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DLCQ, Non-Lorentzian Supergravity, and T-Duality

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We comment on the T-duality relation between non-Lorentzian string theory and the DLCQ of relativistic string theory. Particular focus will be put on the structure of the background geometries. We show how target space supersymmetry constrains the form of the respective supergravity multiplets. In the conclusions, we propose several natural extensions with the aim of eventually establishing a non-Lorentzian web of dualities.

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1. Introduction

The discrete light cone quantization (DLCQ) of M-theory and the related BFSS conjecture [1, 2] propose a radical approach to studying the non-perturbative sector of string theory. By organizing the structure of the theory into non-Lorentzian blocks—i.e., the quantum mechanics of a system of n D0 branes—one hopes to encode fundamental physics in a convenient basis. Around the same time as the matrix conjectures, people have made the observation that the DLCQ of string theory can be described in a dual sigma model, which is manifestly non-relativistic. For the closed string sector, this model [3–5] is referred to as non-relativistic closed string theory, wound string theory, or Gomis-Ooguri string theory interchangeably. The associated duality is known as longitudinal T-duality and can be derived as a worldsheet duality as shown in [6]. Consequently, it also implies a relation between the background geometries in the form of non-relativistic Buscher rules. These relate a background with a lightlike isometry to a so-called torsional string Newton-Cartan geometry. In this note, we will re-derive these rules from a slightly different perspective and use them to make statements about the target space structure of either theory. In particular, we will comment on the role of target space supersymmetry.

The discrete light cone quantization assumes a compact lightlike direction. Locally that implies that there exists a lightlike Killing vector $k = \partial_y$ that constrains the Kaluza-Klein scalar to be zero. This is not a natural condition when considering supergravity multiplets. Instead, one should see the condition of lightlike isometry G_{yy} as part of a multiplet of constraints $C = 0$. This multiplet can be obtained iteratively by varying the null isometry constraint under supersymmetry. We revisit some results from studying the minimal multiplet in ten dimensions [7, 8]. In this article we consider the minimal $\mathcal{N} = (1, 0)$ multiplet in ten dimensions. The primary result is that in this case C consists of **1** bosonic algebraic, **17** fermionic algebraic, and **36** bosonic differential constraints shortening the $\mathcal{N} = (1, 0)$ multiplet to what we refer to as the $\mathcal{N} = (1, 0)_0$ multiplet. We use this example to gain intuition about the structure of this multiplet of constraints and thus the possible DLCQ supergravities in ten dimensions.

When dualizing along the DLCQ circle, the $\mathcal{N} = (1, 0)_0$ multiplet turns out to be T-dual to what we called the DSNC$^-$ multiplet in [8]. The name is short for self-dual dilatation invariant string Newton-Cartan geometry. There, we obtained the multiplet as a singular decoupling limit of the corresponding relativistic one. It realizes 16 supercharges but manifestly does not have Lorentzian symmetries. Instead, it realizes Galilean-type symmetries, as will be explained below. Relatedly, the geometric structure underlying the DSNC$^-$ is manifestly non-Riemannian. One can study equations of motion for these backgrounds [9] and find agreement with the Weyl anomaly cancellation conditions in Gomis-Ooguri string theory [10].

The structure of this article is as follows. We begin by reviewing some properties of DLCQ in section 2. In particular, we define it as an $R_s \to 0$ limit of a conventional circle reduction relation following [11]. We then show that the same limit, applied to the usual T-duality rules, leads to an implicit definition of Gomis-Ooguri decoupling limit. In section 3 we generalize this argument to curved backgrounds and thereby re-derive the target space structure of Gomis-Ooguri string theory. In section 4 we argue how to extend the results of the previous sections to include supersymmetry, referring the reader to [7, 8] for details. We end with conclusions in section 5, speculating about extensions to include maximal supersymmetry and Yang-Mills multiplets. Furthermore, we record some ideas about other dualities and embedding in eleven dimensions.

2. DLCQ and Gomis-Ooguri String Theory

The discrete light cone quantization (DLCQ) is an approach to quantum field theories, string theory, and M theory[1] where one puts the theory in a lightlike box—while at the same time fixing the momentum in the lightlike direction. Here, we are interested in string theory backgrounds where one light-like direction is compact. Introducing $x^{\pm} = 2^{-1/2}(x^0 \pm x^9)$, we can define DLCQ as the condition

$$\left(x^+, x^-, x^{A'}\right) \quad \sim \quad \left(x^+ + 2\pi R,\, x^-, x^{A'}\right). \tag{1}$$

Here and below $A' = 2, \cdots, 9$. It is natural to assign the dual lighlike direction x^- as the new time variable and the Hamiltonian as the associated charge $H = p_-$. From this we see that p_+ is conserved under time evolution since $[H, p_+] = 0$ by virtue of the Poincaré algebra. Hence it is sensible to restrict to a subspace of the full Hilbert space with p_+ fixed. In other words, the lighlike momentum satisfies

$$p_+ = \frac{n}{R}\,, \tag{2}$$

for some non-negative and fixed integer n. In other words, we quantize p_+ to be a discrete value—hence the name DLCQ—in units of $1/R$. The uncompactified theory satisfies the usual mass-shell condition $p^2 + M^2 = 0$ which turns into the dispersion relation for a Galilean particle with mass p_+—that is $H = (2\,p_+)^{-1} p^2_{A'} + E_M$ with $E_M = (2\,p_+)^{-1} M^2$. We thus observe that the physics of DLCQ for some fixed n is that of a non-relativistic system with total mass n/R. This is not just a property of free systems and can, more generally, be seen from the point of view of the Poincaré algebra. Fixing p_+ to a constant is not preserved by boosts in the longitudinal direction, specifically

$$\mathrm{e}^{\sigma\Delta} P_+ \,\mathrm{e}^{-\sigma\Delta} = \mathrm{e}^{\sigma}\, P_+\,, \tag{3}$$

where $\Delta = J_{+-}$. A similar conclusion holds for $J_{-A'}$. This manipulation shows that the DLCQ constraint $p_+ = const$ is only conserved by a subset of the full ten dimensional Poincaré algebra which can be identified as the nine-dimensional Bargmann algebra. This algebra is the Galilei algebra with a central extension p_+—corresponding to particle number conservation in non-relativistic systems. This explains why the DLCQ organizes itself in non-relativistic blocks.

The number of allowed scattering processes grows exponentially as n grows. The idea of the DLCQ proposal is to recover the full relativistic theory by taking the decompactification limit $R \to \infty$ while keeping p_+ finite at the same time—meaning $n \to \infty$. The hope was and is that this re-organization of the relativistic theory into non-relativistic blocks gives a useful regularization of the non-perturbative sector of M-theory. This is the famous matrix theory conjecture [1, 2]. For concrete applications, it can actually be enough to consider large but finite values of n, see the comments in [12]. Instead of worrying about the validity of the $n \to \infty$ limit, we will always take the value of n to be fixed and finite. Hence the physics we describe is going to be non-Lorentzian in nature.

[1]For a useful review of applications in string theory and M theory, see [12].

The validity of DLCQ and the associated $n \to \infty$ limit have been subject to heated debates in the community, see e.g. [13–16] and references therein. The credibility of the DLCQ proposal crucially depends on how to make sense of the compactness condition (1). Seiberg [11] has given a more careful definition of what (1) actually means by regularizing the compact lightlike direction as a compact spatial direction of vanishing radius

$$ds^2 = -2\, dx^+ dx^- + \varepsilon^2 \left(dx^+\right)^2 + \left(dx^{A'}\right)^2 . \tag{4}$$

so that the light cone frame is realized as $\varepsilon \to 0$. For finite ε, however, the coordinate x^+ is spacelike and we can use some of the established results and intuition from relativistic string theory on a compact target space. Let us make this more precise by defining a coordinate redefinition $\bar{x}^- = x^- - \varepsilon^2/2\, x^+$ so that $ds^2 = -2\, dx^+ d\bar{x}^- + (dx^{A'})^2$. The respective coordinates are periodic

$$x^+ \quad \sim \quad x^+ + 2\pi R\,, \qquad\qquad \bar{x}^- \quad \sim \quad \bar{x}^- - \varepsilon^2 \pi R\,. \tag{5}$$

Performing a large longitudinal boost with rapidity $\beta = (1-\varepsilon^2/2)/(1+\varepsilon^2/2)$ leads to a theory that has a compact spacelike direction x^1 with

$$x^0 \quad \sim \quad x^0\,, \qquad\qquad x^1 \quad \sim \quad x^1 + 2\pi\, \varepsilon\, R\,, \tag{6}$$

where $x^0 = 2^{-1/2}(x^+ - \bar{x}^-)$ and $x^1 = 2^{-1/2}(x^+ + \bar{x}^-)$. As was already apparent from the expression for the metric (4) we see that the actual invariant compactification radius is

$$R_s = R \cdot \varepsilon\,, \tag{7}$$

rather than R, which is a convenient parameter but has no invariant meaning since one can always perform a boost to rescale its value. More importantly, we see that the $\varepsilon \to 0$ limit is equivalent to the limit of a vanishing compactification radius $R_s \to 0$. Hence we see that the DLCQ can be seen as the limit of a spacelike reduction where the size of the compact direction shrinks to zero. Using this prescription, one can derive the DLCQ form of the mass formula

$$2n\, H = R\left(p_{A'}^2 + \text{oscillators}\right) \quad + O(\varepsilon^2)\,, \qquad\qquad nw = N - \tilde{N} \quad + O(\varepsilon^2)\,, \tag{8}$$

by using expression for the metric (4) to express $p^2 = -2\, p_+ p_- - \varepsilon^2 p_-^2 + p_{A'}^2$. Here, we have defined $H = p_-/\varepsilon + O(\varepsilon)$. For more details see [16]. Observe that it is crucial to distinguish between $n = 0$ and $n \neq 0$ in the DLCQ dispersion relation.

Let us first assume that $n > 0$. In the strict limit $\varepsilon \to 0$ this leads to the Hamiltonian for a free nine-dimensional Galilean particle of mass n/R and intrinsic energy depending on the oscillator contributions. This demonstrates the observation made above: DLCQ organizes the spectrum into Galilean blocks. Note that the winding number only contributes via the level matching condition.

Secondly, we can consider $n = 0$. In that case, we are led to imposing $p_{A'} = 0$ and appropriate constraints on the oscillator contributions. It was shown in [1] that states with $n = 0$ and $w \neq 0$ do not appear as asymptotic states. However, these states can occur as exotic virtual states in scattering processes involving momentum states. One can obtain an effective nine-dimensional description of these modes by double dimensional reduction [7] which can be recognized as that of a massless

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Galilean particle with zero color, and spin [17, 18]. The case of negative momentum $n < 0$ is excluded since it would lead to negative energies.

T-duality is a non-trivial feature of string theory relating a theory compactified on a small volume to a theory on a large volume. In the case of a reduction on a circle of radius R_s, the duality interchanges winding modes with momentum modes as follows

$$n \quad \leftrightarrow \quad w\,, \qquad\qquad R_s \quad \leftrightarrow \quad \ell_s^2/R_s\,. \tag{9}$$

As mentioned above, one can see the DLCQ of string theory as an $R_s \to 0$ limit of string theory. T-duality, on the other hand, suggests that there should be a dual formulation that derives from the $R_s \to \infty$ limit. The DLCQ is dominated by momentum modes, whereas its T-dual is expected to be dominated by winding modes. We will now construct the T-dual of the DLCQ of string theory directly, which is known as non-relativistic closed string theory. Alternative names include Gomis-Ooguri string theory and wound string theory. The latter is a two-dimensional conformal field theory with non-Lorentzian target space symmetries and thereby makes the non-relativistic character of DLCQ manifest in the sigma model. To do this construction explicitly, let us recall the Buscher rules of relativistic string theory mapping two T-dual NSNS backgrounds to each other:

$$\tilde{G}_{yy} = G_{yy}^{-1}\,, \qquad \mathrm{e}^{-2\tilde{\Phi}} = G_{yy}\,\mathrm{e}^{-2\Phi}\,, \tag{10a}$$
$$\tilde{G}_{yi} = G_{yy}^{-1}\,B_{yi}\,, \qquad \tilde{G}_{ij} = G_{ij} - G_{yy}^{-1}\big(G_{yi}G_{yj} - B_{yi}B_{yj}\big)\,, \tag{10b}$$
$$\tilde{B}_{yi} = G_{yy}^{-1}\,G_{yi}\,, \qquad \tilde{B}_{ij} = B_{ij} + 2\,G_{yy}^{-1}\,B_{y[i}G_{j]y}\,, \tag{10c}$$

where we adapted the coordinates $x^\mu = (y, x^i)$ to a spatial isometry $k = \partial_y$. The first rule maps the Kaluza-Klein modulus to its inverse and thereby realizes the $R_s \leftrightarrow \ell_s^2/R_s$ rule locally. We furthermore notice that the string-coupling modulus $\mathrm{e}^{\Phi_0} = g_s$ is mixed with the radius modulus, leading to the following rule for the loop expansion parameter $g_s \leftrightarrow \tilde{g}_s = g_s\ell_s/R_s$.

Let us now apply the Buscher rules (10) to the regularized DLCQ background (4) with $y = x^+$. That is: $G_{yy} = \varepsilon^2$ and $G_{iy}dx^i = -dx^-$. This leads to the following non-trivial T-dual background

$$d\tilde{s}^2 = \frac{1}{\varepsilon^2}\big(-dx^{-2} + dx^{+2}\big) + dx_{A'}^2\,, \qquad \tilde{B} = \frac{1}{\varepsilon^2}\,dx^- \wedge dx^+\,, \qquad \mathrm{e}^{\tilde{\Phi}} = \frac{g_s}{\varepsilon}\,, \tag{11}$$

which is clearly divergent in the limit $\varepsilon \to 0$. However, a divergence in the background fields does not imply a divergence in the non-linear sigma model. Gomis and Ooguri [4] have shown that the $\varepsilon \to 0$ of the Polyakov model on a background of the form (11) is finite. To tame the divergent contribution $O(\varepsilon^{-2})$ they introduced two additional worldsheet fields $\lambda/\bar{\lambda}$. This leads to the following sigma model

$$S = \frac{1}{4\pi\ell_s^2}\int d^2\sigma\left(\partial X^{A'}\bar{\partial}X_{A'} + \lambda\bar{\partial}X + \bar{\lambda}\partial\bar{X}\right), \tag{12}$$

in conformal gauge. For a recent review of the classical and quantum properties of this non-relativistic string theory, see [19]. The Lagrange multipliers $\lambda/\bar{\lambda}$ are one-forms under residual conformal transformations. It is clear from the splitting into a set of 8 free scalars $X_{A'}$ and a (β, γ) system (λ, X) that the model does not have Lorentzian target space symmetries. In fact, one can already conclude from the spacetime an-isotropic rescaling in (11) that Lorentzian boosts are broken.

In this work we assume closed string boundary condition throughout. Upon quantization one can show that this theory describes a spectrum of string like objects with the manifestly non-relativistic dispersion relation

$$2w\,H = R\left(p_{A'}^2 + \text{oscillators}\right), \qquad nw = N - \tilde{N}\,, \tag{13}$$

where we have inherited the compact direction as $x^+ \sim x^+ + 2\pi R$. It is furthermore not hard to see that the X/λ system contributes a central charge $c_{\lambda X} = 2$. Hence one can see that the critical dimension is $D = 26$ as in relativistic string theory. For superstrings an analogous reasoning leads to a critical dimension $D = 10$, see [20].

For positive winding number $w > 0$, equation (13) is the dispersion relation of a massive Galilean particle in $D = 25/9$ dimensions of mass w/R. Unwound string states with $w = 0$ lead to $p_{A'} = 0$ and are thus not propagating. These modes are somewhat exotic and do not appear as asymptotic states. They do, however, appear as poles of the form $(p_{A'}^2 + \cdots)$ in scattering amplitudes and give a physical interpretation to the instantaneous Newtonian force experienced by the winding states. For this reason, they have been referred to as Galilean gravitons [5] in the past.

We see that the spectrum formula of the DLCQ (8) and that of the Gomis-Ooguri string theory (13) are exchanged by exchanging the momentum and the winding quantum number while at the same time inverting the radius $R \leftrightarrow \ell_s^2/R$. At the level of the physical modes this manifests in the statement that the relevant asymptotic states have $(n \neq 0, w = 0)$ in the DLCQ and $(n = 0, w \neq 0)$ in Gomis-Ooguri string theory. Formally, the T-duality rules are equivalent to expressions (9) with $R_s \to R$. We note that the rule $R \leftrightarrow \ell_s^2/R$ does not have the same level of importance as the analogous rule in relativistic string T-duality. Since the parameter R is not an invariant quantity, one cannot interpret this rule as a statement about physical lengths—and thus the small scale structure of spacetime.

3. Longitudinal T-Duality

In this section, we generalize the longitudinal T-duality relation to arbitrary backgrounds. That is, we will derive T-duality rules between NS backgrounds with a lightlike isometry and torsional string Newton-Cartan geometry on the other side. The latter is the appropriate geometric structure to which non-relativistic string theory couples. It is manifestly non-Lorentzian and can be motivated and derived in many ways. See [9, 21–23] for some approaches involving gauging procedures and singular limits. Here we will take longitudinal T-duality to be fundamental and deduce the geometric properties via the non-relativistic Buscher rules.

To do so we consider an NS background with a spatial isometry $k = \partial_y$ and send the value of the associated Kaluza-Klein scalar to zero parametrically. That is, we introduce a parameter ε and rescale $\tilde{G}_{yy} = (\varepsilon/\mathcal{T})^2$. We take $\mathcal{T}$ to be finite in the limit $\varepsilon \to 0$ which effectively shrinks the size of the compact direction to zero. The T-dual Kaluza-Klein scalar G_{yy}—and thus the T-dual radius—diverge in the $\varepsilon \to 0$ limit. Hence we can see the above as a decompactification limit. For related remarks, see [24]. Apart from the Kaluza-Klein modulus, one should also consider the other fields on both sides of the Buscher rules (10). It is not hard to see that the above prescription leads to unacceptable consequences. As an example, one finds that $\tilde{G}_{yi} \to 0$ and $\tilde{g}_s \to 0$. It seems hard to make sense out of a T-duality relation where the number of classical degrees of freedom does not match on both sides of the duality. Instead, we make an attempt to regularize this limit of a vanishing Kaluza-Klein scalar as follows

$$\begin{aligned} G_{yy} &= \varepsilon^{-2}\mathcal{T}^2\,, & G_{yi} &= -\varepsilon^{-2}\mathcal{T}\,n_i\,, & B_{yi} &= \varepsilon^{-2}\,\mathcal{T}\,\tau_i + m_i\,,\\ \mathrm{e}^{\Phi} &= \varepsilon^{-1}\,\mathrm{e}^{\phi}\,, & G_{ij} &= \varepsilon^{-2}\left(-\tau_i\tau_j + n_i n_j\right) + e_{ij}\,, & B_{ij} &= 2\,\varepsilon^{-2}\,\tau_{[i}n_{j]} + b_{ij}\,. \end{aligned} \tag{14}$$

Note that this should be seen as a convenient redefinition of the fields, that can be iteratively derived by requiring finiteness under $\varepsilon \to 0$. In particular, the number of independent field components is the same in both bases: $(G_{\mu\nu}, B_{\mu\nu}, \Phi) \cong (\tau_i, m_i, n_i, e_{ij}, m_{ij}, \mathcal{T}, \phi)$.[2] It is not hard to see that this choice indeed tames the divergences occurring in the Buscher rules (10) in a convenient parametrization as follows

$$\begin{aligned} \tilde{G}_{yy} &= \left(\varepsilon/\mathcal{T}\right)^2, & \mathrm{e}^{-\tilde{\Phi}} &= \mathcal{T}\,\mathrm{e}^{-\phi}\,,\\ \tilde{G}_{yi} &= \mathcal{T}^{-1}\tau_i + \left(\varepsilon/\mathcal{T}\right)^2 m_i\,, & \tilde{G}_{ij} &= e_{ij} + 2\,\mathcal{T}^{-1} m_{(i}\tau_{j)} + \left(\varepsilon/\mathcal{T}\right)^2 m_i m_j\,,\\ \tilde{B}_{yi} &= -\mathcal{T}^{-1} n_i\,, & \tilde{B}_{ij} &= b_{ij} - 2\,\mathcal{T}^{-1}\,m_{[i}n_{j]}\,. \end{aligned} \tag{15}$$

The tilded fields parametrize a geometry with a spatial isometry of size $\tilde{R}_s \sim \varepsilon$. In the strict limit this turns into a null isometry and maps it to a geometry that is parametrized by $(\mathcal{T}, \tau_i, m_i, n_i, e_{ij}, \phi, b_{ij})$. However, we note that the right-hand side of (15) is developing a Stückelberg symmetry in the strict limit $\varepsilon \to 0$. Concretely, the expressions are invariant when scaling $(\mathcal{T}, \tau_i, n_i, \mathrm{e}^{\phi})$ with the same factor. This indicates that we have overparametrized our fields when choosing the definitions (14). In hindsight, this is not surprising since the $\varepsilon \to 0$ eliminates one field component—namely the Kaluza-Klein scalar itself—and should thus also eliminate one field component on the T-dual side. The theory chooses to realize this by having an additional gauge symmetry emerge in the limit. Conversely, the overparametrization can be taken care of by gauge fixing, e.g., by setting $\mathcal{T} = 1$, yielding the following finite T-duality rules

$$d\tilde{s}^2 = 2\,\tau\left(dy + m\right) + e_{ij}dx^i dx^j\,, \qquad \widetilde{B} = n \wedge \left(dy + m\right) + b\,, \qquad \widetilde{\Phi} = \phi\,. \tag{16}$$

Here, we defined the one-forms $\tau = \tau_i dx^i$, $m = m_i dx^i$, $n = n_i dx^i$, and the two-form $b = 1/2\, b_{ij} dx^i \wedge dx^j$. The gauge fixed result is equivalent to the non-relativistic Buscher rules derived in [22] with $\varepsilon = \omega^{-1}$. The explicit results given there differ from the ones given here due to an overparametrization on the Gomis-Ooguri side. This again leads to a Stückelberg symmetry. Fixing this symmetry by imposing ${m_\mu}^A = 0$ leads to results that are in agreement with eq. (16). It is furthermore not hard to see that the Buscher rules (16) are invariant under transformations of the form

$$\delta m_i = -\lambda_i\,, \qquad \delta e_{ij} = 2\,\tau_{(i}\lambda_{j)}\,, \qquad \delta b_{ij} = 2\,n_{[i}\lambda_{j]}\,. \tag{17}$$

This can be recognized as a local realization of Galilean boosts with $\tau^i \lambda_i = 0$ for consistency. Accordingly, we remark that (τ_i, m_i, e_{ij}) parametrizes a Newton-Cartan type geometry in D=25/9—coupled to matter (n_i, b_{ij}, ϕ). See [7] for more details and references. It is possible to uplift this

[2] For this counting to work out it is important to observe that e_{ij} only has **36** instead of the expected **45** components due to a constraint coming from the invertibility relation of the ten-dimensional metric.

structure to a D=26/10 structure known as a torsional string Newton-Cartan geometry, parametrized by $(\tau_\mu{}^A, e_{(\mu\nu)}, b_{[\mu\nu]}, \phi)$, where $A = 0, 1$. This geometry describes the general background structure of Gomis-Ooguri string theory. It realizes local Galilean boost symmetries $\lambda_\mu{}^A$ as follows

$$\delta e_{\mu\nu} = -2\,\eta_{AB}\tau_{(\mu}{}^A\lambda_{\nu)}{}^B\,, \qquad\qquad \delta b_{\mu\nu} = 2\,\epsilon_{AB}\tau_{[\mu}{}^A\lambda_{\nu]}{}^B\,. \tag{18}$$

Assuming a longitudinal spatial isometry k with $\tau^0(k) = 0$, $\tau^1(k) = 1$, and $e(k,k) = 0$ one can find a Kaluza-Klein like embedding of the Newton-Cartan fields appearing in eq. (16). Part of the higher dimensional boosts are fixed by the Ansatz and $\lambda_i{}^0 = \lambda_i$. For more details, see [7].

4. Supersymmetry

In this section, we will address whether longitudinal T-duality can be extended to include target space supersymmetry. We will consider the simplest setting, i.e., a T-duality relation between the minimal non-relativistic supergravity multiplet presented in [8] and the DLCQ of $\mathcal{N} = (1,0)$ supergravity. The latter denotes the minimal multiplet in ten dimensions, realizing 16 left-handed Majorana-Weyl supercharges. The field content contains the NSNS fields (G, B, Φ), a left-handed gravitino Ψ, and a right-handed dilatino Λ. We will comment on other multiples, including the coupling to vectors and maximal supersymmetry in the conclusions 5. The main point of this section is the following: the presence of a lightlike isometry is not naturally consistent with supersymmetry and leads to a multiplet of constraints. More concretely, this can be seen by noting that a nonzero lightlike Killing vector $k^\mu\partial_\mu = \partial_y$ implies that $G_{yy} = 0$, which is at odds with supersymmetry since

$$\delta_\varepsilon G_{yy} = \bar{\varepsilon}\,\Gamma_+\Psi_y \quad \text{is not zero identically}\,. \tag{19}$$

Requiring consistency with supersymmetry leads to another condition that has two possible solutions: either a breaking of supersymmetry $\Gamma_+\varepsilon = 0$ to eight supercharges or a constraint on the gravitino $\Gamma_+\Psi_y = 0$. Either choice is viable, in principle. In this work, we will focus on the second option. It should be seen as part of a multiplet of constraints $C = 0$ of which $G_{yy} = 0$ is the lowest-lying component. In the following, we will determine the full multiplet of constraints $C = 0$. The resulting multiplet can formally be defined by imposing the constraints on the $\mathcal{N} = (1,0)$ multiplet; schematically

$$\mathcal{N} = \big(1,0\big)_0 \quad \equiv \quad \mathcal{N} = \big(1,0\big)\Big|_{C=0}\,. \tag{20}$$

The existence of an isometry implies that the structure of the multiplet is effectively nine-dimensional. We will establish below that this multiplet is T-dual to the supersymmetric version of the Gomis-Ooguri background, which we derived from a limit in [8]. T-duality in this setting can be defined as a matching of the reduction of the ten-dimensional theories in nine dimensions [25]. This is made manifest by adapting the ten-dimensional fields of the $\mathcal{N} = (1,0)$ multiplet in a so-called null reduction [26–28]. Here, we will not attempt to give explicit results and instead refer the reader to [7, 8] for details. Under the assumption of unbroken supersymmetry, we are led to the fermionic constraints $\Gamma_+\Psi_y = 0$, as shown above. Consequently, one should also vary this under supersymmetry, and vary the result of that, etc. This procedure could stop after any number of steps. This is equivalent to determining the multiplet of constraint C iteratively. A prior, it is not guaranteed whether or not this system becomes overconstrained—i.e., whether or not C turns out to be the full multiplet so that $\mathcal{N} = (1,0)_0$ is an empty theory. In the case of $\mathcal{N} = (1,0)$, it turns out that C is remarkably short:

$$C = \left\{G_{yy}, \Psi_y, Z_{ij}\right\} = 0\,. \tag{21}$$

For more details of this calculation, see [7]. In the above, we have defined

$$Z_{\mu\nu} = \partial_{[\mu}Z_{\nu]}\,, \qquad \text{with} \qquad Z_\mu = k^\nu\left(G_{\nu\mu} - B_{\nu\mu}\right), \tag{22}$$

satisfying $k^\mu Z_\mu = 0$ and hence also $k^\mu Z_{\mu\nu} = 0$. Using this we find that C contains **1** bosonic algebraic, **16** fermionic algebraic, and **36** bosonic differential constraints. It is not hard to verify the multiplet structure of C since the one-form Z_μ turns out to be a supersymmetry singlet.

The differential constraint deserves special attention since it is a bosonic constraint that does, however, not appear in the purely bosonic target space description. Among other things, it implies that not all solutions of NS gravity with a null isometry are also solutions of the supersymmetric $\mathcal{N} = (1,0)_0$ theory. We have shown in [7] that the fundamental string solution solves the constraint—whereas the anti-fundamental string does not. To see that, it is useful to rewrite the constraint $Z_{ij} = 0$ as a condition on the Killing spinor: $(\nabla_\mu\,\delta_\nu{}^\rho + \mathcal{H}_{\mu\nu}{}^\rho)k_\rho = 0$, where $\mathcal{H} = \mathrm{d}B$.

Having spelled out some qualitative features of the $\mathcal{N} = (1,0)_0$ multiplet, let us turn to the T-dual supergravity multiplet. For reasons that will become clear later we will refer to this as DSNC$^-$ supergravity. The bosonic sector is spanned by the background fields of Gomis-Ooguri string theory: $(\tau_\mu{}^A, e_{(\mu\nu)}, b_{[\mu\nu]}, \phi)$ and the fermionic sector is described by two left-handed gravitini $(\psi_{\mu+}, \psi_{\nu-})$ and two right-handed dilatini (λ_+, λ_-). The sub-labels $\pm$ indicate a projection along Γ_{01} that turns out to give a convenient representation of boosts $\lambda_\mu{}^A$. For more details and conventions, see [8]. There, we constructed the DSNC$^-$ multiplet via a limiting procedure analogous to the one defined by (14). It turns out that this leads to a non-zero term at order $O(\varepsilon^{-2})$ in the expansion of the supersymmetry rules. Naively this implies that the non-relativistic limit $\varepsilon \to 0$ is ill-defined. This inconsistency can be lifted by

1. Imposing a constraint on the intrinsic torsion of the geometry $T^\rho_{\mu\nu}\tau_\rho{}^- = 0$, or

$$\tau^- \wedge \mathrm{d}\tau^- = 0\,, \tag{23}$$

which corresponds to **36** independent first order differential constraints on the longitudinal Vielbein $\tau_\mu{}^-$. Alternatively, it can be expressed as $\tau_{A'B'}{}^- = 0$ and $\tau_{A'}{}^{--} = 0$. Torsional string Newton-Cartan geometries [9, 23] with this additional constraint are referred to as self-dual dilatation invariant string Newton-Cartan geomtries—or, DSNC$^-$ for short. This constraint is consistent with all the symmetries, in particular with supersymmetry since $\delta_\epsilon\tau_\mu{}^- = 0$. In [10], the same constraint has played a central role when studying the (bosonic) n-loop quantum effective action of Gomis-Ooguri string theory.

2. Including additional symmetries δ_D, δ_S, and δ_T, which we refer to as anisotropic dilatations and fermionic S- and T-symmetries, respectively. All of these additional symmetries act as local shifts on some fields. These additional symmetries give **1** bosonic and **8** + **8** fermionic gauge symmetries. They are absent in the relativistic parent theory and imply that the

multiplet is effectively smaller since the local shift symmetries can be gauge fixed by setting some field components to constant values.

The fact that the divergent limit can be regularized and made sense of is highly non-trivial and a genuine feature of ten-dimensional $\mathcal{N} = (1,0)$ supergravity. We have shown in [8] that the same limit is well-defined at the level of the equations of motion—which are conjectured to capture the universal sector of the beta functions of non-relativistic superstring theory.

Let us remark that the additional ingredients are in one-to-one correspondence to the multiplet of constraints (21) in the $\mathcal{N} = (1,0)_0$ theory. In particular, the additional gauge symmetries are T-dual to the vanishing of certain (fermionic) field components:

$$\begin{array}{lll} & \text{DSNC}^- & \mathcal{N} = (1,0)_0 \\ \mathbf{1}: & \delta_D & G_{yy} = 0\,, \\ \mathbf{8}+\mathbf{8}: & \delta_S + \delta_T & \Psi_y = 0\,, \\ \mathbf{36}: & \tau^- \wedge \mathrm{d}\tau^- = 0\,, & Z_{ij} = 0\,. \end{array} \tag{24}$$

These observations show that the T-duality between the DSNC$^-$ and the $\mathcal{N} = (1,0)_0$ multiplets can be extended beyond the bosonic sector. For a more explicit derivation, see [7].

5. Conclusions

In this article, we have reviewed some recent developments in non-relativistic string theory. Rather than defining the target space theory through some decoupling limit as in [4, 6], we have taken the T-duality relation to the discrete lightcone quantization of string theory to be fundamental. Defining the DLCQ of string theory as a regularized limit of a vanishing Kaluza-Klein scalar, we have re-derived the longitudinal Buscher rules of [6]. Furthermore, we have argued how to extend these results to include target space supersymmetry. The existence of a lightlike isometry is a constraint on the supergravity multiplet. For this to be consistent with supersymmetry, it has to be extended to a multiplet of constraints C. This multiplet can be found iteratively by varying the null-isometry condition $G_{yy} = 0$. The resulting supergravity is denoted as $\mathcal{N} = (1,0)_0$. We have also sketched some of the non-trivial properties of the T-dual multiplet [8] known as DSNC$^-$ supergravity.

So far, we have only studied the minimal supergravity setting in ten dimensions. This is unsatisfactory when having string theory applications in mind since it does not describe the low-energy dynamics of any superstring theory. Relatedly, it is unsatisfactory since $\mathcal{N} = (1,0)$ supergravity has gravitational anomalies [29]. For these reasons, one should see the results presented in this work (and [8]) as an intermediate step toward non-Lorentzian heterotic and type IIA/B supergravity. Phrased more positively, one can see the results presented here as the common sector of all possible multiplets in ten dimensions. Following the logic of this article, we propose the following approach for constructing non-Lorentzian versions of the five supergravities in ten dimensions corresponding to analogous superstring theories, as follows:

1. Analyze the respective multiplets of constraints of which $G_{yy} = 0$ is the leading component. We do not expect the structure of the constraint multiplet to change significantly when coupling to Yang-Mills, that is $C_{HET} \sim C$ as given in (21). Maximal supersymmetry, on

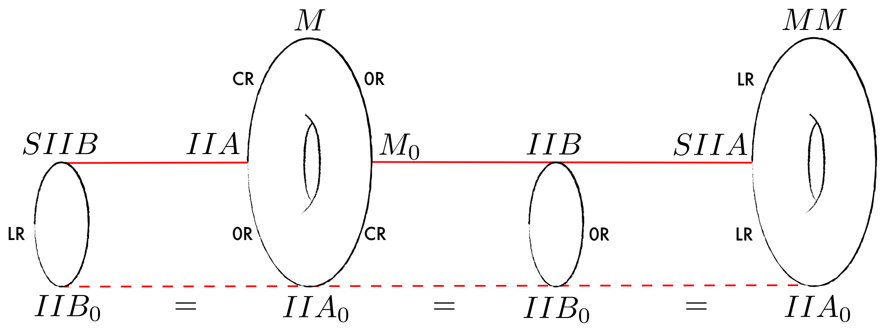

Figure 1: Schematic diagram of possible embeddings in eleven dimensions. The central two-torus has one lightlike circle. Correspondingly, one can either reduce over a spatial circle (CR) or the null circle (0R). Accordingly, one can get either one of two theories in ten dimensions: IIA and M_0. Both are T-dual in the sense that they reduce to the same nine-dimensional theory, namely the DLCQ of IIA theory which we denote here as IIA_0. The theory can be uplifted to give two different ten-dimensional theories. First, one should be able to uplift it to the IIB theory along the lines of [25]. Here, we implicitly assume that there is a unique type II theory IIA_0 =IIB_0. Secondly, and more interestingly, one should be able to uplift the theory in a longitudinal circle (LR) on a Gomis-Ooguri background that we tentatively call stringy IIB theory. Starting from the DLCQ of the type IIB theory, on the other hand (denoted as IIB_0), we can apply longitudinal T-duality to obtain a IIA version of Gomis-Ooguri backgrounds, here denoted as SIIA. It is conceivable that this supergravity multiplet can be uplifted over a longitudinal spatial circle (LR) to eleven dimensions. This theory is denoted as MM (for membrane M-theory) and has a rank-3 distribution. The bosonic sector of this theory was studied in [33].

the other hand, is expected to lead to an enlarged set of constraints $C_{IIA/B}$. In particular, we expect more than double the number of components in (21). The DLCQ of type IIA/B would then be formally defined as constrained multiplets $\text{IIA/B}_0 = \text{IIA/B}|_{C_{IIA/B}=0}$. An iterative approach to determining $C_{IIA/B}$ should, in principle, be possible—albeit tedious. It would be interesting to see whether the multiplets of constraints can be determined directly in a superspace formalism along the lines of [30, 31] by finding a superspace version of $G_{yy} = 0$.

2. Use the longitudinal T-duality rules to determine the structure of the supersymmetric version of the Gomis-Ooguri backgrounds. The correct rules for the Ramond-Ramond fields can, for example, be derived by considering the non-relativistic limit of D-brane actions [32]. Alternatively, one can perform the null-reduction to nine dimensions and uplift the result to a Gomis-Ooguri background with a longitudinal spatial circle, following the approach of [25]. This idea should also be applicable to heterotic supergravity.

These steps seem straightforward in principle, and we hope to present an explicit construction soon. Assuming that the above works out, one can construct the non-Lorentzian versions of the five supergravities in ten dimensions. It is then natural to ask about duality relations between the different theories. As shown above, the respective multiplets of constraints encode crucial information about the supergravity multiplets. One question that comes to mind is how these different multiplets fit together. For example, we speculated above that $C_{IIA/B} \supset C \sim C_{HET}$ and expect that C_{IIA} is related to C_{IIB} through a T-duality relation. It would be fascinating to figure out in detail how the information about the multiplets of constraints is realized in a non-Lorentzian web of dualities. This is interesting from the point of view of non-Lorentzian string theory—but at the same time, it also asks fundamental questions about the consistency of DLCQ prescription in the presence of target space supersymmetry.

Relatedly, it is tempting to wonder about an embedding in an eleven-dimensional theory. Note that we see type IIA superstrings/supergravity as the dimensional reduction over a circle. Furthermore, we define the DLCQ of IIA superstrings/supergravity as the lightlike reduction to nine dimensions. Hence, the DLCQ of type IIA is formally defined as the torus compactification from eleven to nine dimensions where one direction is lightlike. This way of phrasing the DLCQ of IIA suggests that there should also be another definition: consider the DLCQ of eleven-dimensional supergravity and reduce this theory over another spatial circle. See the tentative diagram 1. It would be very interesting to see whether this intuitive construction can be made precise.

The DLCQ of the IIA theory is conjectured to be T-dual to a stringy version of the type IIB theory. Similarly, we expect the DLCQ of the type IIB multiplet to be T-dual to a stringy version of the type IIA multiplet. It would be interesting to see whether the latter can be embedded as a membrane geometry in eleven dimensions, relating to the work of [33]. An exciting and complementary view on many of these questions is provided by the study of D-brane actions in [32].

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