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Neutrino mixing, entanglement and the gauge paradigm in quantum field theory

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We shortly review the quantum field theory formulation of neutrino mixing and oscillations extensively studied in literature. The canonical formalism requires a non-trivial condensate structure of the flavor vacuum, which appears to be an entangled coherent state. Within such a frame, mixing is described as resulting from the interaction of the neutrino field with a non-abelian gauge field.

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1. Introduction

Neutrino mixing and oscillations are efficiently described in the Pontecorvo formulation in quantum mechanics (QM) formalism [1, 2].

There are, however, few important problems left open. One is that Pontecorvo states are not eigenstates of the flavor neutrino charges, with consequent violation of the conservation of leptonic charge in the neutrino production vertices [3, 4]. Also, neutrinos are field operators properly handled in the quantum field theory (QFT) formalism on which the Standard Model (SM) is based. The study of neutrino mixing and oscillations cannot be carried on by ignoring specific features of QFT. One of these is at very same basis of the SM formulation, namely the fact that in the SM the physical vacuum is singled out of the set of the infinitely many vacua of the representations of the canonical commutation (anticommutation) relations (CCR or CAR), i.e. the vacuum providing the expectation value for the Higgs field, able to produce the experimentally observed electron and other particle masses [5–7].

Thus, by treating neutrino field operators in their natural frame of QFT, it appears immediately that flavor fields ν_σ, $\sigma = e, \mu, \tau$, are properly defined on a space which is a representation of the CAR unitarily inequivalent to the representation where the massive neutrino fields ν_i, $i = 1, 2, 3$, are defined [8]. In the following Sections, we shortly review this result which has been extensively studied in a number of publications [9–30] and also extended to boson mixing [31–33]. For a recent review see [34].

One very important 'ingredient' enters in our discussion, namely the fact that QFT is a 'canonical' theory. The canonical structure cannot be violated, it constitutes a constraint of primary relevance in the discussion of the problem under study. We will see that it requires a non-trivial condensate structure of the flavor vacuum state.

These general considerations are necessary in order to understand the limits and the meaningfulness of the QM Pontecorvo mixing formalism. It is indeed well known [35] that the von Neumann theorem in QM states that all the representations of the CCR (or CAR) are unitarily equivalent, and therefore physically equivalent. This theorem does not hold in QFT, where infinitely many unitarily inequivalent representations (uir) of the CCR (or CAR) exist. It is clear that one could not even think to formulate the SM in a QM framework where it would not be possible to make the proper "choice" of the physical vacuum.

In this paper we also review the gauge field formalism for neutrino mixing, where the mixing is described as resulting from the interaction of the neutrino field with a non-abelian gauge field [28], acting as an external field. We also discuss the issue of Poincaré invariance for flavor neutrino states.

2. Neutrino mixing transformations

For simplicity, we limit ourselves to the case of two Dirac neutrino mixing. Our considerations can be however extended also to the case of three neutrinos. The Lagrangian density written in terms of neutrino field operators $\nu_e(\mathbf{x}, t)$ and $\nu_\mu(\mathbf{x}, t)$ and two massive neutrino field operators

$\nu_1(\mathbf{x},t)$ and $\nu_2(\mathbf{x},t)$ is

$$\begin{aligned}\mathcal{L} &= \bar{\nu}_e\,(i\,\not{\partial} - m_e)\,\nu_e + \bar{\nu}_\mu\,\left(i\,\not{\partial} - m_\mu\right)\nu_\mu \;-\; m_{e\mu}\,\left(\bar{\nu}_e\,\nu_\mu \;+\; \bar{\nu}_\mu\,\nu_e\right), &(1)\\ &= \bar{\nu}_1\,(i\,\not{\partial} - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\not{\partial} - m_2)\,\nu_2, &(2)\end{aligned}$$

respectively, where for notational simplicity we have used $\nu_i \equiv \nu_i(\mathbf{x},t)$, $i = 1,2$, and $\nu_\sigma \equiv \nu_\sigma(\mathbf{x},t)$, $\sigma = e,\mu$. In the usual notation, the field operators ν_1 and ν_2 are:

$$\nu_j(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u^r_{\mathbf{k},j}(t)\alpha^r_{\mathbf{k},j} + v^r_{-\mathbf{k},j}(t)\beta^{r\dagger}_{-\mathbf{k},j}\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad j = 1,2, \tag{3}$$

where $u^r_{\mathbf{k},j}(t) = u^r_{\mathbf{k},j}e^{-i\omega_{\mathbf{k},j}t}$, $v^r_{-\mathbf{k},j}(t) = v^r_{-\mathbf{k},j}e^{i\omega_{\mathbf{k},j}t}$, and $\omega_{\mathbf{k},j} = \sqrt{\mathbf{k}^2 + m_j^2}$. The CAR are:

$$\begin{aligned}\{\nu^a_i(x), \nu^{b\dagger}_j(y)\}_{t_x=t_y} &= \delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}\delta_{ij}, &(4)\\ \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}; \qquad &\{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}, \;\; i,j = 1,2, &(5)\end{aligned}$$

with $a,b = 1,\ldots,4$. All other CAR are zero. The orthonormality and completeness relations are: $u^{r\dagger}_{\mathbf{k},j}u^s_{\mathbf{k},j} = v^{r\dagger}_{\mathbf{k},j}v^s_{\mathbf{k},j} = \delta_{rs}$, $u^{r\dagger}_{\mathbf{k},j}v^s_{-\mathbf{k},j} = v^{r\dagger}_{-\mathbf{k},j}u^s_{\mathbf{k},j} = 0$, $\sum_r(u^r_{\mathbf{k},j}u^{r\dagger}_{\mathbf{k},j} + v^r_{-\mathbf{k},j}v^{r\dagger}_{-\mathbf{k},j}) = 1$.

The operators $\alpha^r_{\mathbf{k},j}$ and $\beta^r_{-\mathbf{k},j}$, $j = 1,2$, $r = 1,2$ are the annihilation operators for the vacuum state $|0\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},j}|0\rangle_{1,2} = 0$; $\beta^r_{-\mathbf{k},j}|0\rangle_{1,2} = 0$. By operating with the creation operators $\alpha^{r\dagger}_{\mathbf{k},j}$ and $\beta^{r\dagger}_{-\mathbf{k},j}$, $j = 1,2$, $r = 1,2$, on the vacuum state $|0\rangle_{1,2}$ the full tower of many particle states is generated and the Hilbert space $\mathcal{H}_{1,2}$ is constructed.

Of course, equations and relations among operators, e.g. the CAR, are well defined only when it is specified the space $\mathcal{H}_{1,2}$ on which the field operators ν_1, ν_2, $\alpha^r_{\mathbf{k},j}$ and $\beta^r_{-\mathbf{k},j}$, $j = 1,2$, are defined. This is a non-trivial condition since in QFT infinitely many unitarily non-equivalent representations of the CAR exist. In quantum mechanics (QM) such a problem does not exist since there the von Neumann theorem holds, stating that all the representations of the CAR are unitarily (and thus physically) equivalent.

The Lagrangian in Eq. (1) is formally diagonalized, leading to the Lagrangian (2) for the neutrino filed operators ν_1 and ν_2, by using the *mixing transformation*

$$\begin{aligned}\nu_e &= \nu_1\cos\theta + \nu_2\sin\theta &(6)\\ \nu_\mu &= -\nu_1\sin\theta + \nu_2\cos\theta, &(7)\end{aligned}$$

provided that the following constraints between the mixing angle θ, the masses m_i, m_σ, $i = 1,2$; $\sigma = e,\mu$, and the 'strenght' of the mixing terms $m_{e\mu}$ in Eq. (1) are satisfied:

$$\sin 2\theta = \frac{2m_{e\mu}}{m_2 - m_1}, \qquad \cos 2\theta = \frac{m_\mu - m_e}{m_2 - m_1}, \qquad m_e + m_\mu = m_1 + m_2, \tag{8}$$

where $m_1 \neq m_2$, namely, $m_e = m_1\cos^2\theta + m_2\sin^2\theta$, $m_\mu = m_1\sin^2\theta + m_2\cos^2\theta$, $m_{e\mu} = (1/2)(m_2 - m_1)\sin 2\theta$.

The mixing transformations are *canonical* transformations since they preserve the CAR among the neutrino field operators. Of course, preserving the canonical formalism is essential in order to allow the particle description of the theory and the meaningfulness of the Lagrangian and Hamiltonian formal apparatus.

In the following, our analysis proceeds within the standard QFT framework [6, 7, 35].

The total Hamiltonian H and the momentum operator P^i, $i = 1, 2, 3$, are:

$$H = \int d^3\mathbf{x}\, T^{00} = \int d^3\mathbf{x}\, \nu_1^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_1\right) \nu_1 + \int d^3\mathbf{x}\, \nu_2^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_2\right) \nu_2, \qquad (9)$$

$$P^i = \int d^3\mathbf{x}\, T^{0i} = i\int d^3\mathbf{x}\, \nu_1^\dagger \partial^i \nu_1 + i\int d^3\mathbf{x}\, \nu_2^\dagger \partial^i \nu_2, \qquad i = 1, 2, 3, \qquad (10)$$

respectively. They are obtained from the canonical energy-momentum tensor $T_{\rho\sigma}$ derived from the Lagrangian:

$$\begin{aligned} T_{\rho\sigma} &= \bar{\nu}_e i\gamma_\rho \partial_\sigma \nu_e - \eta_{\rho\sigma}\bar{\nu}_e(i\gamma^\lambda\partial_\lambda - m_e)\nu_e + \bar{\nu}_\mu i\gamma_\rho\partial_\sigma\nu_\mu - \eta_{\rho\sigma}\bar{\nu}_\mu(i\gamma^\lambda\partial_\lambda - m_\mu)\nu_\mu \\ &\quad + \eta_{\rho\sigma} m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) \qquad (11) \\ &= \bar{\nu}_1 i\gamma_\rho\partial_\sigma\nu_1 - \eta_{\rho\sigma}\bar{\nu}_1(i\gamma^\lambda\partial_\lambda - m_1)\nu_1 + \bar{\nu}_2 i\gamma_\rho\partial_\sigma\nu_2 - \eta_{\rho\sigma}\bar{\nu}_2(i\gamma^\lambda\partial_\lambda - m_2)\nu_2, \qquad (12) \end{aligned}$$

with the Minkowskian metric tensor $\eta_{\rho\sigma} = \mathrm{diag}(+1, -1, -1, -1)$. We see that, when expressed in terms of ν_i, $i = 1, 2$, the Hamiltonian and the momentum operators are the sum of the respective contributions from these two field operators, $H = H_1 + H_2$, and $P^i = P_1^i + P_2^i$, respectively.

3. The canonical algebraic structure and the mixed field operators

The analysis of the canonical structure of the theory shows [17] that the total charge Q associated to the global $U(1)$ symmetry can be computed in terms of two conserved Noether charges for ν_i, $Q = Q_1 + Q_2$:

$$Q_i = \int d^3\mathbf{x}\, \nu_i^\dagger(x)\, \nu_i(x)\,, \qquad i = 1, 2. \qquad (13)$$

Moreover, from the Lagrangian Eq.(1) the (non conserved) time-dependent flavor charges can be derived [17]:

$$Q_\sigma(t) = \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x)\,, \quad \sigma = e, \mu. \qquad (14)$$

It is $Q_e(t) + Q_\mu(t) = Q$. These flavor charges are relevant physical quantities describing the neutrino oscillation phenomenon, as we will see in the following. $Q_\sigma(t)$ are given, in terms of the charges Q_i and field operators ν_i, by

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \frac{1}{2}\sin 2\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right], \qquad (15)$$

$$Q_\mu(t) = \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \frac{1}{2}\sin 2\theta \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x) + \nu_2^\dagger(x)\nu_1(x)\right], \qquad (16)$$

where, in the terms proportional to the mixing coefficient $m_{e\mu}$ on the r.h.s. (recall that $m_1 \neq m_2$), the first of Eqs. (8) has been used.

The (broken) $SU(2)$ symmetry, to which the $Q_\sigma(t)$, $\sigma = e, \mu$, charges are associated, can be easily recognized by denoting $2S_3 \equiv Q_1 - Q_2$, $S_+ \equiv \int d^3\mathbf{x} \left[\nu_1^\dagger(x)\nu_2(x)\right]$, $S_- = S_+^\dagger \equiv \int d^3\mathbf{x} \left[\nu_2^\dagger(x)\nu_1(x)\right]$. Then, use of the CAR (5) shows that the $SU(2)$ algebra is satisfied: $[S_+, S_-] =$

$2S_3$, $[S_3, S_\pm] = \pm S_\pm$. The Casimir operator, which commutes with the $SU(2)$ generators, $S_\pm$ and S_3, is given by $S_0 \equiv (1/2)(Q_1 + Q_2) = (1/2)Q$.

The relevance of the $SU(2)$ algebraic structure is recognized by noticing that $S_+ - S_-$ is the generator of the mixing transformations Eqs. (6) - (7). Indeed, at finite volume, the operator

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right], \tag{17}$$

with $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$, generates the mixing transformations (6) - (7) , which thus can be written as

$$\nu_\sigma(x) = G_\theta^{-1}(t)\nu_j(x)G_\theta(t)\,, \qquad (\sigma, j) = (e,1), (\mu,2), \tag{18}$$

as it can be shown by use of the CAR (5). The annihilation and creation operators for the mixed $\nu_\sigma(x)$, $\sigma = e, \mu$, field operators are thus derived:

$$\alpha^r_{\mathbf{k},\sigma}(t) \;\;\equiv\;\; G^{-1}_\theta(t)\; \alpha^r_{\mathbf{k},j}\; G_\theta(t), \qquad (\sigma, j) = (e,1), (\mu,2), \tag{19}$$

and similarly for $\beta^{r\dagger}_{-\mathbf{k},\sigma}$ and their h.c. In the reference frame where $\mathbf{k} = (0,0,|\mathbf{k}|)$, the electron neutrino annihilation operator obtained from (19) is given by:

$$\alpha^r_{\mathbf{k},e}(t) \;=\; \cos\theta\;\alpha^r_{\mathbf{k},1} \;+\; \sin\theta\;\left(U_\mathbf{k}(t)\;\alpha^r_{\mathbf{k},2} \;+\; \epsilon^r\; V_\mathbf{k}(t)\;\beta^{r\dagger}_{-\mathbf{k},2}\right), \tag{20}$$

and similar expressions are derived for the other (electron and muon) ladder operators [8]. Here $\epsilon^r = (-1)^r$, and the notation is

$$U_\mathbf{k}(t) \;\equiv\; u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t) = v^{r\dagger}_{-\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t), \tag{21}$$

$$V_\mathbf{k}(t) \;\equiv\; \epsilon^r\; u^{r\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t) = -\epsilon^r\; u^{r\dagger}_{\mathbf{k},2}(t)v^r_{-\mathbf{k},1}(t), \tag{22}$$

where $U_k(t) = |U_k|\; e^{i(\omega_{k,2}-\omega_{k,1})t}$, $V_k(t) = |V_k|\; e^{i(\omega_{k,2}+\omega_{k,1})t}$, with $|U_k|^2 \;+\; |V_k|^2 \;=\; 1$.

We remark that both the linearly independent oscillating frequency terms appear in (20), i.e. the $(\omega_{k,2} - \omega_{k,1})$ and $(\omega_{k,2} + \omega_{k,1})$ terms, consistently with mathematical completeness.

The $\nu_\sigma(x)$, $\sigma = e, \mu$, mixed (flavor) field operators, expanded in the same $\{u^r_{\mathbf{k},i}(t), v^r_{-\mathbf{k},i}(t)\}$ bases as the fields ν_i, are thus:

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u^r_{\mathbf{k},j}(t)\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},j}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right] e^{i\mathbf{k}\cdot\mathbf{x}},\; (\sigma, j) = (e,1), (\mu,2), \tag{23}$$

In the following we show that expansions in different bases are also possible.

We notice that Eq. (20) describes a rotation transformation nested with a Bogoliubov transformation; e.g. in (20) the 'Bogoliubov transformation' is the one in the round brackets $(U_\mathbf{k}(t)\;\alpha^r_{\mathbf{k},2} \;+\; \epsilon^r\; V_\mathbf{k}(t)\;\beta^{r\dagger}_{-\mathbf{k},2})$ (see [8, 29] for details). We also observe that $|U_\mathbf{k}|^2 \longrightarrow 1$ and $|V_\mathbf{k}|^2 \longrightarrow 0$ for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ (the relativistic limit). In such a limit, Eq. (20) reduces to the "rotation" of the $\alpha^r_{\mathbf{k},1}$ operator (and similarly for the other ladder operators). One then obtains the usual "Pontecorvo formalism" (and oscillation formula).

4. The vacuum for the flavor fields operators

Clearly, the operator $\alpha^r_{\mathbf{k},e}(t)$ in (20) does not annihilate the vacuum $|0\rangle_{1,2}$, due to the presence of the creation operator $\beta^{r\dagger}_{-\mathbf{k},2}$, in the $V_{\mathbf{k}}(t)$-term. Such a term appears there because the mixing transformations are canonical transformations, i.e. they preseve the CAR for the ν_σ, $\sigma = e, \mu$, operator fields and for their annihilation and creation operators (the equal time CAR for ν_σ operators are immediately obtained by use of (6), (7) and (4)). As already observed, canonicity cannot be violated since it expresses the physical meaningfulness of the theory.

For example, by using Eq. (20) (and its hermitian conjugate), the equal time CAR for $\alpha^r_{\mathbf{k},e}(t)$ and $\alpha^{r\dagger}_{\mathbf{k},e}(t)$ is $\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t)\} = \cos^2\theta + |U_{\mathbf{k}}|^2 \sin^2\theta + |V_{\mathbf{k}}|^2 \sin^2\theta = 1$, in agreement with the canonical structure of the mixing transformations.

In the case of the V-term dropped out, the result is $\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t)\} = \cos^2\theta + |U_{\mathbf{k}}|^2 \sin^2\theta = 1 - |V_{\mathbf{k}}|^2 \sin^2\theta \neq 1$, in violation of the mixing transformations (6) and (7) one starts with. In order to recover the correct canonical result, one *must add* the quantity $+|V_{\mathbf{k}}|^2 \sin^2\theta$, which comes in fact from the omitted β-anticommutator term (i.e. $+|V_{\mathbf{k}}|^2 \sin^2\theta$).

As a further example, compute ${}_{1,2}\langle\alpha^r_{\mathbf{k},e}(t)|\alpha^r_{\mathbf{k},e}(t)\rangle_{1,2}$, which at initial time $t = 0$ must be of course 1. Use (20), the implied CAR $\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t)\} = 1$ and put $\alpha^{r\dagger}_{\mathbf{k},e}(t)|0\rangle_{1,2} = |\alpha^r_{\mathbf{k},e}(t)\rangle_{1,2}$. Since from (20) we have ${}_{1,2}\langle 0|\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},e}(t)|0\rangle_{1,2} = |V_{\mathbf{k}}|^2 \sin^2\theta$, then, ${}_{1,2}\langle 0|\alpha^r_{\mathbf{k},e}(t)\alpha^{r\dagger}_{\mathbf{k},e}(t)|0\rangle_{1,2} = -{}_{1,2}\langle 0|\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},e}(t)|0\rangle_{1,2} + {}_{1,2}\langle 0|\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t)\}|0\rangle_{1,2} = -|V_{\mathbf{k}}|^2 \sin^2\theta + 1$, which is not acceptable since, as said, the computed probability amplitude must be 1 at the initial time $t = 0$.

In this case, the unwanted result ($\neq 1$ at time $t = 0$) is due to the fact that the vacuum $|0\rangle_{1,2}$ has been used and the V-terms with the $\beta\beta^\dagger$ operator on the vacuum $|0\rangle_{1,2}$ has contributed with the quantity $|V_{\mathbf{k}}|^2 \sin^2\theta$. In the following we will see that such a quantity has a precise physical meaning.

In conclusion, in order to get the correct result, either one needs to violate the CAR, by putting $\{\alpha^r_{\mathbf{k},e}(t), \alpha^{r\dagger}_{\mathbf{k},e}(t)\} = +|V_{\mathbf{k}}|^2 \sin^2\theta + 1 \neq 1$, or the vacuum $|0\rangle_{1,2}$ must not be used, changing it to a 'different' vacuum. This last option, however, would not be allowed in QM since, as mentioned, the Stone-von Neumann theorem states that in QM all the representations of the CAR are unitarily equivalent (so there are no 'different' vacua in QM). The option is instead possible in QFT where infinitely many unitarily inequivalent representations do exist.

As a matter of fact, Eq. (19) shows that the state (the vacuum state) annihilated by $\alpha^r_{\mathbf{k},\sigma}(t)$, $\sigma = e, \mu$, is $G^{-1}_\theta(t)\; 0\rangle_{1,2}$, which, at finite volume V, is denoted by

$$|0(\theta, t)\rangle_{e,\mu} = G^{-1}_\theta(t)\; |0\rangle_{1,2}\; . \tag{24}$$

Thus, $|0(\theta, t)\rangle_{e,\mu}$ is the vacuum for the Hilbert space $\mathcal{H}_{e,\mu}$ at time t, which we will refer to as the flavor vacuum and the flavor Hilbert space, respectively. In the infinite volume limit, at each time t, Eq. (24) gives that $|0(t)\rangle_{e,\mu}$ is orthogonal to the vacuum $|0\rangle_{1,2}$ [8, 19, 20]:

$$\lim_{V\to\infty}\; {}_{1,2}\langle 0|0(\theta, t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \ln\left(1-\sin^2\theta\, |V_{\mathbf{k}}|^2\right)^2} = 0 \tag{25}$$

and

$$\lim_{V\to\infty}\; {}_{e,\mu}\langle 0(\theta, t')|0(\theta, t)\rangle_{e,\mu} = 0, \qquad t \neq t'. \tag{26}$$

This shows that $\mathcal{H}_{e,\mu}$ and $\mathcal{H}_{1,2}$, are unitarily inequivalent representations of the CAR at each time t and that at each $t \neq t'$ the corresponding flavor Hilbert spaces are also unitarily inequivalent. We will come back to this point in the following.

The vacuum $|0(\theta,t)\rangle_{e,\mu}$ is normalized to 1, ${}_{e,\mu}\langle 0(\theta,t)|0(\theta,t)\rangle_{e,\mu} = 1$, and turns out to be a generalized $SU(2)$ coherent state [8]. Explicitly, it is given by

$$|0(\theta,t)\rangle_{e,\mu} = \prod_{\mathbf{k},r}\Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\cos\theta\, V_{\mathbf{k}}(t)\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right) \tag{27}$$

$$+\,\epsilon^r \sin^2\theta\, V_{\mathbf{k}}(t)\left(U^*_{\mathbf{k}}(t)\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - U_{\mathbf{k}}(t)\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right) + \sin^2\theta\, V^2_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2},$$

which shows its condensate structure in terms of zero momentum pairs of operators $\alpha^r_{\mathbf{k},i}$ and $\beta^r_{-\mathbf{k},i}$, $i = 1,2$, for any $\mathbf{k}$. Note that such a condensate structure is generated by the Bogoliubov part of the G^{-1}_θ transformation (cf. e.g. Eq. (20)) characterized by the coefficients $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$, for any $\mathbf{k}$. Eq. (25) shows the nonperturbative character of the flavor vacuum, i.e. that the expansion (27) of $|0(\theta,t)\rangle_{e,\mu}$ in terms of states of $\mathcal{H}_{1,2}$ is meaningless in the infinite volume limit; thus, Eq. (27) has to be understood as a formal relation, holding at finite volume. The infinite volume limit has to be taken after the computations have been done.

Use of $|0(\theta,t)\rangle_{e,\mu}$ in the amplitude discussed above leads to correct result. Since now $\alpha^r_{\mathbf{k},e}(t)|0(\theta,t)\rangle_{e,\mu} = 0$, we have ${}_{e,\mu}\langle 0(\theta,t)|\alpha^r_{\mathbf{k},e}(t)\alpha^{r\dagger}_{\mathbf{k},e}(t)|0(\theta,t)\rangle_{e,\mu} = 1$.

Note that the quantity $|V_{\mathbf{k}}|^2\sin^2\theta$ which in our previous discussion was giving us problems, is nothing but the massive neutrino condensate content of the flavor vacuum. In fact we have:

$${}_{e,\mu}\langle 0(\theta,t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}|0(\theta,t)\rangle_{e,\mu} = |V_{\mathbf{k}}|^2\sin^2\theta, \quad i = 1,2, \tag{28}$$

and similar expressions for the $\beta^r_{\mathbf{k},i}$ number operators.

In Appendix A the computation of the propagator functions for the $\nu_e \to \nu_e$ transition amplitude $\mathcal{P}_{ee}(\mathbf{k},t)$ (and for other processes $\nu_\sigma \to \nu_\tau$, with $\sigma,\tau = e,\mu$) is briefly summarized. There, the flavor charge conservation in the production vertex is also commented and it is shown that the use of the flavor vacuum Eq. (27) avoids wrong results obtained by using the vacuum $|0\rangle_{1,2}$.

The expectation values of the flavor charges on the flavor state give the flavor oscillation formulas. We have ${}_{e,\mu}\langle 0(\theta,0)| :: Q_\sigma(t) :: |0(\theta,0)\rangle_{e,\mu} = 0$. The symbol $:: \ldots ::$ denotes normal ordering with respect to the flavor vacuum $|0(\theta,0)\rangle_{e,\mu}$, namely $:: A :: \equiv A - {}_{e,\mu}\langle 0(\theta,0)|A|0(\theta,0)\rangle_{e,\mu}$ for a generic operator A. We obtain [14]:

$$\begin{aligned}
\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t) &= \langle \nu^r_{\mathbf{k},e}| :: Q_e(t) :: |\nu^r_{\mathbf{k},e}\rangle \\
&= 1 - \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) + |V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right], \qquad (29)\\
\mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t) &= \langle \nu^r_{\mathbf{k},e}| :: Q_\mu(t) :: |\nu^r_{\mathbf{k},e}\rangle = 1 - \mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t). \qquad (30)
\end{aligned}$$

Here, the state $|\nu^r_{\mathbf{k},e}\rangle$ denotes the flavored neutrino state at $t = 0$

$$|\nu^r_{\mathbf{k},e}\rangle = \alpha^{r\dagger}_{\mathbf{k},e}(0)|0(\theta,0)\rangle_{e,\mu}, \quad \text{at } t = 0. \tag{31}$$

The Pontecorvo oscillation formulas are recovered in the relativistic limit: $|\mathbf{k}| \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

The flavor vacuum $|0(\theta,t)\rangle_{e,\mu}$ is an entangled state of the neutrino pairs $(\alpha^r_{\mathbf{k},i}, \beta^r_{-\mathbf{k},j})$ condensed in it. The linear correlation coefficient $J(N_a, N_b)$ [36] provides a measure of such an entanglement:

$$J(N_a, N_b) = \frac{cov(N_a, N_b)}{\sqrt{\langle(\Delta N_a)^2\rangle \, \langle(\Delta N_b)^2\rangle}}, \tag{32}$$

where a and b denote α and/or β quanta. N_a, N_b are number operators; the symbol $\langle \ldots \rangle$ denotes expectation value in $|0\rangle_{e,\mu}$; $\langle(\Delta N)^2\rangle \equiv \langle(N - \langle N\rangle)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$ is the variance and $cov\,(N_a, N_b) \equiv \langle N_a N_b\rangle - \langle N_a\rangle \langle N_b\rangle$ is the covariance.

The definition (32) implies that values of θ and $|V_\mathbf{k}|^2$ such that $\langle(\Delta N_a)^2\rangle$ and/or $\langle(\Delta N_b)^2\rangle$ are zero, are excluded, for any $\mathbf{k}$, from the existence domain of $J(N_a, N_b)$. Since $\langle(\Delta N_{\alpha^r_{\mathbf{k},i}})^2\rangle = \sin^2\theta\,|V_\mathbf{k}|^2(1 - \sin^2\theta\,|V_\mathbf{k}|^2)$ (and a similar expression is found for $\langle(\Delta N_{\beta^r_{\mathbf{k},i}})^2\rangle$), $i = 1, 2$, one must exclude the values $\theta = 0$ and $|V_\mathbf{k}|^2 = 0$ ($|U_\mathbf{k}|^2 = 1$), solutions of $\sin^2\theta\,|V_\mathbf{k}|^2 = 0$. Solutions of $1 - \sin^2\theta\,|V_\mathbf{k}|^2 = 0$ have to be also excluded, namely $\theta = \pi/2$ and $|V_\mathbf{k}|^2 = 1$ ($|U_\mathbf{k}|^2 = 0$). Note that otherwise $\sin^2\theta = 1/|V_\mathbf{k}|^2$ is never satisfied since for $|V_\mathbf{k}|^2 \neq 1$ it is $|V_\mathbf{k}|^2 < 1$.

Note that $J = 0$ for non-correlated modes, since in such a case $\langle N_a N_b\rangle = \langle N_a\rangle \langle N_b\rangle$, and $cov\,(N_a, N_b)$ is zero. Also, $J = 1$ in the case $a = b$, since then $cov(N_a, N_a) = \langle(\Delta N_a)^2\rangle$.

For any $\mathbf{k}$, and $i, j = 1, 2$, within its existence domain, we find [34]

$$J\left(N_{\alpha^r_{\mathbf{k},i}}, N_{\beta^r_{-\mathbf{k},j}}\right) = \frac{1}{1 + \tan^2\theta\,|U_\mathbf{k}|^2}, \qquad i \neq j, \tag{33}$$

and for the pairs $(\alpha^r_{\mathbf{k},i}, \beta^r_{-\mathbf{k},i})$, $i = 1, 2$,

$$J\left(N_{\alpha^r_{\mathbf{k},i}}, N_{\beta^r_{-\mathbf{k},i}}\right) = \frac{|U_\mathbf{k}|^2\,\tan^2\theta}{1 + \tan^2\theta\,|U_\mathbf{k}|^2}. \tag{34}$$

We obtain $J = 0$ for the pairs $(\alpha^r_{\mathbf{k},i}, \alpha^r_{\mathbf{k},j})$ since then the covariance is zero because $\langle N_{\alpha^r_{\mathbf{k},i}} N_{\alpha^r_{\mathbf{k},j}}\rangle = \sin^4\theta\,|V_\mathbf{k}|^4 = \langle N_{\alpha^r_{\mathbf{k},i}}\rangle \langle N_{\alpha^r_{\mathbf{k},j}}\rangle$, and similarly for $(\beta^r_{\mathbf{k},i}, \beta^r_{\mathbf{k},j})$ pairs, for $i \neq j; i, j = 1, 2$.

In Appendix B, we shortly review the static and dynamic entanglement for single particle states associated to the variances of the flavor charges (see [37–44]).

5. Non-abelian gauge theory and neutrino mixing

The Euler–Lagrange equations derived from the Lagrangian (1) are

$$i\partial_0 \nu_e = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_e)\nu_e + \beta m_{e\mu}\nu_\mu \tag{35}$$

$$i\partial_0 \nu_\mu = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_\mu)\nu_\mu + \beta m_{e\mu}\nu_e. \tag{36}$$

We choose the following representation for the α_i, $i = 1, 2, 3$ and β Dirac matrices:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{37}$$

with σ_i denoting the Pauli matrices. $\mathbb{I}$ is the 2×2 identity matrix and we will also use $\gamma^0 = \gamma_0 = \beta$; $(\gamma^0)^2 = 1$; $\gamma^i = \gamma^0\alpha^i$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{I}$. $\nu_f = (\nu_e, \nu_\mu)^T$ denotes the flavor doublet and M_d

the diagonal mass matrix $M_d = \mathrm{diag}(m_e, m_\mu)$. Eqs. (35) and (36) are written in the compact form [28]:

$$i\gamma^0 D_0 \nu_f = (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + M_d)\nu_f, \tag{38}$$

where the (non-abelian) covariant derivative D_0 has been defined as:

$$D_0 = \partial_0 + i\,\gamma_0 m_{e\mu}\,\sigma_1, \tag{39}$$

with $m_{e\mu} = (1/2)\,\tan 2\theta\,\delta m = (1/2)\,g\,\delta m$, putting $g = \tan 2\theta$, $\delta m = m_\mu - m_e$ (cf. Eq. (8)). The covariant derivative is then

$$D_\mu = \partial_\mu + i\,g\,A_\mu, \tag{40}$$

with $D_i = \partial_i$; $i = 1, 2, 3$, and $D_0 = \partial_0 + i\,gA_0 = \partial_0 + i\,g\gamma_0\delta m\sigma_1/2$. Note that $g = \tan 2\theta$ plays the role of the "charge", i.e. the coupling constant of the neutrino field operators with the gauge field.

Eqs. (38) and (39) formally express the flavor mixing contribution in terms of the coupling of the flavor field operators with the $SU(2)$ gauge field:

$$A_\mu \quad = \quad A^a_\mu \tau_a\,, \tag{41}$$

with $\tau_a = \sigma_a/2$, $A^a_\mu = (A^1_0 = \gamma_0\delta m, 0, 0, 0)$, whose only non-zero component, $A^1_0 = \gamma_0\delta m$, is the temporal component in space-time and the first component in the $SU(2)$ space.

Consider the transformation $U(t)\nu_f = e^{ig\lambda_1(t)\,\tau_1}\nu_f = \nu'_f$, with time dependent parameter $\lambda_1(t)$, we have

$$U(t)D_\mu\nu_f = D'_\mu U(t)\nu_f = D'_\mu\nu'_f, \tag{42}$$

with $D'_i = U(t)D_iU^{-1}(t) = D_i$ and $D'_0 = U(t)D_0U^{-1}(t) = \partial_t + i\,g\,\,(A^1_0 + \partial_t\lambda_1(t))\tau_1$ as required for a covariant derivative [45]. In conclusion, the Lagrangian density (1) may be written in the form

$$\mathcal{L} = \bar{\nu}_f(i\gamma^\mu D_\mu - M_d)\nu_f, \tag{43}$$

describing a doublet of Dirac field operators in interaction with an external Yang-Mills field. The field equations have the manifestly covariant form:

$$(i\gamma^\mu D_\mu - M_d)\nu_f = 0. \tag{44}$$

The Lagrangian (43) is not invariant under the $SU(2)$ transformation

$$\nu'_f = e^{i\theta_1\tau_1}\nu_f. \tag{45}$$

The current

$$j^\rho_{f,1} = \bar{\nu}_f\gamma^\rho\tau_1\nu_f = \frac{1}{2}(\bar{\nu}_e\gamma^\rho\nu_\mu + \bar{\nu}_\mu\gamma^\rho\nu_e), \tag{46}$$

associated to the $\mathcal{L}$ variation $\delta\mathcal{L} = -\theta_1\,\partial_\mu j^\mu_{f,1}$ [17], has only one component in group space.

Since A^1_0 is constant and is the only non-zero component of A^a_μ, the field strength $F^a_{\mu\nu}$ vanishes identically for any μ, ν and a:

$$F^a_{\mu\nu} = \epsilon^{abc}A^b_\mu A^c_\nu = 0, \tag{47}$$

with $a, b, c = 1, 2, 3$. Remarkably, although $F^a_{\mu\nu}$ is identically zero, the gauge field has physical effect.

The energy momentum tensor associated with the flavor neutrino fields in interaction with the external gauge field can be obtained [45] as

$$\widetilde{T}_{\rho\sigma} = \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma}\bar{\nu}_f(i\gamma^\lambda D_\lambda - M_d)\nu_f. \tag{48}$$

Comparison with the canonical energy momentum tensor (12) shows that they differ just for the mixing terms in the 00 component, i.e. $T_{00} - \widetilde{T}_{00} = m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e)$. We also have $T_{0i} = \widetilde{T}_{0i}$, $T_{ij} = \widetilde{T}_{ij}$, and

$$\partial^\rho \widetilde{T}_{\rho i} = 0, \qquad \partial^\rho \widetilde{T}_{\rho 0} \neq 0. \tag{49}$$

Non-conservation is due to $[\gamma_\mu, D_0] \neq 0$, which in turn follows from the presence of the γ_0 matrix in D_0. The conserved 3−momentum operator is:

$$\begin{aligned}
\widetilde{P}^i &= \int d^3\mathbf{x}\, \widetilde{T}^{0i} = i\int d^3\mathbf{x}\, \nu_f^\dagger \partial^i \nu_f \\
&= i\int d^3\mathbf{x}\, \nu_e^\dagger \partial^i \nu_e + i\int d^3\mathbf{x}\, \nu_\mu^\dagger \partial^i \nu_\mu \\
&\equiv \widetilde{P}^i_e(t) + \widetilde{P}^i_\mu(t), \qquad\qquad i = 1, 2, 3
\end{aligned} \tag{50}$$

and the Hamiltonian operator:

$$\begin{aligned}
\widetilde{P}^0(t) \equiv \widetilde{H}(t) &= \int d^3\mathbf{x}\, \widetilde{T}^{00} = \int d^3\mathbf{x}\, \bar{\nu}_f \left(i\gamma_0 D_0 - i\gamma^\mu D_\mu + M_d\right)\nu_f \\
&= \int d^3\mathbf{x}\, \nu_e^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \gamma_0 m_e\right)\nu_e + \int d^3\mathbf{x}\, \nu_\mu^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \gamma_0 m_\mu\right)\nu_\mu \\
&\equiv \widetilde{H}_e(t) + \widetilde{H}_\mu(t),
\end{aligned} \tag{51}$$

which is not conserved and does not generate time translations. These are generated by $H = \int d^3\mathbf{x}\, T^{00}$, i.e. the Hamiltonian including the interaction term. Eqs. (50) and (51) provide the momentum and the Hamiltonian operators, respectively, for each flavor field. They split in two contributions, for the electron and the muon flavor separately.

6. On the Poincaré structure and flavor states

We have seen in Eq. (23) that flavor field operators ν_σ, $\sigma = e, \mu$, may be expanded in the same basis as the ν_i, $i = 1, 2$, field operators [8]. The flavor states, $|\nu^r_{\mathbf{k},\sigma}\rangle = \alpha^{r\dagger}_{\mathbf{k},\sigma}|0(\theta,0)\rangle_{e,\mu}$, are eigenstates of the momentum operators $P^i = \int d^3\mathbf{x}\, T^{0i}$ and of the flavor charge operators Q_σ, at a given time. These states do not have, however, definite energy since the Q_σ do not commute with the Hamiltonian operator H [17, 24].

Time-dependence of the flavor vacuum breaks the Lorentz invariance, so that flavor states do not belong to irreducible representations of the Poincaré group[30, 46]. The problem has attracted much attention and has been also studied in the frame of nonlinear realizations of the Poincaré group [47–50] and of its spontaneous symmetry breaking in the flavor vacuum [46].

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In such a connection, it is interesting to show that, within the frame of the mixing as a gauge theory, it possible to expand the ν_σ field operators in a different spinors basis. Consider the Bogoliubov transformation [10]

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} = J^{-1}_\mu(t) \begin{pmatrix} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} J_\mu(t), \tag{52}$$

with generator

$$J_\mu(t) = \prod_{\mathbf{k},r} \exp\left\{ i \sum_{(\sigma,j)} \xi^{\mathbf{k}}_{\sigma,j} \left[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) + \beta^r_{-\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) \right] \right\}, \tag{53}$$

where $(\sigma, j) = (e, 1), (\mu, 2)$, and $\xi^{\mathbf{k}}_{\sigma,j} = (\chi_\sigma - \chi_j)/2$, $\chi_\sigma = \arctan(\mu_\sigma/|\mathbf{k}|)$, $\chi_j = \arctan(m_j/|\mathbf{k}|)$. From (52) we get:

$$\begin{pmatrix} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix} = \begin{pmatrix} \rho^{\mathbf{k}}_{\sigma,j} & i\lambda^{\mathbf{k}}_{\sigma,j} \\ i\lambda^{\mathbf{k}}_{\sigma,j} & \rho^{\mathbf{k}}_{\sigma,j} \end{pmatrix} \begin{pmatrix} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{pmatrix}, \qquad (\sigma, j) = (e, 1), (\mu, 2), \tag{54}$$

with $\rho^{\mathbf{k}}_{\sigma,j} = \cos\xi^{\mathbf{k}}_{\sigma,j}$ and $\lambda^{\mathbf{k}}_{\sigma,j} = \sin\xi^{\mathbf{k}}_{\sigma,j}$. Flavor vacua, associated to the masses (μ_e, μ_μ) in the χ_σ parameter, are then obtained:

$$|\widetilde{0}(\theta, t)\rangle_{e\mu} = J^{-1}_\mu(\theta, t)|0(t)\rangle_{e\mu}. \tag{55}$$

The original masses (m_1, m_2) are associated to $|0(\theta, t)\rangle_{e\mu}$.

Flavor field operators are then expanded as follows:

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u^r_{\mathbf{k},\sigma}(t)\widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},\sigma}(t)\widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \sigma = e, \mu. \tag{56}$$

The tilde operators correspond to the masses (m_e, m_μ) and $u^r_{\mathbf{k},\sigma}(t) = u^r_{\mathbf{k},\sigma}e^{-i\omega_{\mathbf{k},\sigma}t}$, $v^r_{-\mathbf{k},\sigma}(t) = v^r_{-\mathbf{k},\sigma}e^{i\omega_{\mathbf{k},\sigma}t}$ are solutions of the equations:

$$(-\alpha \cdot \mathbf{k} + m_\sigma\beta)u^r_{\mathbf{k},\sigma} = \omega_{\mathbf{k},\sigma}u^r_{\mathbf{k},\sigma} \tag{57}$$

$$(-\alpha \cdot \mathbf{k} + m_\sigma\beta)v^r_{-\mathbf{k},\sigma} = -\omega_{\mathbf{k},\sigma}v^r_{-\mathbf{k},\sigma}, \tag{58}$$

with $\omega_{\mathbf{k},\sigma} = \sqrt{\mathbf{k}^2 + m^2_\sigma}$. Time dependence of the creation and destruction operators is due to the interaction with the external field. They are defined on states at same time:

$$|\widetilde{\nu}^r_{\mathbf{k},\sigma}(\theta, t)\rangle = \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)|\widetilde{0}(\theta, t)\rangle_{e\mu}. \tag{59}$$

These single particle states are eigenstates of both the Hamiltonian and the momentum operator:

$$\begin{pmatrix} \widetilde{H}_\sigma(t) \\ \widetilde{\mathbf{P}}_\sigma(t) \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(\theta, t)\rangle = \begin{pmatrix} \omega_{\mathbf{k},\sigma} \\ \mathbf{k} \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(\theta, t)\rangle, \tag{60}$$

where the Hamiltonian and momentum operators Eqs.(50),(51), in terms of the tilde flavor operators, read:

$$\widetilde{\mathbf{P}}_\sigma(t) = \sum_r \int d^3\mathbf{k}\, \mathbf{k} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) + \widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t) \right), \tag{61}$$

$$\widetilde{H}_\sigma(t) = \sum_r \int d^3\mathbf{k}\, \omega_{\mathbf{k},\sigma} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\, \widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) - \widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t)\, \widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t) \right). \tag{62}$$

The Bogoliubov transformations (52) leave invariant the flavor charges (and the oscillation formulae) [15], $\widetilde{Q}_\sigma = Q_\sigma$:

$$\widetilde{Q}_\sigma(t) = \sum_r \int d^3\mathbf{k} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k}\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k}\sigma}(t) - \widetilde{\beta}^{r\dagger}_{-\mathbf{k}\sigma}(t)\widetilde{\beta}^{r}_{-\mathbf{k}\sigma}(t) \right). \tag{63}$$

Since the interaction term is absent in $\widetilde{H}$ (cf. Eq. (51)), we have $[\widetilde{Q}_\sigma(t), \widetilde{H}(t)] = 0$, and

$$[\widetilde{Q}_\sigma(t), \widetilde{H}_{\sigma'}(t)] = 0, \qquad \sigma, \sigma' = e, \mu. \tag{64}$$

The flavor states (59) are thus eigenstates also of the flavor charges:

$$\widetilde{Q}_\sigma(t)|\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle = |\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle. \tag{65}$$

A common set of eigenstates of these operators thus exists .

In addition to the generators given in Eqs.(50) and (51), the Lorentz generators are also obtained:

$$\widetilde{M}^{\lambda\rho}(t) = \int d^3\mathbf{x} \left(\widetilde{T}^{0\rho}x^\lambda - \widetilde{T}^{0\lambda}x^\rho \right) + \frac{1}{2}\int d^3\mathbf{x}\, \nu_f^\dagger \sigma^{\lambda\rho}\nu_f = \widetilde{M}^{\lambda\rho}_e(t) + \widetilde{M}^{\lambda\rho}_\mu(t), \tag{66}$$

with $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The (equal-time) commutators for the two sets of operators (for $\sigma, \sigma' = e, \mu$) are

$$[\widetilde{P}^\mu_\sigma, \widetilde{P}^\nu_{\sigma'}] = 0, \qquad [\widetilde{M}^{\mu\nu}_\sigma, \widetilde{P}^\lambda_{\sigma'}] = i\delta_{\sigma\sigma'}\left(\eta^{\mu\lambda}\widetilde{P}^\nu_\sigma - \eta^{\nu\lambda}\widetilde{P}^\mu_\sigma \right),$$

$$[\widetilde{M}^{\mu\nu}_\sigma, \widetilde{M}^{\lambda\rho}_{\sigma'}] = i\delta_{\sigma\sigma'}\left(\eta^{\mu\lambda}\widetilde{M}^{\nu\rho}_\sigma - \eta^{\nu\lambda}\widetilde{M}^{\mu\rho}_\sigma - \eta^{\mu\rho}\widetilde{M}^{\nu\lambda}_\sigma + \eta^{\nu\rho}\widetilde{M}^{\mu\lambda}_\sigma \right), \quad \sigma, \sigma' = e, \mu, \tag{67}$$

where for simplicity we have omitted the notation of time dependence. The algebra is thus the direct sum of two Poincaré algebras and Poincaré invariance holds at each t. A different Poincaré structure exists at each t.

7. Concluding remarks

In this paper we have reviewed some features of neutrino mixing and oscillations in QFT.

Mixing transformations are physically meaningful only provided that the *canonical* structure of the QFT formalism is not violated. This implies that the action of flavor neutrino field operators ν_σ, $\sigma = e, \mu$, is properly defined on a representation of the CAR which is unitarily inequivalent to the one where massive neutrino fields ν_i, $i = 1, 2$, are defined. Wrong results, incompatible with the canonical structure, are obtained by omitting to consider the contributions from the condensate structure of the flavor vacuum.

Within such a QFT framework, mixing can be seen as the result of the interaction of the neutrino fields with a suitably defined background gauge field. The coupling of neutrino fields with the gauge field appears then to be responsible of the flavor oscillations, suggesting that the flavor vacuum, with its condensation content, may behave as a refractive medium (see e.g. [51]). Such a scenario also suggests analogies with the case of photons in the vacuum acting as a medium with refractive properties due to quantum gravity fluctuations [52]. In this line, neutrino mixing and oscillations have been proposed to be seen in connection to quantum gravity effects [53–58].

In the gauge theory frame, flavor neutrino fields ν_e and ν_μ can be defined to be on-shell fields with masses m_e and m_μ, and thus viewed as fundamental fields. This allows the existence at each time t of the Poincaré structure for the flavor states. Time evolution then describes trajectories through uir of the CAR, each one endowed with a Poincaré structure (a similar phenomenon occurs in QFT in curved background [59]). In this respect, the comparison with the description of unstable particles is very enlightening [60, 61], and naturally emerges from the study of flavor-energy uncertainty relations [24, 26]. In fact, in the case of unstable particles, the inequivalence of representations at different times is an inescapable and well-understood feature [60–63]. The Hamiltonian operator $\widetilde{H}$ that commutes with the flavor charges can be defined and flavor states are simultaneous eigenstates of flavor charges.

A phenomenological consequence of this is that in the charged current processes with neutrino creation at the interaction vertex, the neutrino is actually ν_e or ν_μ (as allowed by the corresponding lepton number conservation law), not ν_1 or ν_2. Lepton number conservation is not violated in the vertex only provided that the flavor vacuum state is used (App. A and B).

Another phenomenological consequence, that might be tested experimentally, may be seen in the beta decay, say $A \longrightarrow B + e^- + \bar{\nu}_e$. A and B are two nuclei, e.g. tritium ^{3}H and ^{3}He. If flavor neutrinos manifest as fundamental fields according to the above scheme, we have the neutrino mass m_e and the end point of the beta decay is the maximal kinetic energy K that the electron can assume, $K_{max} = Q - m_e$, with $Q = E_A - E_B - m \approx m_A - m_B - m$; m is the electron mass. In the case of tritium decay, $Q = 18.6$ KeV.

Of course, for massless neutrinos, $m_\nu = 0$ and $K_{max} = Q$. For massive neutrinos $K_{max} = Q - m_j$, or $K_{max} = Q - m_e$, in the two cases, for mass eigenstate $m_\nu = m_j$, $j = 1, 2$, or flavor fundamental field $m_\nu = m_e$, respectively.

The spectrum is proportional to the phase volume factor $EpE_e p_e$, with $E = m + K$ and $p = \sqrt{E^2 - m^2}$ the electron energy and momentum, and $E_e = Q - K$:

$$\frac{dN}{dK} = CEp\, E_e \sqrt{E_e^2 \, - \, m_e^2}\, \Theta(E_e - m_e). \tag{68}$$

$\Theta(E_e - m_j)$ is the Heaviside step function. If neutrinos are the massive ones with m_j, $j = 1, 2$, one would have:

$$\frac{dN}{dK} = CEp\, E_e \sum_j |U_{ej}|^2 \sqrt{E_e^2 - m_j^2}\, \Theta(E_e - m_j), \tag{69}$$

where $U_{ej} = (\cos\theta, \sin\theta)$. The end point is at $K = Q - m_1$ and the spectrum has an inflexion at $K \simeq Q - m_2$. For these different scenarios see Fig.(1), where the spectrum for a massless neutrino is also plotted.

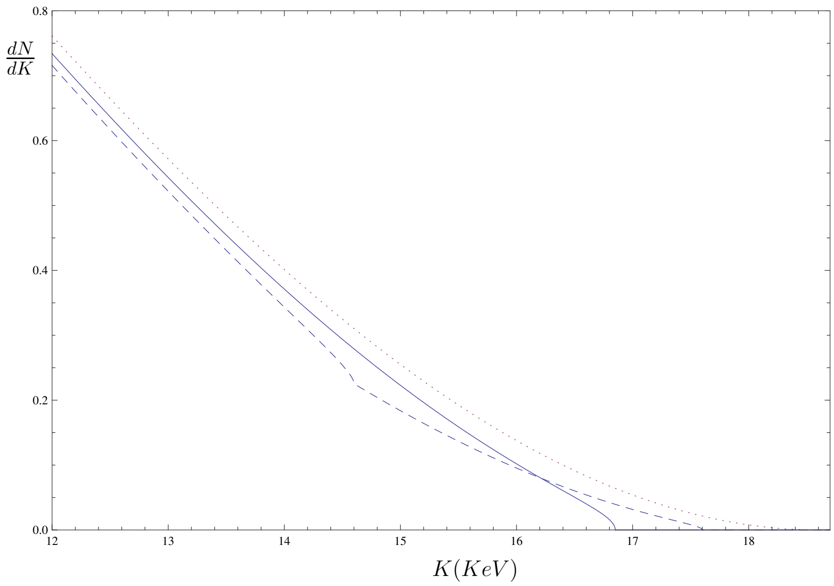

Figure 1: Tail of the tritium β spectrum: - a massless neutrino (dotted line); - fundamental flavor states (continuous line); - superposed prediction for 2 mass states (short-dashed line): the inflexion in the spectrum is where the most massive state switches off. Parameters as in Ref.[28].

Finally, we observe that the Hamiltonian operator $\widetilde{H}$, that does not include the energy "frozen" in the mixing (which cannot be turned off), is the sum of the kinetic energies available for scattering processes of the flavor neutrinos. $\widetilde{H}$ may be thus interpreted as the "free" energy $F \equiv \widetilde{H}$. Then, the entropy associated with flavor mixing can be defined:

$$H - F = TS, \tag{70}$$

with the "temperature" T identified with the coupling constant $g = \tan 2\theta$, i.e.:

$$S \;=\; \int d^3\mathbf{x}\, \bar{\nu}_f A_0 \nu_f \;=\; \frac{1}{2}\,\delta m \int d^3\mathbf{x}\, (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e). \tag{71}$$

The rôle played by the gauge field in Eq. (71) appears to be consistent with an interpretation of the gauge field as a reservoir [64]. From such a perspective, each of the two flavor neutrinos might be viewed as an open system, with cyclic dissipation occurring in the oscillation dynamics and might be studied in the frame of quantum dissipative systems [62] in the finite temperature QFT [6]. In the approximate context of QM, it has been shown that the difference of the muon and electron free energies at a given time, is less than the total initial energy of the flavor neutrino state and is proportional to the expectation value of the entropy (see details in Appendix B of [28]).

An interesting question is the one of the connection of the entropy in Eq. (71) to the time dependent entanglement entropy associated with neutrino mixing and oscillations [37–40].

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A. Two-point Green's functions for flavor fields

Consider the creation of a ν_e neutrino at $t = 0$. Suppose its flavor is observed unchanged at time $t > 0$. The amplitude of the process computed by using the vacuum state $|0\rangle_{1,2}$ is [14]:

$$\mathcal{P}^{>}_{ee}(\mathbf{k},t) \;=\; i\, u^{r\dagger}_{\mathbf{k},1}\, e^{i\omega_{\mathbf{k},1}t}\, S^{>}_{ee}(\mathbf{k},t)\,\gamma^0\, u^r_{\mathbf{k},1}\,, \tag{A.1}$$

with $S^{>}_{ee}(\mathbf{k},t)$ the Fourier transform of the Wightman function

$$S^{>}_{ee}(t,\mathbf{x};0,\mathbf{y}) = {}_{1,2}\langle 0|\nu_e(t,\mathbf{x})\,\overline{\nu}_e(0,\mathbf{y})|0\rangle_{1,2}. \tag{A.2}$$

We obtain

$$\mathcal{P}^{>}_{ee}(\mathbf{k},t) \;=\; \cos^2\theta \;+\; \sin^2\theta\,|U_{\mathbf{k}}|^2\, e^{-i(\omega_{\mathbf{k},2}-\omega_{\mathbf{k},1})t}\,. \tag{A.3}$$

This is an unacceptable result because at $t = 0$ it is not 1:

$$\mathcal{P}^{>}_{ee}(\mathbf{k},0^+) \;=\; \cos^2\theta \;+\; \sin^2\theta|U_{\mathbf{k}}|^2 \;<\; 1\,. \tag{A.4}$$

Such an unconsistency is eliminated by using the flavor vacuum $|0(\theta,t)\rangle_{e,\mu}$. Indeed, it can be shown [14] that, in such a case, one obtains:

$$\mathcal{P}^{>}_{ee}(\mathbf{k},t) \;=\; i\, u^{r\dagger}_{\mathbf{k},1}\, e^{i\omega_{\mathbf{k},1}t}\, \mathcal{G}^{>}_{ee}(\mathbf{k},t)\,\gamma^0\, u^r_{\mathbf{k},1}\,, \tag{A.5}$$

where $\mathcal{G}^{>}_{ee}(\mathbf{k},t)$ is the Fourier transform of the Wightman function computed in $|0(\theta,t)\rangle_{e,\mu}$. We have:

$$\mathcal{P}^{>}_{ee}(\mathbf{k},t) \;=\; \cos^2\theta \;+\; \sin^2\theta\left(|U_{\mathbf{k}}|^2\, e^{-i(\omega_{\mathbf{k},2}-\omega_{\mathbf{k},1})t} + |V_{\mathbf{k}}|^2\, e^{i(\omega_{\mathbf{k},1}+\omega_{\mathbf{k},2})t}\right)\,, \tag{A.6}$$

satisfying the correct condition at $t = 0$,

$$\mathcal{P}^{>}_{ee}(\mathbf{k},0^+) \;=\; 1\,. \tag{A.7}$$

Also, it is

$$|\mathcal{P}^{>}_{ee}(\mathbf{k},t)|^2 \;+\; |\mathcal{P}^{>}_{\mu\mu}(\mathbf{k},t)|^2 \;+\; |\mathcal{P}^{>}_{e,\mu}(\mathbf{k},t)|^2 \;+\; |\mathcal{P}^{>}_{\mu e}(\mathbf{k},t)|^2 \;=\; 1\,, \tag{A.8}$$

with $\mathcal{P}^{>}_{\rho\sigma}$, $\rho,\sigma = e,\mu$, defined in a similar way as $\mathcal{P}^{>}_{ee}$.

By using retarded propagators the difference between propagators on mass and flavor vacuum does not appear: $S^{ret}(\mathbf{k},t) \;=\; \mathcal{G}^{ret}(\mathbf{k},t)$, with

$$S^{ret}(t,\mathbf{x};0,\mathbf{y}) \;=\; \theta(t)\,{}_{1,2}\langle 0|\left\{\nu_\rho(t,\mathbf{x}),\overline{\nu}_\sigma(0,\mathbf{y})\right\}|0\rangle_{1,2}\,, \tag{A.9}$$

$$\mathcal{G}^{ret}(t,\mathbf{x};0,\mathbf{y}) \;=\; \theta(t)\,{}_{e,\mu}\langle 0(\theta,0)|\left\{\nu_\rho(t,\mathbf{x}),\overline{\nu}_\sigma(0,\mathbf{y})\right\}|0(\theta,0)\rangle_{e,\mu}\,, \quad \rho,\sigma = e,\mu\,. \tag{A.10}$$

However, computing the oscillation probability as

$$\mathcal{Q}_{\nu_\rho\to\nu_\sigma}(\mathbf{k},t) \;=\; \mathrm{Tr}\left[\mathcal{G}^{ret}_{\sigma\rho}(\mathbf{k},t)\mathcal{G}^{ret\dagger}_{\sigma\rho}(\mathbf{k},t)\right]\,, \tag{A.11}$$

Equations (29) and (30) are re-obtained.

B. Lepton number conservation in the vertex

At tree level, in the production and detection vertices, flavor oscillation can be neglected and lepton number has to be conserved (violation of the lepton number in loop diagrams are negligible in our following discussion). Consider for example the process $W^+ \to e^+ + \nu_e$ and use in the computation of the amplitude the Pontecorvo states:

$$\mathcal{A}^P_{W^+\to e^+\,\nu_e} \;=\; {}_P\langle \nu^r_{\mathbf{k},e}| \otimes \langle e^s_{\mathbf{q}}| \left[-i \int_{t_{in}}^{t_{out}} \mathrm{d}^4x\, \mathcal{H}^e_{int}(x)\right] |W^+_{\mathbf{p},\lambda}\rangle\,, \tag{B.1}$$

where

$$\mathcal{H}^e_{int}(x) \;=\; -\frac{g}{2\sqrt{2}}\, W^+_\mu(x)\, J^\mu_e(x) + h.c.\,, \tag{B.2}$$

is the interaction Hamiltonian density, with the current $J^\mu_e(x)$ given by

$$J^\mu_e(x) \;=\; \bar{\nu}_e(x)\,\gamma^\mu\,(1-\gamma^5)e(x)\,. \tag{B.3}$$

Due to the flavor oscillations, it is not meaningful to consider flavor fields in the asymptotic limits $t_{in/out} \to \infty$. Short-time behavior of the amplitude, around the production time $t = 0$, has to be evaluated. One then obtains [18]:

$$\begin{aligned}\mathcal{A}^P_{W^+\to e^+\,\nu_e} \;&=\; \frac{ig}{2\sqrt{4\pi}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \\ &\times\; \sum_{j=1}^{2} U^2_{ej} \int_{t_{in}}^{t_{out}} \mathrm{d}t\, e^{-i\omega_{\mathbf{k},j}\,t_{out}}\, \bar{u}^r_{\mathbf{k},j}\,\gamma^\mu(1-\gamma^5)\, v^s_{\mathbf{q},e}\, e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},j})t}.\end{aligned} \tag{B.4}$$

The energy and the polarization vector of W^+ are $E^W_{\mathbf{p}}$ and $\varepsilon_{\mathbf{p},\mu,\lambda}$, respectively. The positron wave function is $v^s_{\mathbf{q},e}$. Consider a Δt such that $\tau_W \ll \Delta t \ll t_{osc}$, with τ_W the W^+ lifetime, and t_{osc} the oscillation time. Take then $t_{in} = -\Delta t/2$ and $t_{out} = \Delta t/2$. The amplitude expansion at the leading order in Δt, gives:

$$\mathcal{A}^P_{W^+\to e^+\,\nu_e} \;\approx\; \frac{ig}{2\sqrt{4\pi}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\Delta t\, \sum_{j=1}^{2} U^2_{ej}\;\bar{u}^r_{\mathbf{k},j}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,. \tag{B.5}$$

The flavor violating amplitude

$$\mathcal{A}^P_{W^+\to e^+\,\nu_\mu} \;=\; {}_P\langle \nu^r_{\mathbf{k},\mu}| \otimes \langle e^s_{\mathbf{q}}| \left[-i \int_{t_{in}}^{t_{out}} \mathrm{d}^4x\, \mathcal{H}^e_{int}(x)\right] |W^+_{\mathbf{p},\lambda}\rangle\,. \tag{B.6}$$

is evaluated in a similar way:

$$\begin{aligned}\mathcal{A}^P_{W^+\to e^+\,\nu_\mu} \;&=\; \frac{ig}{2\sqrt{4\pi}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k}) \\ &\times\; \sum_{j=1}^{2} U_{\mu j}\,U_{ej} \int_{t_{in}}^{t_{out}} \mathrm{d}t\, e^{-i\omega_{\mathbf{k},j}\,t_{out}}\, \bar{u}^r_{\mathbf{k},j}\,\gamma^\mu(1-\gamma^5)\, v^s_{\mathbf{q},e}\, e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},j})t}\,,\end{aligned} \tag{B.7}$$

and it gives in the short time limit the non-zero unacceptable result:

$$\mathcal{A}^{P}_{W^+\to e^+\,\nu_\mu} \approx \frac{ig}{2\sqrt{4\pi}}\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\Delta t\,\sum_{j=1}^{2} U_{\mu j}\,U_{ej}\;\bar{u}^r_{\mathbf{k},j}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,. \tag{B.8}$$

Such a wrong result, violating flavor conservation at production vertex, is not obtained when using the QFT flavor states introduced in our discussion (cf. Eq. (31)). The amplitudes of the decay process $W^+ \to e^+\,\nu_e$ is found to be [18]:

$$\begin{aligned}
\mathcal{A}_{W^+\to e^+\,\nu_e} &= \frac{ig}{2\sqrt{2}(2\pi)^{\frac{3}{2}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\int_{t_{in}}^{t_{out}}\mathrm{d}t\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\\
&\times\;\Big\{\cos^2\theta\,e^{-i\omega_{\mathbf{k},1}t_{in}}\,\bar{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},1})t}\\
&+\;\sin^2\theta\Big[|U_{\mathbf{k}}|\,e^{-i\omega_{\mathbf{k},2}t_{in}}\,\bar{u}^r_{\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},2})t}\\
&+\;\epsilon^r|V_{\mathbf{k}}|\,e^{i\omega_{\mathbf{k},2}t_{in}}\,\bar{v}^r_{-\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q+\omega_{\mathbf{k},2})t}\Big]\Big\}\,,
\end{aligned} \tag{B.9}$$

where $\epsilon^r \equiv (-1)^r$. The amplitude for the 'flavor non-conserving' process $W^+ \to e^+\,\nu_\mu$ is:

$$\begin{aligned}
\mathcal{A}_{W^+\to e^+\,\nu_\mu} &= \sin\theta\,\cos\theta\,\frac{ig}{2\sqrt{2}(2\pi)^{\frac{3}{2}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\int_{t_{in}}^{t_{out}}\mathrm{d}t\,\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\\
&\times\;\Big\{e^{-i\omega_{\mathbf{k},2}t_{in}}\,\bar{u}^r_{\mathbf{k},2}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},2})t}\\
&-\;\Big[|U_{\mathbf{k}}|\,e^{-i\omega_{\mathbf{k},1}t_{in}}\,\bar{u}^r_{\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q-\omega_{\mathbf{k},1})t}\\
&+\;\epsilon^r|V_{\mathbf{k}}|\,e^{i\omega_{\mathbf{k},1}t_{in}}\,\bar{v}^r_{-\mathbf{k},1}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,e^{-i(E^W_{\mathbf{p}}-E^e_q+\omega_{\mathbf{k},1})t}\Big]\Big\}\,.
\end{aligned} \tag{B.10}$$

These expressions give for $\tau_W \ll \Delta t \ll t_{osc}$ [18]:

$$\begin{aligned}
\mathcal{A}_{W^+\to e^+\,\nu_e} &\approx \frac{ig}{2\sqrt{2}(2\pi)^{\frac{3}{2}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\frac{\varepsilon_{\mathbf{p},\mu,\lambda}}{\sqrt{2E^W_{\mathbf{p}}}}\,\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{k})\,\Delta t\\
&\times\;\Big\{\cos^2\theta\,\bar{u}^r_{\mathbf{k},2}+\sin^2\theta\Big[|U_{\mathbf{k}}|\bar{u}^r_{\mathbf{k},2}+\epsilon^r|V_{\mathbf{k}}|\,\bar{v}^r_{-\mathbf{k},2}\Big]\Big\}\,\gamma^\mu(1-\gamma^5)\,v^s_{\mathbf{q},e}\,,
\end{aligned} \tag{B.11}$$

and

$$\mathcal{A}_{W^+\to e^+\,\nu_\mu} \approx 0\,, \tag{B.12}$$

which is the result fitting the experiments.

C. Static and dynamic neutrino entanglement

The two-qubit representation of neutrinos ν_1 and ν_2 is obtained by writing, in obvious notation, [7]

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle. \tag{C.1}$$

The flavor states are then described as the entanglement of the ν_j, $j = 1, 2$, neutrinos, and this is called [7] the *static* entanglement.

The *dynamical* entanglement arises from time evolution of flavor states (flavor oscillation) [38] and the two-qubit states are

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu, \qquad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu, \tag{C.2}$$

with $|0\rangle_\sigma$ and $|1\rangle_\sigma$ denoting absence and presence of a σ-flavored neutrino, respectively, $\sigma = e, \mu$.

The variances of charges Q_{ν_j} (Eq. (13)), provide a measure of the static entanglement occurring in the states $|\nu^r_{\mathbf{k},\sigma}\rangle$ [41]:

$$\sigma^2_{Q_j} \equiv \left(\Delta Q_{\nu_j}\right)^2 = \langle Q^2_{\nu_j}\rangle_\rho - \langle Q_{\nu_j}\rangle^2_\rho = \frac{1}{4}\sin^2(2\theta)\,, \tag{C.3}$$

which is the same result obtained for quantum mechanics (QM) entanglement [41]. Similarly the variances of the flavor charges measure the *dynamic* entanglement:

$$\sigma^2_Q \equiv \left(\Delta Q_{\nu_\rho}\right)^2 = \langle Q^2_{\nu_\rho}(t)\rangle_\rho - \langle Q_{\nu_\rho}(t)\rangle^2_\rho = \mathcal{Q}_{\rho\to\rho}(t)\left(1 - \mathcal{Q}_{\rho\to\rho}(t)\right)\,. \tag{C.4}$$

In these results, through the contributions from the vacuum condensate appearing in the oscillation formulas Eqs. (29) and (30), the differences can be traced with respect to the QM (the relativistic limit) result. In this connection, we finally observe that the Leggett-Garg inequalities for temporal correlations appear to be more strongly violated in QFT than in QM [40].

It is to be remarked that the static and the dynamical entanglement have origin in the unitarily inequivalence between the representations (Hilbert spaces) of the qubit states $|\nu_i\rangle$ (Eq. (25)) and the representation to which the qubit states $|\nu_\sigma(t)\rangle$ belong, at time t, for the static case; while, for the dynamical case, where the qubits are flavor states at time $t = 0$, the unitarily inequivalence is among representations at different times (Eq. (26)).

D. The gauge field and the flavor vacuum as a refractive medium

In Sec. 7, we have observed that the description of the mixing in terms of the neutrino field coupling with the gauge field suggests that the flavor vacuum may be seen as a refractive medium.

To see this in a simple way [65], assume the propagation of two waves of same frequency f, $\omega = 2\pi f$, propagating with speed $\mathrm{v}_0 = \lambda f$ in the vacuum. Let us represent the two waves as two degenerate states, say $|1\rangle$ and $|2\rangle$ (in our oversimplified discussion, these are generic states, not necessarily neutrino states). Their time evolution is given by

$$\begin{pmatrix} |1(t)\rangle \\ |2(t)\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \begin{pmatrix} |1(0)\rangle \\ |2(0)\rangle \end{pmatrix}. \tag{D.1}$$

Suppose now that the degeneracy is broken since the propagation medium presents different refraction indexes, $n_i = \mathrm{v}_0/\mathrm{v}_i$ for $|i\rangle$, $i = 1, 2$, respectively, with v_i the respective propagation (phase) speed. The propagation over a path of length L occurs then in different times, t_1 and t_2 for $|1\rangle$ and $|2\rangle$, respectively:

$$t_i = \frac{L}{\mathrm{v}_i} = n_i \frac{L}{\mathrm{v}_0} = n_i\, t\,, \qquad i = 1, 2, \tag{D.2}$$

where $t = L/\mathrm{v}_0$. Then, the time evolution phase factors are $e^{-i\omega\, t_i} = e^{-i\omega_i t}$, $i = 1, 2$, for the two states, respectively, where $\omega\, t_i = 2\pi\, f\, n_i\, t = 2\pi\, f_i\, t = \omega_i\, t$, with $f_i \equiv f\, n_i$, i.e. $\lambda_i\, f = \mathrm{v}_i$ and $\lambda_i\, f_i = \mathrm{v}_0$.

Under the mixing transformation (cf. Eq. (6)) the states $|1(t)\rangle$ and $|2(t)\rangle$ give

$$\begin{pmatrix} |\phi(t)\rangle \\ |\psi(t)\rangle \end{pmatrix} = e^{-i\omega_1 t} \begin{pmatrix} \cos\theta & e^{-i(\omega_2-\omega_1)t}\sin\theta \\ -\sin\theta & e^{-i(\omega_2-\omega_1)t}\cos\theta \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}, \tag{D.3}$$

with $\omega_1 \neq \omega_2$. For $\theta \neq \frac{\pi}{4} + \frac{n\,\pi}{2}$, by inverting Eq. (D.3), the matrix elements of the time derivative $i\partial_t$ show that, the effect of the propagation through the refractive medium is equivalent to the one coming from the coupling to the gauge field $A_0^{(1)} = \frac{1}{2}(\omega_2 - \omega_1)\, \cos 2\theta = \frac{1}{2}\omega(n_2 - n_1)\, \cos 2\theta$, which of course is vanishing for $n_1 = n_2$.

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