PROCEEDINGS OF SCIENCE

Fuzzy field theories in the string modes formalism

Harold C. Steinacker[a] **and Juraj Tekel**[b,*]

[a] *Faculty of Physics, University of Vienna*
Boltzmanngasse 5, A-1090 Vienna, Austria

[b] *Department of Theoretical Physics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava,*
Mlynská Dolina, 842 48 Bratislava, Slovakia

E-mail: harold.steinacker@univie.ac.at, juraj.tekel@fmph.uniba.sk

We review the formulation of the scalar field theory on the fuzzy sphere in terms of the string modes – functions optimally localized in position and momentum space. We show how this greatly simplifies the computation of loop contributions in position space and provides some new insights into the structure of the effective action.

Corfu Summer Institute 2021 "School and Workshops on Elementary Particle Physics and Gravity"
29 August - 9 October 2021
Corfu, Greece

*Speaker

PoS(CORFU2021)244

1. Introduction

The matter content of the universe is exceptionally well described by quantum theory. Since matter is coupled to the dynamics of spacetime via the Einstein equations of general relativity, we can not get around the issue of quantizing gravity as well. The straightforward quatization of general relativity leads to a non-renormalizable theory and several approaches have been proposed to circumvent this issue, with various degrees of success, or lack thereof.

However, in all these cases we are led to believe that in the eventually successful quantum theory of gravity, the continuous spacetime manifold is going to be replaced by some kind of discrete structure [1]. Some reasons to believe this is the case are the following: Heisenberg gravitational microscope – where measurement of sufficiently small distances creates a black hole which hides the result of the experiment, instability of quantum vacuum against creation of black holes from very energetic fluctuations [2], emergence of spacetime from more fundamental degrees of freedom in matrix models [3].

One setting where the consequences of such a discrete structure can be studied is the case of fuzzy spaces – finite mode approximations to compact manifolds, which retain the symmetries of the original space. Field theories on such spaces are described in terms of matrices and can be studied analytically [4–15] or numerically [16–28]. The standard approach using the momentum eigenstates leads to the most important consequence of the fuzzy structure – the UV/IR mixing. The short distance UV fluctuations leave their imprint in the large distance IR physics of the theory. At the technical level, the continuum limit of the one-loop quantum effective action of the theory is non-local and very different from the strictly continuum case.

In this contribution, we summarize a different approach to physics on fuzzy spaces, presented in [29]. We first construct the string modes in section 2, a basis for the fuzzy functions derived from the coherent states on the fuzzy sphere. We then express the operators on fuzzy sphere in terms of these new modes in section and finally in the section 3 we show how this formulation leads to some new insight into fuzzy field theories.

We will deal only with the fuzzy sphere case [30, 31], but most of what we say naturally generalizes to any more general setting where the coherent states can be constructed [32, 33].

2. String modes

In this section, we will summarize some preliminary information about the fuzzy sphere and the coherent states on the fuzzy sphere. We will then construct the string modes, introduce their properties and use them to construct representation of operators of fuzzy functions.

2.1 Fuzzy sphere

The fuzzy sphere is defined by the following deformation of the usual sphere coordinate functions

$$\hat{x}_i \hat{x}_i = r^2 \quad , \quad \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i = i\theta \varepsilon_{ijk} \hat{x}_k \ , \quad i, j, k = 1, 2, 3 \ . \tag{1}$$

Such $\hat{x}$'s generate a non-commutative algebra and the fuzzy sphere S_F^2 is whatever object has this as the algebra of functions. The above commutation relations have a straightforward realization in the $N = 2s + 1$ dimensional spin-s representation of $su(2)$

$$\hat{x}_i = \frac{2r}{\sqrt{N^2-1}}\hat{L}_i \quad , \quad \theta = \frac{2r}{\sqrt{N^2-1}} \sim \frac{2}{N} \quad , \quad \hat{x}_i\hat{x}_i = \frac{4r^2}{N^2-1}s(s+1) = r^2 \; , \tag{2}$$

where $\hat{L}_i$'s are the corresponding generators. The group $SU(2)$ still acts on $\hat{x}_i$'s – this space enjoys a full rotational symmetry and this motivates the particular form of the commutation relations (1). Moreover, in the limit $N \to \infty$ we recover the original sphere, as the non-commutativity parameter θ vanishes. Intuitively, nonzero commutator between a pair of coordinates leads to an inability of resolving points in arbitrary precision and thus the space is blurred at scales at the order of $\sqrt{\theta} \sim 1/\sqrt{N}$.[1]

Functions on the fuzzy sphere are elements of the algebra generated by (2), which is the N^2 dimensional algebra of $N \times N$ hermitian matrices. We can introduce a basis in this space T_{lm}, which respects the structure of the $SU(2)$-module, as follows

$$M = \sum_{l=0}^{N-1} \sum_{m=-l}^{m=l} c_{lm} T_{lm} \; . \tag{3}$$

T_{lm}'s are often referred to as polarization tensors or fuzzy spherical harmonics. Note that this is reminiscent of an algebra of functions on the regular sphere S^2 cut off at maximal angular momentum $L = N - 1$, which further shows that the large N limit is the limit of the regular sphere. A subtlety is though that such algebra would not be closed under the point-wise multiplication of functions, which is achieved by the non-commutative matrix multiplication.

2.2 Coherent states

The auxiliary Hilbert space $\mathcal{H}$ which the matrices M in the previous section acted on has a natural basis in terms of the spin states $|i\rangle$. We can however come up with a different set of states that serve the same purpose. For any $x \in S^2$ with radius 1, choose some $g_x \in SO(3)$ such that $x = g_x \cdot p$, where p is the north pole on S^2, then define [32]

$$|x\rangle = g_x \cdot |s\rangle \; , \; g_x \in SU(2) \; , \tag{4}$$

where $|s\rangle$ is the highest weight state, and call the set of all $|x\rangle$ the coherent states. We will set the radius of the sphere $r = 1$ for the rest of the text. Such a vector is located around x, since it minimizes the uncertainty

$$\sum_i \left\langle (\hat{x}_i)^2 \right\rangle - \langle x_i \rangle^2 \approx \frac{2}{N} \; , \tag{5}$$

and is the fuzzy analogue of a point x. This formula also supports the idea of the coherent states as objects localized in area with spatial extent $\sim 1/\sqrt{N}$. However, the coherent state is not a function

[1]This is most easily seen around the north pole, where $\hat{x}^3 \sim 1$, but thanks to the symmetry of the space needs to hold anywhere.

on S^2_F, since it is an element of $\mathcal{H}$. The set of all coherent states is over-complete in the sense[2]

$$\mathbf{1} = \frac{N}{4\pi} \int d^2x \, |x\rangle \langle x| \ , \tag{6}$$

but states for distinct x and y are orthogonal only in the large N limit

$$|\langle x|y\rangle|^2 = \left(\frac{1 + x \cdot y}{2}\right)^{N-1} \ . \tag{7}$$

Coherent states can be used to map (quantize) functions on S^2 on matrices

$$\phi(x) \to M = \int d^2x \, \phi(x)|x\rangle\langle x| \ . \tag{8}$$

and matrices on functions (de-quantize)

$$M \to \phi(x) = \langle x| \, M \, |x\rangle \ . \tag{9}$$

The quantization map maps the spherical harmonics Y_{lm} onto the polarization tensors T_{lm}, up to a normalization defined by

$$T_{lm} = c_l \int d^2x \, Y_{lm}(x) \, |x\rangle \, \langle x| \ . \tag{10}$$

The normalization can be shown to be [29]

$$c_l^2 = \frac{1}{4\pi} \frac{(N-1-l)!(N+l)!}{((N-1)!)^2} \sim \frac{N}{4\pi} e^{\frac{l^2}{N}} \ . \tag{11}$$

Even though this is rather technical, it highlights and important feature of the quantization/de-quantization maps. For $l \ll \sqrt{N}$ the coefficients c_l are approximately constant, quantization and de-quantization are inverse of each other and there are no issues. However for $l > \sqrt{N}$ the coefficient c_l grows extremely fast and de-quantized matrices are misleading. This already suggests that some kind of crossover behaviour is lurking at momenta $l \approx \sqrt{N}$, but let us keep the tension for a little further and proceed to define the main protagonist of our discussion.

2.3 String modes and their properties

Spin states produce also a natural basis in the algebra of fuzzy functions M given simply by $|i\rangle \langle j|$. Similarly, we can express the matrix M in using the coherent states

$$M = \left(\frac{N}{4\pi}\right)^2 \int d^2x \, d^2y \, \phi(x,y) \, |x\rangle \, \langle y| \ . \tag{12}$$

[2]In expressions like this we use normalization $\int d^2x = 4\pi$.

The objects [29, 34, 35]

$$|x\rangle\langle y| =: \left|{x \atop y}\right) \tag{13}$$

thus form a natural way for expansion of functions on the fuzzy sphere and we will call them the **string modes**. They are over-complete similarly to the coherent states they stem from and we have

$$\left({x' \atop y'}\middle|\middle|{x \atop y}\right) = \mathrm{tr}\left(|y\rangle\langle x|' \, |x\rangle\langle y|\right) = \langle x|x'\rangle\langle y|y'\rangle \ , \tag{14}$$

where we denote the trace over $\mathcal{H}$ by $\mathrm{tr}(\cdot)$, reserving the upper case trace for the operator trace over $\mathrm{End}(\mathcal{H})$. Clearly, such a representation of matrix M with a bi-local function $\phi(x, y)$ is not unique. But it can be shown [29] that the derivatives of $\phi(x, y)$ are bounded from above by $\sqrt{N}$, which means that the Fourier modes are restricted by $l_x, l_y \le \sqrt{N}$. Or in different words, any information that the function $\phi(x, y)$ tries to carry over to the matrix M in (12) at scales $1/\sqrt{N}$ or shorter gets averaged over by the finite extent of the coherent states over area $\sim 4\pi/N$. The functional representations of matrices now have very mild changes and have no rapid oscillations, but the price we have to pay is the non-local character of the string modes (13).

We will distinguish between two regimes:

- **Short string modes** for $|x - y| < 1/\sqrt{N}$, which are given by two points withing the same Planck cell of the fuzzy sphere and thus there is no distinction of them from the fuzzy point of view. It can be shown [29] that such modes correspond to wave packets, localized around the two points and with momentum $\sim N|x - y|$. They thus correspond to the standard commutative field theory degrees of freedom and form the classical regime.

  As an extreme example, state with $x = y$ represents object as localized as is allowed by the fuzziness of the space and thus is the fuzzy analogue of the δ-function. As a check, we can expand the matrix $|x\rangle\langle x|$ in terms of the basis T_{lm} and see that we obtain a cut-off expansion of $\delta(x)$ in terms of the spherical harmonics[3].

- **Long string modes** for $|x - y| > 1/\sqrt{N}$, which are proper, non-local and non-commutative objects with no classical analogue. And as we will shortly see, they are the source of all sorts of troublesome effects.

2.4 Operators in the string modes basis

As we did with the matrices in (12), we can use the string modes to expand the operators $\mathcal{O} \in \mathrm{End}(\mathrm{End}(\mathcal{H}))$ as

$$\mathcal{O} = \left(\frac{N}{4\pi}\right)^4 \int d^2x\, d^2x'\, d^2y\, d^2y' \left|{x \atop y}\right) \mathcal{O}(x, y; x', y') \left({x' \atop y'}\right| \ . \tag{15}$$

[3]This is again most easily done around the north pole, but thanks to the symmetry of the space needs to hold anywhere.

Operator traces can be calculated as follows

$$\mathrm{Tr}\, O = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle| O \middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right) , \tag{16}$$

formula which has already led to interesting results in the context of supersymmetric matrix models [36].

We can insist on a local representation of the operator

$$O = \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|\begin{smallmatrix}x\\x\end{smallmatrix}\right) O^L(x,y) \left(\begin{smallmatrix}y\\y\end{smallmatrix}\right| , \tag{17}$$

which is essentially the standard integral kernel of an operator. However, when expressed in this way the general operator will have typically a very oscillatory and ill-behaved representation similarly to the symbol map (9). And as was the case before, non-local representations are going to behave much better.

In what follows, we will be primarily interested in the propagator of the field theory $(\Box + m^2)^{-1}$. In the large N limit and in the non-local regime $|x-y| > 1/\sqrt{N}$ it is approximately diagonal

$$\frac{1}{\Box + m^2} \approx \left(\frac{N}{4\pi}\right)^2 \int d^2x\, d^2y \left|\begin{smallmatrix}x\\y\end{smallmatrix}\right) O_P^D(x,y) \left(\begin{smallmatrix}x\\y\end{smallmatrix}\right| ,\ O_P^D(x,y) = \left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle| \frac{1}{\Box + m^2} \middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right) \tag{18}$$

and we can obtain the following formula for a general function of $\Box$

$$\left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle| f(\Box) \middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right) = \frac{1}{N}\sum_{k,l}(2k+1)(2l+1)(-1)^{l+k+2s} f\big(k(k+1)\big) \begin{Bmatrix} l & s & s \\ k & s & s \end{Bmatrix} e^{-l^2/N} P_l(\cos\vartheta) , \tag{19}$$

where P_l are the standard Legendre polynomials and the objects $\{\ldots\}$ are the Wigner 6j-symbols. This then leads to [29]

$$\left(\begin{smallmatrix}x\\y\end{smallmatrix}\middle| \frac{1}{\Box + m^2} \middle|\begin{smallmatrix}x\\y\end{smallmatrix}\right) \approx \frac{1}{\frac{N^2}{4}|x-y|^2 + m^2} . \tag{20}$$

3. Fuzzy field theory in string modes formalism

In this section we will first shortly summarize the construction of fuzzy field theories and then introduce the main antagonist of our discussion – the UV/IR mixing on the fuzzy sphere. We will then reformulate the fuzzy scalar field theory in terms of the above defined string modes and show the consequences this has.

3.1 Fuzzy field theory and UV/IR mixing

To define the fuzzy field theory, we take the standard Euclidean field theory action on the sphere and use the commutative-to-fuzzy dictionary to translate the quantities to obtain

$$S(M) = \frac{4\pi r^2}{N}\mathrm{tr}\left(\frac{1}{2}M\frac{1}{r^2}[L_i,[L_i,M]] + \frac{1}{2}m^2M^2 + V(M)\right) . \tag{21}$$

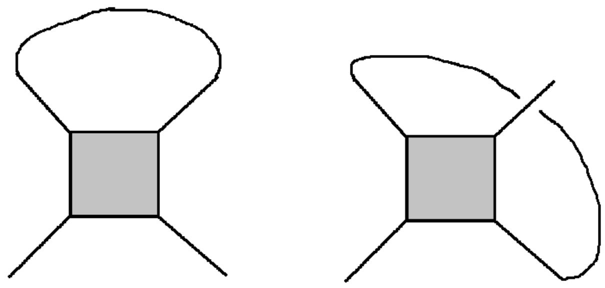


Figure 1: The two different contractions of a vertex without a permutation symmetry between the external legs.

We then define the theory using the functional integral correlation functions

$$\langle F \rangle = \frac{\int dM\, F(M) e^{-S(M)}}{\int dM\, e^{-S(M)}} \tag{22}$$

and proceed with the standard calculations using the appropriate version of the Feynman rules [37, 38]. In what follows we will consider the quartic case $V(x) = gx^4/4!$. This leads to several important consequences, among which one will play a prime role in our discussion – the UV/IR mixing [39, 41]. As a consequence of non-locality of the non-commutative theory, the very energetic UV regime of the theory has consequences for processes at low IR energies. For non-compact non-commutative spaces, such as a non-commutative plane $\mathbb{R}^2_\theta$, this leads to divergences of the one-loop diagrams at zero external momentum, for compact fuzzy spaces the contribution of such diagrams stays finite, but leads to non-local terms in the one-loop effective actions [41].

From the technical point of view, the problem stems from lack of permutation symmetry of the fuzzy interaction vertex. The fuzzy modes (3) can be exchanged under the trace in (21) only in a cyclic fashion and the two contractions depicted in the figure 1 yield a different result. Namely

$$I^P = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2}\,, \tag{23}$$

$$I^{NP} = \sum_{j=0}^{N-1} \frac{2j+1}{j(j+1)+m^2} N(-1)^{l+j+N-1} \left\{ \begin{array}{ccc} l & s & s \\ j & s & s \end{array} \right\}\,,\ s = \frac{N-1}{2}\,, \tag{24}$$

where the crucial property of the two diagrams is (non)planarity. Note, how the non-planar contribution depends on the external momentum l. One can then write the corresponding one-loop

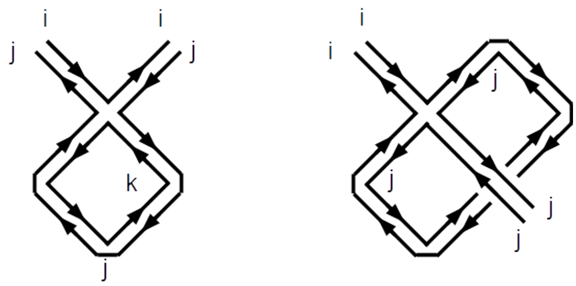


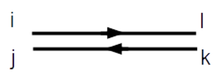
Figure 2: Two examples of diagrams for kinetic-term-less theory (21).

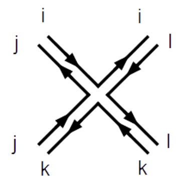
effective action and look at the result in the commutative limit, which is [41]

$$S_{\text{one loop}} = S_0 + \frac{1}{2}\int d^2x\,\phi(x)^2\delta m^2 - \frac{g}{12\pi}\int d^2x\,\phi(x)h(\tilde{\Delta})\phi(x)\,, \tag{25}$$
$$\tilde{\Delta}Y_{lm} = l\,Y_{lm}\,,\ \ h(n) = \sum_{k=1}^{n}\frac{1}{k}\,.$$

We have obtained something different than the effective action of a commutative theory. Even though the underlying fuzzy sphere became the standard continuous sphere, the theory that was first defined on S_F^2 remembers the fuzzy origin through the last, strange and non-local, contribution to the effective action.

Expressing M in terms of T_{lm} diagonalizes the kinetic term of the action (21) and leaves us to struggle with the interaction term. A different approach is offered by treating the theory as a random matrix model [12, 42]. If we drop the kinetic term for a second, we can treat the matrix model as a field theory with degrees of freedom M_{ij} and propagator and vertex given by



$$= \left\langle M_{ij}M_{kl}\right\rangle \sim \frac{1}{m^2}\delta_{il}\delta_{jk}\,,$$


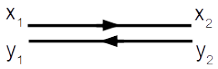
$$\sim g\,. \tag{26}$$

Lines in the Feynman graphs of such theory are then doubled, two examples are shown in figure 2 and are referred to as ribbon graphs or fat graphs. Matrix models are a huge body of research [43] and we will observe just two properties of the diagrams relevant to us. The doubled lines take care of the smaller symmetry of the interaction vertex and the labels running in the loops of the diagrams are the matrix indices, instead of momenta.

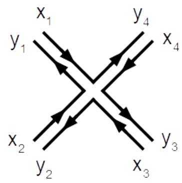
However once we include the kinetic term the situation is much more complicated. The propagator is no longer diagonal and most of the standard matrix model techniques fail. Something can be said about such models numerically and in certain approximations, as mentioned in the introduction, but complete analytical results are lacking.

3.2 Feynmann rules in string modes formalism

Writing the matrix model Feynman rules in the above fashion is using the $|i\rangle\langle j|$ basis for the fuzzy functions. But we can use the string modes basis (13) as well [29] to obtain similar looking rules



$$= \left(\begin{smallmatrix}x_2\\y_2\end{smallmatrix}\right| \frac{N}{4\pi}\frac{1}{\Box + m^2} \left|\begin{smallmatrix}x_1\\y_1\end{smallmatrix}\right) \approx \frac{N}{4\pi}\frac{1}{\frac{N^2}{4}|x-y|^2+m^2}\langle x_2|x_1\rangle\langle y_2|y_1\rangle \ , \tag{27}$$



$$= \frac{4\pi}{N}\frac{g}{4!}\langle y_1|x_2\rangle\langle y_2|x_3\rangle\langle y_3|x_4\rangle\langle y_4|x_1\rangle \ . \tag{28}$$

The labels in the diagrams are now points on the sphere and thanks to (18,7) they are also preserved along the lines in the large N limit. An external line starting with point x and ending in point y produces a factor $\langle y|x\rangle$. Each internal line with no endpoints leaves a position label to be integrated over.

The string modes bring the best from the two worlds. They diagonalize the kinetic term and at the same time keep the structure of the vertices simple. All the calculations are done in position space, which is possible also in standard field theories, but here we do not encounter any singularities thanks to the non-commutative cutoff and blurred fuzzy structure of the space. Clearly, this sounds too good to be true. And as is always the case, there is a catch. The price we will have to pay are the non-local modes $\left|\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ running in the diagrams or equivalently a non-local theory of a bi-local field $\phi(x,y)$ characterizing the matrix M in (12).

3.3 One-loop effective action and beyond

We can now use the rules of the previous section to analyze loop contributions to the effective action of the theory (21). As we will see, this leads to a much clearer interpretation of the UV/IR-mixing contribution in the effective action (25). The two diagrams relevant at one-loop are shown in the figure 3 and in terms of the field $\phi(x,y)$ the contribution to the effective action reads

$$\frac{g}{3}\left(\frac{N}{4\pi}\right)^4 \int d^2x_1 d^2y_1 d^2x_2 d^2y_2\, \phi(x_1,y_1)\phi(x_2,y_2)\times$$
$$\times\left[\langle y_2|x_1\rangle\langle y_1|x_2\rangle\frac{m_N^2}{N} + \frac{1}{2}\langle y_2|x_2\rangle\langle y_1|x_1\rangle\frac{1}{\frac{N^2}{4}|x_1-y_2|^2+m^2}\right] , \tag{29}$$

where the mass renormalization m_N^2 is independent of the external positions x, y but includes a integral over loop position z.

If we now consider the case of a local field $\phi(x,y) = \phi(x)\langle x|y\rangle$ we obtain one-loop effective

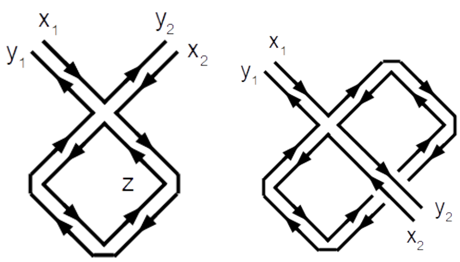


Figure 3: The one-loop diagrams contributing to the effective action of the scalar field theory on the fuzzy sphere. The diagrams are essentially the same as the pure matrix model diagrams 2 and the difference is only in their interpretation.

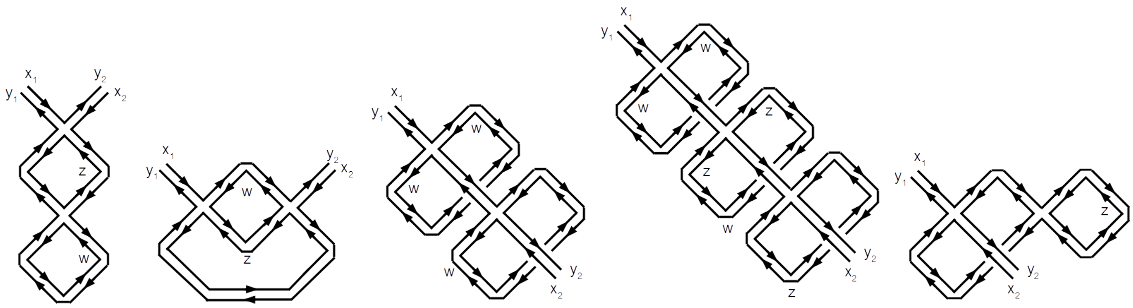


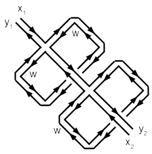
Figure 4: Several of the higher loop diagrams contributing to the effective action (29). They all contribute either to the $\langle y_2|x_1\rangle\,\langle y_1|x_2\rangle$ or to the $\langle y_2|x_2\rangle\,\langle y_1|x_1\rangle$ term, second of which leads to the non-local part of the effective action in the commutative limit.

action

$$S_{\text{eff}} = \int d^2x\,\phi(x)\frac{1}{2}(\Box + m^2)\phi(x) + \frac{g}{4!}\int d^2x\,\phi(x)^4 + \frac{g}{3}\frac{1}{4\pi}\int d^2x\,\phi(x)^2 m_N^2 + \\ + \frac{g}{6}\left(\frac{N}{4\pi}\right)^2\int d^2x\,d^2y\,\phi(x)\phi(y)\frac{1}{\frac{N^2}{4}|x-y|^2 + m^2}\,. \tag{30}$$

One can show [35] that the last non-local term in the above action reproduces the non-local term in (25). But here, it has a straightforward interpretation as an exchange of a virtual long string mode. Even though the local part of the field theory, described by the short string modes with momentum up to $\sqrt{N}$, reproduces the degrees of freedom of the commutative theory in the large N limit, the non-local modes are still present and can be exchanged in the loops of diagrams and enter the commutative limit of the fuzzy theory through quantum corrections.

To conclude, let us comment on higher loop diagrams, examples of which we show in the figure 4. They all lead to geometric integrals and are thus much more tractable than higher loop expression in the standard formalism, which include complicated group theoretical factors such as in (24). Let us concentrate on the middle diagram, which leads to



$$\approx \langle y_1|x_1\rangle \langle x_2|y_2\rangle \left(\frac{g}{4!}\right)^2 \frac{N}{4\pi} \left[\frac{N}{4\pi} \int dw\, O_P^D(\cos\vartheta_{x_1 w}) O_P^D(\cos\vartheta_{w y_1})\right] O_P^D(1)\ . \tag{31}$$

When taken for local external fields, this yields another non-local contribution to the term $\phi(x)\phi(y)$ in (30). The integral can not be evaluated analytically, but we can compute it numerically as a function of the separation betwee points x and y. And it can be checked that this contribution is more non-local than that of the non-planar diagram from figure 2. This confirms the expectation from the original analysis that higher loops make the UV/IR mixing worse.

4. Discussion and conclusions

We have presented the description of the functions on the fuzzy sphere in terms of string modes (13). We have shown how the space of functions separates into the short modes, which correspond to the classical functions, and long modes, which are purely non-commutative. Most interestingly, an exchange of the non-local long modes produces a non-local term in the effective action for the local regime of the theory, which survives the commutative limit and leads to the UV/IR-mixing.

The first line of further research would be to investigate the full expression for the field theory propagator. The diagonal expression (18) we have used is valid in the non-local regime, but numerical calculations at finite N suggest that it should have a reasonable form also for the full expression (15).

Second, it would be very interesting to study whether the ribbon graph structure of the diagrams in the string mode formalism leads to a similar hierarchy of graphs as in the pure matrix models, and to a simplifications or new insights into the higher-loop structure of the fuzzy field theory.

Finally, the above approach has already been used to obtain the Einstein-Hilbert action from the one-loop effective action of the maximally supersymmetric IKKT or IIB matrix model [36], where the stringy regime and UV/IR-mixing are suppressed by SUSY. The string modes approach thus clearly has potential to bring many new results both within and beyond fuzzy field theory.

Acknowledgments

The work of HS is supported by the Austrian Science Fund (FWF) grant P32086, the work of JT is supported by VEGA 1/0703/20 grant *Quantum structure of spacetime*.

References

[1] S. Hossenfelder, *Minimal Length Scale Scenarios for Quantum Gravity* Living Rev. Rel. **16** (2013), 2, [arXiv:1203.6191 [gr-qc]].

[2] S. Doplicher, K. Fredenhagen, J.E. Roberts, *The quantum structure of spacetime at the Planck scale and quantum fields*, Comm. Math. Phys. **172** (1995), no. 1, 187-220.

[3] H. Steinacker, *Emergent Geometry and Gravity from Matrix Models: an Introduction*, Class. Quant. Grav. **27** (2010), 133001, [arXiv:1003.4134 [hep-th]].

[4] H. Steinacker, *A Non-perturbative approach to non-commutative scalar field theory*, J. High Energ. Phys. **0503**, 075 (2005), [arXiv:hep-th/0501174].

[5] D. O'Connor, C. Sämann, *Fuzzy Scalar Field Theory as a Multitrace Matrix Model*, J. High Energ. Phys. **0708**, 066 (2007), [arXiv:hep/th 0706.2493].

[6] C. Sämann, *The Multitrace Matrix Model of Scalar Field Theory on Fuzzy* $\mathbb{C}P^n$, SIGMA **6** (2010), [arXiv:1003.4683 [hep-th]].

[7] V.P. Nair, A.P. Polychronakos, J. Tekel, *Fuzzy spaces and new random matrix ensembles*, Phys. Rev. D **85**, 045021 (2012), [arXiv:0706.2493 [hep-th]].

[8] A.P. Polychronakos, *Effective action and phase transitions of scalar field on the fuzzy sphere*, Phys. Rev. D **88**, 065010 (2013), [arXiv:1306.6645 [hep-th]].

[9] C. Sämann, *Bootstrapping fuzzy scalar field theory*, J. High Energ. Phys. **1504** (2015) 044, [arXiv:1412.6255 [hep-th]].

[10] S. Rea, Ch. Sämann, *The Phase Diagram of Scalar Field Theory on the Fuzzy Disc*, J. High Energ. Phys. **2015**, 115 (2015), [arXiv:1507.05978 [hep-th]].

[11] J. Tekel, *Asymmetric hermitian matrix models and fuzzy field theory*, Phys. Rev. D **97**, 125018 (2018), [arXiv:1711.02008 [hep-th]].

[12] M. Šubjaková, J. Tekel, *Fuzzy field theories and related matrix models*, PoS CORFU2019 (2020) 189, [arXiv:2006.12605 [hep-th]].

[13] M. Šubjaková, J. Tekel, *Second moment fuzzy-field-theory-like matrix models*, J. High Energ. Phys. **06** (2020) 088, [arXiv:2002.02317 [hep-th]].

[14] B. Ydri, R. Ahmim *Wilsonian Matrix Renormalization Group*, arXiv: 2008.09564 [hep-th].

[15] M. Šubjaková and J. Tekel, *Beyond second-moment approximation in fuzzy-field-theory-like matrix models*, J. High Energ. Phys. **22** (2020), 065, [arXiv:2109.03363 [hep-th]].

[16] J. Medina, W. Bietenholz, D. O'Connor, *Probing the fuzzy sphere regularization in simulations of the 3d* $\lambda\phi^4$ *model*, J. High Energ. Phys. **04** (2008) 041, [arXiv:0712.3366 [hep-th]].

[17] F. García-Flores, X. Martin, D. O'Connor, *Simulation of a scalar field on a fuzzy sphere*, Internat. J. Modern Phys. A **24** (2009), 3917–3944, [0903.1986 [hep-lat]].

[18] F. Lizzi, B. Spisso, *Noncommutative Field Theory: Numerical Analysis with the Fuzzy Disc*, Int. J. Mod. Phys. A27 (2012) 1250137, [arXiv:1207.4998 [hep-th]].

[19] J. L. Karczmarek, P. Sabella-Garnier, *Entanglement entropy on the fuzzy sphere*, J. High Energ. Phys. 1403 (2014) 129, [arXiv:1310.8345 [hep-th]].

[20] B. Ydri, *New Algorithm and Phase Diagram of Noncommutative Φ^4 on the Fuzzy Sphere*, J. High Energ. Phys. **03**, 065 (2014), [arXiv:1401.1529 [hep-th]].

[21] H. Mejía-Díaz, W. Bietenholz, M. Panero, *The Continuum Phase Diagram of the 2d Non-Commutative $\lambda\phi^4$ Model*, J. High Energ. Phys. **1410** (2014) 56, [arXiv:1403.3318 [hep-lat]].

[22] B. Ydri, K. Ramda, A. Rouag, *Phase diagrams of the multitrace quartic matrix models of noncommutative Φ^4 theory*, Phys. Rev. D **93** 065056 (2016), [arXiv:1509.03726 [hep-th]].

[23] S. Okuno , M. Suzuki, A. Tsuchiya, *Entanglement entropy in scalar field theory on the fuzzy sphere*, PTEP 2016 (2016) no.2, 023B03, [arXiv:1512.06484 [hep-th]].

[24] H. Z. Chen, Joanna L. Karczmarek, *Entanglement entropy on a fuzzy sphere with a UV cutoff*, PTEP 2016 (2016) no.2, 023B03, [arXiv:1712.09464 [hep-th]].

[25] S. Kováčik, D. O'Connor, *Triple Point of a Scalar Field Theory on a Fuzzy Sphere*, J. High Energ. Phys. **2018**, 10 (2018), [arXiv:1805.08111 [hep-th]].

[26] K. Hatakeyama, A. Tsuchiya, K. Yamashiro, *Renormalization on the fuzzy sphere*, PTEP 2018 (2018) no.6, 063B05, [arXiv:1811.10806 [hep-lat]].

[27] D. Prekrat, K. N. Todorović-Vasović, D. Ranković *Detecting scaling in phase transitions on the truncated Heisenberg algebra*, J. High Energ. Phys. **2021**, 197 (2021). [arXiv:2002.05704 [hep-th]].

[28] D. Prekrat, *Renormalization footprints in the phase diagram of the Grosse-Wulkenhaar model*, Phys. Rev. D **104**, 114505 (2021), [arXiv: 2104.00657 [hep-th]].

[29] H. C. Steinacker and J. Tekel, *String modes, propagators and loops on fuzzy spaces* arXiv:2203.02376 [hep-th].

[30] J. Madore, *The Fuzzy sphere* Class. Quant. Grav. **9** (1992), 69-88.

[31] J. Hoppe, *Quantum Theory of a Relativistic Surface*, Ph. D. Thesis (Advisor: J. Goldstone), MIT, 1982. http://www.aei.mpg.de/ hoppe/

[32] A. M. Perelomov, *Generalized coherent states and their applications*, Springer 1986.

[33] H. C. Steinacker, *Quantum (Matrix) Geometry and Quasi-Coherent States*, J. Phys. A **54** (2021) no.5, 055401, [arXiv:arXiv:2009.03400 [hep-th]].

[34] S. Iso, H. Kawai and Y. Kitazawa, *Bilocal fields in noncommutative field theory*, Nucl. Phys. B **576** (2000), 375-398, [arXiv:hep-th/0001027 [hep-th]].

[35] H. C. Steinacker, *String states, loops and effective actions in noncommutative field theory and matrix models*, Nucl. Phys. B **910** (2016), 346-373, [arXiv:1606.00646 [hep-th]].

[36] H. C. Steinacker, *Gravity as a Quantum Effect on Quantum Space-Time*, Phys.Lett.B **827** (2022) 136946, [arXiv:2110.03936 [hep-th]].

PoS(CORFU2021)244

[37] A.P. Balachandran, S. Kurkcuoglu and S. Vaidya, *Lectures on Fuzzy and Fuzzy SUSY Physics*, World Scientific, 2007, [hep-th/0511114].

[38] B. Ydri, *Lectures on Matrix Field Theory*, Springer, 2017, [arXiv:1603.00924 [hep-th]].

[39] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics* J. High Energ. Phys. **02** (2000), 020 doi:10.1088/1126-6708/2000/02/020 [arXiv:hep-th/9912072 [hep-th]].

[40] H. C. Steinacker, *On the quantum structure of space-time, gravity, and higher spin in matrix models*, Class. Quant. Grav. **37** (2020) no.11, 113001, [arXiv:1911.03162 [hep-th]].

[41] C. S. Chu, J. Madore and H. Steinacker, *Scaling limits of the fuzzy sphere at one loop*, J. High Energ. Phys. **08** (2001), 038, [arXiv:hep-th/0106205 [hep-th]].

[42] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, *Planar Diagrams*, Commun. Math. Phys. **59** (1978), 35.

[43] D. Anninos and B. Mühlmann, *Notes on matrix models (matrix musings)*, J. Stat. Mech. (2020) 083109, [arXiv:2004.01171 [hep-th]].