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Gravitating Nielsen-Olesen vortices in an AdS_3 and Minkowski background

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We consider Nielsen-Olesen vortices under Einstein gravity in an AdS_3 background. We find numerically non-singular solutions characterized by three parameters: the cosmological constant Λ, the winding number n and the vacuum expectation value (VEV) labeled by v. The mass (ADM mass) of the vortex is expressed in two ways: one involves subtracting the value of two metrics asymptotically and the other is expressed as an integral over matter fields. The latter shows that the mass has an approximately $n^2\,v^2$ dependence and our numerical results corroborate this. We also observe that as the magnitude of the cosmological constant increases the core of the vortex becomes slightly smaller and the mass increases. We then embed the vortex under gravity in a Minkowski background and obtain numerical solutions for different values of Newton 's constant. There is a smooth transition from the non-singular origin to an asymptotic conical spacetime with angular deficit that increases as Newton's constant increases. We end by stating that the well-known logarithmic divergence in the energy of the vortex in the absence of gauge fields can be seen in a new light with gravity: it shows up in the metric as a $2+1$ Newtonian logarithmic potential leading to a divergent ADM mass.

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1. Introduction

The Nielsen-Olesen vortex [1] is a $2+1$ dimensional non-perturbative solution composed of complex scalar fields and gauge fields that have finite energy and are stable due to its topology. These topological objects have been known for a while but how they act under gravity has been studied only more recently. Black hole vortex solutions were investigated in [2] in an AdS_3 background. The scalar field profile for the vortex had a singularity at the origin $r = 0$ and approached zero asymptotically. They obtained an analytical expression for the black hole mass M in terms of the scalar charge c and the winding number n. Later, how vortices affect the tunneling decay of a symmetry-breaking false vacuum in $2+1$ dimensional Einstein gravity was studied in [3]. There have also been recent studies of gravitating magnetic monopoles [4].

In this work, we embed the vortex under Einstein gravity in an AdS_3 background. We find non-singular numerical solutions in contrast to the singular black-hole vortex discussed above. Our solutions depend on three parameters: the cosmological constant Λ, the winding number n and the vacuum expectation value (VEV) v. The mass (ADM mass) of the vortex is obtained in two ways: by subtracting the asymptotic values of two different metrics and via an integral formula over matter fields. The two results must match and they do. The integral formula shows that the mass has an approximately $n^2 v^2$ dependence and our numerical results show that the cases with $n = 2$ or $v = 2$ have masses that are significantly larger than their $n = 1$, $v = 1$ counterparts. We observe that the core of the vortex becomes more compressed and its mass increases as the magnitude of the cosmological constant increases. The vortex is then embedded under gravity in a Minkowski background. There is no singularity at the origin and the spacetime transitions smoothly to an asymptotic conical spacetime with a given angular deficit [5]. As Newton's constant increases, the angular deficit increases whereas the mass of the vortex hardly changes. We end by mentioning that the well-known logarithmic divergence in the energy of the vortex in the absence of gauge fields shows up in the metric as the $2+1$ dimensional Newtonian logarithmic potential when gravity is included.

2. Nielsen-Olesen vortex under Einstein gravity with cosmological constant

The Lagrangian density for the vortex embedded in Einstein gravity with cosmological constant is given by

$$\mathcal{L} = \sqrt{-g}\Big(\alpha\,(R-2\Lambda) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{\lambda}{4}(|\phi|^2 - v^2)^2\Big)\,. \tag{1}$$

The covariant derivatives are defined in the usual fashion by $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$ where A_μ is the gauge field and e is the coupling constant. $F_{\mu\nu}$ is the electromagnetic field tensor, Λ is the cosmological constant, R is the Ricci scalar, v is the VEV of the scalar field, λ is a coupling constant and $\alpha = 1/(16\pi G)$ where G is Newton's constant. We consider circularly symmetric static solutions. The metric takes on the form

$$ds^2 = -B(r)\,dt^2 + \frac{1}{A(r)}\,dr^2 + r^2 d\theta^2 \tag{2}$$

where $B(r)$ and $A(r)$ are functions. The ansatz for the scalar and gauge fields are

$$\phi(\mathbf{x}) = f(r)e^{in\theta} \;\text{ and }\; A_j(\mathbf{x}) = \epsilon_{jk}\hat{x}^k\,\frac{a(r)}{er} \tag{3}$$

where n is the winding number and f and a are functions of r associated with the scalar and gauge field respectively . When (2) and (3) is substituted into (1) this yields

$$\mathcal{L} = \sqrt{B/A}\, r \left(\alpha(R - 2\Lambda) - (\lambda/4)(f^2 - v^2)^2 - \frac{(f')^2 A}{2} - \frac{(n-a)^2 f^2}{2r^2} - \frac{A(a')^2}{2e^2 r^2} \right) . \tag{4}$$

The prime means derivative with respect to r. The Ricci scalar can be expressed in terms of the metric functions $A(r)$ and $B(r)$. The Lagrangian density contains four functions of r: A, B, f and a. The four equations of motion can be reduced to three by expressing B in terms of the other functions. The three equations are

$$2e^2 (n-a)^2 f^2 - 2e^2 r^2 v^2 \lambda f^2 + e^2 r^2 \lambda f^4 + e^2 r \left(r v^4 \lambda + 8 r \alpha \Lambda + 4\alpha A' \right) + 2A \left((a')^2 + e^2 r^2 (f')^2 \right) = 0 \tag{5}$$

$$-2(n-a)^2 f + 2r^2 v^2 \lambda f - 2r^2 \lambda f^3 + \frac{r f'}{4e^2 \alpha} \Big(-e^2 r^2 \left(v^4 \lambda + 8\alpha\Lambda \right) - 2e^2 (n-a)^2 f^2 + 2e^2 r^2 v^2 \lambda f^2 - e^2 r^2 \lambda f^4 + 2A \left((a')^2 + e^2 r^2 (f')^2 \right) \Big) + r \left(r A' f' + 2A \left(f' + r f'' \right) \right) = 0 \tag{6}$$

$$2e^2 r (n-a) f^2 - 2A a' + r a' A' + \frac{a'}{4e^2\alpha} \Big(-e^2 r^2 \left(v^4 \lambda + 8\alpha\Lambda \right) - 2e^2 (n-a)^2 f^2 + 2e^2 r^2 v^2 \lambda f^2 - e^2 r^2 \lambda f^4 + 2A \left((a')^2 + e^2 r^2 (f')^2 \right) \Big) + 2rAa'' = 0 . \tag{7}$$

The metric in vacuum (subscript '0') is obtained by setting $f = v$ and $a = n$ identically in Eq. (5) and yields

$$A_0(r) = -\Lambda r^2 + C \tag{8}$$

where the integration constant C sets the initial conditions at $r = 0$ (in this work we set $C = 1$). The metric function B in vacuum turns out to be equal to $A_0(r)$ i.e. $B_0(r) = A_0(r)$. Let R be the computational boundary representing formally infinity. With matter (the vortex) we have that $f \to v$ and $a \to n$ as $r \to R$. The asymptotic form of the metric function $A(r)$ in the presence of matter is then given by $A(R) = -\Lambda R^2 + D$. The constant D differs from the constant C in (8) because matter is now present.

3. Expressions for the ADM mass

The ADM mass of the vortex suitably generalized to 2 + 1 dimensions is given by [6]

$$M = -2\alpha \lim_{C_t \to R} \oint_{C_t} (k - k_0) \sqrt{\sigma} N(R) d\theta \tag{9}$$

where C_t is the circle at spatial infinity, $N(R)$ is the lapse given by $[B_0(R)]^{1/2} = [A_0(R)]^{1/2}$, the metric on C_t is σ_{AB} and k and k_0 are the extrinsic curvatures of C_t embedded on the two-dimensional spatial surface of the metric (2) and AdS_3 respectively. A straightforward calculation yields

$$M = 4\pi\alpha \left(A_0(R) - [A_0(R) A(R)]^{1/2} \right) . \tag{10}$$

Since $A_0(R) >> (D - C)$ the above formula simplifies to $M = 2\pi\alpha\,(C - D)$ (with a completely negligible error) so that the mass in an AdS$_3$ background can be effectively written as

$$M_{AdS_3} = 2\pi\alpha\,(A_0(R) - A(R))\,. \tag{11}$$

The value of $A_0(R) = -\Lambda R^2 + C$ can be calculated directly for a given R (and $C = 1$) and the quantity $A(R)$ is obtained by solving the equations of motion numerically.

The ADM mass formula (10) applies to asymptotically flat spacetime where $\Lambda = 0$. In that case we obtain $A_0(R) = C$ and $A(R) = D$ so that

$$M_{flat} = 4\pi\alpha\left(C - (C\,D)^{1/2}\right) = 4\pi\alpha\left(1 - D^{1/2}\right) \tag{12}$$

where we used $C = 1$. In asymptotically flat spacetime, $A_0(r) = B_0(r) = 1$ whereas $A(r)$ starts at unity at $r = 0$ and then decreases with r until it reaches a plateau at a positive value of D that is obtained numerically. In this asymptotic region the spacetime is conical and the angular deficit is given by $\delta = 2\pi(1 - D^{1/2})$ [5].

The second and third equations of motion (6) and (7) can be solved for $A(r)$ in terms of matter fields. This can then be substituted into the first equation (5) to obtain $A'(r)$ in term of matter only. One then obtains the following integral representation for the ADM mass (11) of the vortex in an AdS$_3$ background

$$M_{AdS_3} = n^2\,v^2\,F \tag{13}$$

where F is the integral given by

$$F = \frac{\pi}{2}\int_0^{R_1}\frac{1}{u}\Bigg[u^2\frac{\lambda}{e^2} + f_1^4 u^2\frac{\lambda}{e^2} + 2f_1^2\big((-1+a_1)^2 - u^2\frac{\lambda}{e^2}\big) + \frac{2f_1\big((a_1')^2 + (f_1')^2u^2\big)\Big((1-a_1)f_1 f_1' u^2 + a_1'\big((-1+a_1)^2 + (-1+f_1^2)u^2\frac{\lambda}{e^2}\big)\Big)}{u(2a_1' f_1' - a_1'' f_1' u + a_1' f_1'' u)}\Bigg]du\,. \tag{14}$$

Here $u = \frac{e\,v}{n}\,r$, $f_1(u) = f(u)/v$, $a_1(u) = a(u)/n$, $R_1 = \frac{e\,v}{n}R$. Derivatives are with respect to u. The formula (13) does not necessarily imply that the mass grows exactly quadratically with v and n because the matter profiles also change with v and n. However, the mass should still increase significantly if, for example, we double v or n. This is what is observed numerically.

Using the equations of motion, the ADM mass (12) for the Minkowski background can similarly be expressed as an integral over matter fields

$$M_{flat} = 4\pi\alpha\left(1 - \sqrt{1 - \frac{n^2\,v^2\,F}{2\pi\,\alpha}}\right) \tag{15}$$

where F is integral (14) evaluated using the matter field profiles in the flat case. Note that the matter field profiles are going to be different for the AdS$_3$ and Minkowski cases yielding different values for the integral F.

4. Numerical results for AdS$_3$ and Minkowski background

We solve the equations of motion for non-singular profiles of the metric $A(r)$, the gauge field $a(r)$ and the scalar field $f(r)$. The boundary conditions are: $f(0) = 0$; $a(0) = 0$; $f(R) =$

v ; $a(R) = n$; $A(0) = 1$ where R is the computational boundary representing formally infinity. We obtain the profiles by adjusting $f'(r)$ and $a'(r)$ near the origin to give the final boundary conditions at R where f and a plateau to v and n respectively. Our numerical simulation for an AdS_3 background involve six parameters: Λ, n, v, λ, α and e. We set $\lambda = 1$, $\alpha = 1$ and $e = 3$. We ran five different cases determined by the values of the three parameters (n, v, Λ). We calculate the mass of the vortex in the two ways: one using the metric and the other using an integral over matter fields. The masses are listed in Table 1 and labeled M_{metric} and $M_{integral}$ respectively. The two values match to within two or three decimal places. Cases I, II and III have the same value of $n = 1$ and $v = 1$ but differ in their value of Λ which are -1, -2 and -3 respectively. In Table 1, the mass increases as one goes from case I to case III i.e. as Λ becomes more negative. The core of the vortex gets smaller (more compressed) and the vortex gains positive energy. The mass $M = 6.53$ of case IV with $v = 2$ and the mass $M = 7.14$ of case V with $n = 2$ are significantly greater than the mass of the previous three cases. This is in accord with the $n^2\, v^2$ coefficient in the integral mass formula (13). As already pointed out, the masses are not exactly four times greater because the matter profiles f_1 and a_1 that enter (13) change with v and n.

(n,v,Λ)	A_0[10]	A[10]	M_{metric}	$M_{integral}$
I=(1,1,-1)	101	100.588	2.589	2.587
II=(1,1,-2)	201	200.534	2.928	2.931
III=(1,1,-3)	301	300.489	3.211	3.210
IV=(1,2,-2)	201	199.961	6.529	6.529
V=(2,1,-2)	201	199.863	7.142	7.142

Table 1: Table with values of the metric A_0 and A at $r = 10$, M_{metric} evaluated using A_0 and A in Eq. (11) and $M_{integral}$ evaluated using the integral mass formula (13). The two masses match (agree to two decimal places and sometimes at three decimal places).

We also obtained numerical results with gravity in a Minkowski background ($\Lambda = 0$) for different values of $\alpha = 1/(16\,\pi\, G)$. The spacetime has curvature but no singularity at the origin and it transitions smoothly to an asymptotic conical spacetime with angular deficit δ. We looked at three different cases: $\alpha = 1, \alpha = 5$ and $\alpha = 10$ (keeping the following parameters fixed: $e = 1$, $\lambda = 1$, $v = 1$, $n = 1$). The metric function $A(r)$ starts at unity at $r = 0$ and plateaus to the value D asymptotically. Table 2 contains the values of D, the angular deficit $\delta = 2\pi(1 - D^{1/2})$ converted in degrees and the masses of the vortex calculated using the metric (Eq. 12) and the matter profiles (Eq. 15). The angular deficit has a strong dependence on α: it basically increases tenfold from $\alpha = 10$ to $\alpha = 1$. In contrast, the mass of the vortex hardly changes with α.

We end with an insight that we gain when gravity is included in the vortex. In the absence of gauge fields, it is well-known in the non-gravity case that one cannot construct a stable vortex because it has a logarithmic divergence in its energy [7]. With gravity, this logarithmic divergence does not disappear but shows up in the metric [8] in the form of the $2 + 1$ Newtonian logarithmic potential $G\, m\, ln(r)$ (the mass m here is equal to $8\pi\, n^2\, v^2$). Therefore, one way to see that General Relativity cannot have a Newtonian limit in $2+1$ dimensions is that it would lead to a logarithmically divergent ADM mass.

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	D	δ	M_{metric}	$M_{integral}$
$\alpha = 1$	0.501	105.2 deg	3.672	3.674
$\alpha = 5$	0.887	20.9 deg	3.643	3.643
$\alpha = 10$	0.943	10.4 deg	3.638	3.638

Table 2: The quantity D is where the metric function $A(r)$ plateaus to asymptotically and δ is the angular deficit quoted in degrees. The mass M_{metric} is obtained using the metric and calculated via (12) and $M_{integral}$ is evaluated via (15) as an integral over the matter profiles. The two masses should match and they do.

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