirmed in the Euclidean IKKT model, in which one indeed obtains three extended directions [27, 28].

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References

- [1] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *A Large N reduced model as superstring*, *Nucl. Phys.* **B498** (1997) 467–491, [[hep-th/9612115](#)].
- [2] M. Fukuma, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *String field theory from IIB matrix model*, *Nucl. Phys. B* **510** (1998) 158–174, [[hep-th/9705128](#)].
- [3] K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. K. Papadoudis, and A. Tsuchiya, *Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model*, in *38th International Symposium on Lattice Field Theory*, 12, 2021. [arXiv:2112.15368](#).
- [4] K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. K. Papadoudis, and A. Tsuchiya, *Complex Langevin studies of the emergent space-time in the type IIB matrix model*, in *East Asia Joint Symposium on Fields and Strings 2021*, 1, 2022. [arXiv:2201.13200](#).
- [5] S. Brahma, R. Brandenberger, and S. Laliberte, *Emergent cosmology from matrix theory*, *JHEP* **03** (2022) 067, [[arXiv:2107.11512](#)].
- [6] H. C. Steinacker, *Gravity as a quantum effect on quantum space-time*, *Phys. Lett. B* **827** (2022) 136946, [[arXiv:2110.03936](#)].
- [7] F. R. Klinkhamer, *Towards a Numerical Solution of the Bosonic Master-Field Equation of the IIB Matrix Model*, *Acta Phys. Polon. B* **53** (2022), no. 1 1, [[arXiv:2110.15309](#)].
- [8] K. Anagnostopoulos, T. Azuma, K. Hatakeyama, M. Hirasawa, J. Nishimura, S. K. Papadoudis, and A. Tsuchiya, work in progress.
- [9] K. Hatakeyama, A. Matsumoto, J. Nishimura, A. Tsuchiya, and A. Yosprakob, *The emergence of expanding space-time and intersecting D-branes from classical solutions in the Lorentzian type IIB matrix model*, *PTEP* **2020** (2020), no. 4 043B10, [[arXiv:1911.08132](#)].
- [10] S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions*, *Phys. Rev. Lett.* **108** (2012) 011601, [[arXiv:1108.1540](#)].

- [11] Y. Ito, S.-W. Kim, Y. Koizuka, J. Nishimura, and A. Tsuchiya, *A renormalization group method for studying the early universe in the Lorentzian IIB matrix model*, *PTEP* **2014** (2014), no. 8 083B01, [[arXiv:1312.5415](#)].
- [12] Y. Ito, J. Nishimura, and A. Tsuchiya, *Power-law expansion of the Universe from the bosonic Lorentzian type IIB matrix model*, *JHEP* **11** (2015) 070, [[arXiv:1506.04795](#)].
- [13] Y. Ito, J. Nishimura, and A. Tsuchiya, *Universality and the dynamical space-time dimensionality in the Lorentzian type IIB matrix model*, *JHEP* **03** (2017) 143, [[arXiv:1701.07783](#)].
- [14] T. Azuma, Y. Ito, J. Nishimura, and A. Tsuchiya, *A new method for probing the late-time dynamics in the Lorentzian type IIB matrix model*, *PTEP* **2017** (2017), no. 8 083B03, [[arXiv:1705.07812](#)].
- [15] T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura, and A. Tsuchiya, *On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model*, *PTEP* **2019** (2019), no. 9 093B03, [[arXiv:1904.05914](#)].
- [16] J. Nishimura and A. Tsuchiya, *Complex Langevin analysis of the space-time structure in the Lorentzian type IIB matrix model*, *JHEP* **06** (2019) 077, [[arXiv:1904.05919](#)].
- [17] M. Hirasawa, K. Anagnostopoulos, T. Azuma, K. Hatakeyama, Y. Ito, J. Nishimura, S. K. Papadoudis, and A. Tsuchiya, *A new phase in the Lorentzian type IIB matrix model and the emergence of continuous space-time*, in *38th International Symposium on Lattice Field Theory*, 12, 2021. [arXiv:2112.15390](#).
- [18] G. Parisi, *On Complex Probabilities*, *Phys. Lett.* **B131** (1983) 393–395.
- [19] J. R. Klauder, *Coherent State Langevin Equations for Canonical Quantum Systems With Applications to the Quantized Hall Effect*, *Phys. Rev.* **A29** (1984) 2036–2047.
- [20] G. Aarts, E. Seiler, and I.-O. Stamatescu, *The Complex Langevin method: When can it be trusted?*, *Phys. Rev.* **D81** (2010) 054508, [[arXiv:0912.3360](#)].
- [21] G. Aarts, F. A. James, E. Seiler, and I.-O. Stamatescu, *Complex Langevin: Etiology and Diagnostics of its Main Problem*, *Eur. Phys. J.* **C71** (2011) 1756, [[arXiv:1101.3270](#)].
- [22] J. Nishimura and S. Shimasaki, *New insights into the problem with a singular drift term in the complex Langevin method*, *Phys. Rev.* **D92** (2015), no. 1 011501, [[arXiv:1504.08359](#)].
- [23] K. Nagata, J. Nishimura, and S. Shimasaki, *Justification of the complex Langevin method with the gauge cooling procedure*, *PTEP* **2016** (2016), no. 1 013B01, [[arXiv:1508.02377](#)].
- [24] K. Nagata, J. Nishimura, and S. Shimasaki, *Argument for justification of the complex Langevin method and the condition for correct convergence*, *Phys. Rev.* **D94** (2016), no. 11 114515, [[arXiv:1606.07627](#)].

- [25] W. Krauth, H. Nicolai, and M. Staudacher, *Monte Carlo approach to M theory*, *Phys. Lett.* **B431** (1998) 31–41, [[hep-th/9803117](#)].
- [26] P. Austing and J. F. Wheeler, *Convergent Yang-Mills matrix theories*, *JHEP* **04** (2001) 019, [[hep-th/0103159](#)].
- [27] K. N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura, T. Okubo, and S. K. Papadoudis, *Complex Langevin analysis of the spontaneous breaking of 10D rotational symmetry in the Euclidean IKKT matrix model*, *JHEP* **06** (2020) 069, [[arXiv:2002.07410](#)].
- [28] J. Nishimura, T. Okubo, and F. Sugino, *Systematic study of the $SO(10)$ symmetry breaking vacua in the matrix model for type IIB superstrings*, *JHEP* **10** (2011) 135, [[arXiv:1108.1293](#)].
- [29] H. C. Steinacker, *Cosmological space-times with resolved Big Bang in Yang-Mills matrix models*, *JHEP* **02** (2018) 033, [[arXiv:1709.10480](#)].
- [30] M. Hanada, H. Kawai, and Y. Kimura, *Describing curved spaces by matrices*, *Prog. Theor. Phys.* **114** (2006) 1295–1316, [[hep-th/0508211](#)].
- [31] S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory*, *Phys. Rev.* **D86** (2012) 027901, [[arXiv:1110.4803](#)].
- [32] S.-W. Kim, J. Nishimura, and A. Tsuchiya, *Late time behaviors of the expanding universe in the IIB matrix model*, *JHEP* **10** (2012) 147, [[arXiv:1208.0711](#)].
- [33] H. C. Steinacker, *Quantized open FRW cosmology from Yang-Mills matrix models*, *Phys. Lett.* **B782** (2018) 176–180, [[arXiv:1710.11495](#)].
- [34] M. Sperling and H. C. Steinacker, *Covariant cosmological quantum space-time, higher-spin and gravity in the IKKT matrix model*, *JHEP* **07** (2019) 010, [[arXiv:1901.03522](#)].
- [35] A. Chaney, L. Lu, and A. Stern, *Matrix Model Approach to Cosmology*, *Phys. Rev.* **D93** (2016), no. 6 064074, [[arXiv:1511.06816](#)].
- [36] A. Chaney, L. Lu, and A. Stern, *Lorentzian Fuzzy Spheres*, *Phys. Rev.* **D92** (2015), no. 6 064021, [[arXiv:1506.03505](#)].
- [37] A. Chaney and A. Stern, *Fuzzy CP^2 spacetimes*, *Phys. Rev.* **D95** (2017), no. 4 046001, [[arXiv:1612.01964](#)].
- [38] A. Stern and C. Xu, *Signature change in matrix model solutions*, *Phys. Rev.* **D98** (2018), no. 8 086015, [[arXiv:1808.07963](#)].
- [39] F. Attanasio and B. Jäger, *Dynamical stabilisation of complex Langevin simulations of QCD*, *Eur. Phys. J. C* **79** (2019), no. 1 16, [[arXiv:1808.04400](#)].
- [40] J. Nishimura and G. Vernizzi, *Spontaneous breakdown of Lorentz invariance in IIB matrix model*, *JHEP* **04** (2000) 015, [[hep-th/0003223](#)].