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Investigating quark confinement from the viewpoint of lattice gauge-scalar models

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In this talk, first, we show that the color N-dependent area law falloffs of the double-winding Wilson loop averages for the $SU(N)$ lattice gauge model are reproduced from the Z_N lattice Abelian gauge model due to the center group dominance in quark confinement. Next, we discuss lattice gauge-scalar models which allow analytic continuation for gauge invariant operators between confinement region and Higgs region. Applying the cluster expansion, we try to understand non-trivial contribution from scalar field in quark confinement mechanism. In order to understand quark confinement further, moreover, we study double-winding Wilson loop averages in the analytical region on the phase diagram.

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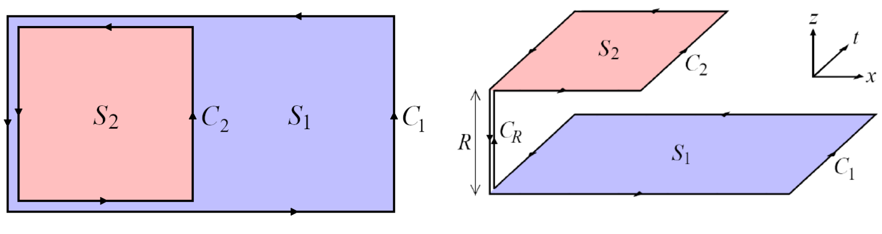
1. Introduction

In the lattice gauge theory, a double-winding Wilson loop operator $W(C_1 \cup C_2)$ has been introduced in [1] to examine the possible mechanisms for quark confinement. The double-winding Wilson loop operator is defined as a trace of the path-ordered product of gauge link variables U_ℓ along a closed loop C composed of two loops C_1 and C_2:

$$W(C_1 \cup C_2) \equiv \mathrm{tr}\left[\prod_{\ell \in C_1 \cup C_2} U_\ell\right] . \tag{1}$$

The double-winding Wilson loop is called *coplanar* if the two loops C_1 and C_2 lie in the same plane, while it is called *shifted* if the two loops C_1 and C_2 lie in planes parallel to the x-t plane, but are displaced from one another in the transverse z-direction by distance R, and are connected by lines running parallel to the z-axis to keep the gauge invariance. See Fig.1. Note that the double-winding Wilson loop operators are defined as a gauge invariant operator.



(a) **(b)**

Figure 1: (a) a "coplanar" double-winding Wilson loop, (b) a "shifted" double-winding Wilson loop.

The area dependence of the expectation value $\langle W(C_1 \cup C_2)\rangle$ has been first investigated in [1] to show that the coplanar double-winding Wilson loop average obeys the " difference-of-areas law " in the lattice $SU(2)$ Yang-Mills model by using the strong coupling expansion and the numerical simulations:

$$\langle W(C_1 \cup C_2)\rangle_{R=0} \simeq \exp[-\sigma||S_1| - |S_2||] , \tag{2}$$

where S_1 and S_2 are respectively the minimal areas bounded by loops C_1 and C_2.

In the continuum $SU(N)$ Yang-Mills model, general multiple-winding Wilson loops have been investigated in [2] to show that there is a novel " max-of-areas law " which is neither difference-of-areas law nor sum-of-areas law for multiple-winding Wilson loop average, provided that the string tension obeys the Casimir scaling for quarks in the higher representations.

In the lattice $SU(N)$ Yang-Mills model, it has been shown in [3] that the coplanar double-winding Wilson loop average has the N-dependent area law falloff in the strong coupling region: " difference-of-areas law " for $N = 2$, " max-of-areas law " for $N = 3$ and " sum-of-areas law " for $N \geqslant 4$:

$$\langle W(C_1 \cup C_2)\rangle_{R=0} \simeq \begin{cases} \exp[-\sigma||S_1| - |S_2||] & (N = 2) \\ \exp[-\sigma \max(|S_1|, |S_2|)] & (N = 3) \\ \exp[-\sigma(|S_1| + |S_2|)] & (N \geqslant 4) \end{cases} . \tag{3}$$

Moreover, a shifted double-winding Wilson loop average as a function of the distance R in a transverse direction has the long distance behavior which does not depend on N, while the short distance behavior depends on N.

In our investigation in [4], we examine the center group dominance for a double winding Wilson loop average. It has been shown in [5] that the ordinary single-winding Wilson loop average in the non-Abelian lattice gauge theory with the gauge group G is bounded from above by the same Wilson loop average in the Abelian lattice gauge theory with the center gauge group $Z(G)$:

$$|\langle W_{R(G)}(C)\rangle_G(\beta)| \leq 2\mathrm{tr}(\mathbf{1})\langle W_{R(Z(G))}(C)\rangle_{Z(G)}(2\dim(G)\beta) . \tag{4}$$

We have extended the above statement to the double winding Wilson loop average, beyond the case of the ordinary single-winding Wilson loop average:

$$|\langle W_{R(G)}(C_1 \cup C_2)\rangle_G(\beta)| \leq 2\mathrm{tr}(\mathbf{1})\langle W_{R(Z(G))}(C_1 \cup C_2)\rangle_{Z(G)}(2\dim(G)\beta) . \tag{5}$$

From this point of view, we introduce the *character expansion* to the weight $e^{S_G[U]}$ coming from the action and perform the group integration, in order to estimate the expectation value in the Z_N lattice gauge model. We evaluate the double-winding Wilson loop average up to the leading contribution to show that the N-dependent area law falloff in the $SU(N)$ lattice gauge model can be reproduced by using the (Abelian) Z_N lattice gauge model. By taking the limit $N \to \infty$, we show the center group dominance for a double-winding Wilson loop average in the $U(N)$ lattice gauge model through the $U(1)$ lattice gauge model.

Finally, we extend the above arguments for the lattice gauge-scalar model on the “ analytic region ”. For this purpose, we estimate the area law falloff, the string tension, and the mass gap by using the *cluster expansion*.

2. Lattice Z_N gauge model

First, we consider the lattice Z_N gauge model with the coupling constant defined by $\beta := 1/g^2$ on a D-dimensional lattice Λ with unit lattice spacing, which is specified by the action

$$S_G[U] = \beta \sum_{p \in \Lambda} \mathrm{Re}\, U_p , \quad U_p := \prod_{\ell \in \partial p} U_\ell , \tag{6}$$

where ℓ labels a link, p labels an elementary plaquette. To examine this Z_N gauge model analytically, we introduce the *character expansion* to the weight $e^{S_G[U]}$ to obtain the expanded form of the expectation value of an operator $\mathscr{F}$:

$$\langle \mathscr{F} \rangle_\Lambda := Z_\Lambda^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \, e^{S_G[U]} \mathscr{F} = Z_\Lambda^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \prod_{p \in \Lambda} \sum_{n=0}^{N-1} b_n(\beta) U_p^n \mathscr{F} , \tag{7}$$

$$Z_\Lambda := \int \prod_{\ell \in \Lambda} dU_\ell \, e^{S_G[U]} , \tag{8}$$

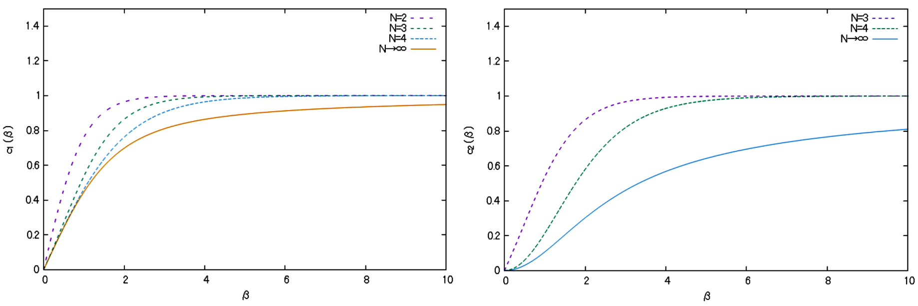
where the coefficients $b_n(\beta)$ is defined by

$$b_n(\beta) := \frac{1}{N} \sum_{\zeta \in Z_N} \zeta^{-n} e^{\beta \mathrm{Re}\, \zeta} . \tag{9}$$

We define $c_n(\beta) := b_n(\beta)/b_0(\beta)$. For $N = 2, 3, 4$ and ∞, $c_1(\beta)$ and $c_2(\beta)$ are written in the form

$$\begin{aligned} c_1(\beta) &= \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \quad (N = 2) , \qquad c_1(\beta) = \frac{e^{\beta} - e^{-\beta/2}}{e^{\beta} + 2e^{-\beta/2}} = c_2(\beta) \quad (N = 3) , \\ c_1(\beta) &= \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + 2 + e^{-\beta}}, \quad c_2(\beta) = \frac{e^{\beta} - 2 + e^{-\beta}}{e^{\beta} + 2 + e^{-\beta}} \quad (N = 4) , \\ c_1(\beta) &= \frac{I_1(\beta)}{I_0(\beta)}, \quad c_2(\beta) = \frac{I_2(\beta)}{I_0(\beta)} \quad (N = \infty) . \end{aligned} \tag{10}$$

Note that $b_{N-n}(\beta) = b_n(\beta)$ and $0 \leqslant c_n(\beta) < 1$ for $0 \leqslant \beta < \infty$. For $N = 2, 3, 4$ and ∞ , the behavior of $c_1(\beta)$ and $c_2(\beta)$ as functions of β are indicated in Fig.2. We find that $c_1(\beta) \sim O(\beta)$ $(N \geqslant 2)$ and $c_2(\beta) \sim O(\beta^2)$ $(N \geqslant 4)$ for $\beta \ll 1$.



(a) $c_1(\beta)$ $(N = 2, 3, 4, \infty)$ **(b)** $c_2(\beta)$ $(N = 3, 4, \infty)$

Figure 2: The character expansion coefficient as a function of β, (a) $c_1(\beta)$, (b) $c_2(\beta)$

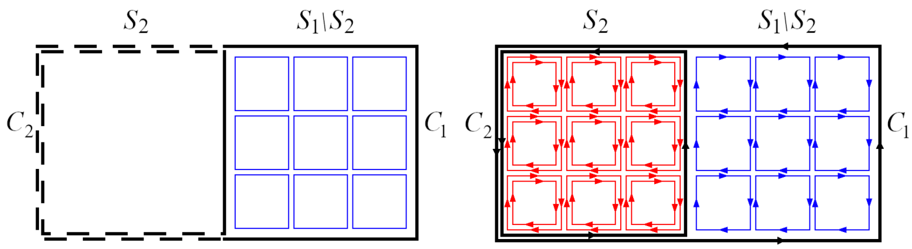
Next, we evaluate the expectation value of a coplanar double-winding Wilson loop in the lattice Z_N pure gauge model. The leading contribution to a coplanar double-winding Wilson loop average is given by the tiling of a planar set of plaquettes, as shown in the Fig.3. (These result are exact for all β when $D = 2$, while valid for $\beta \ll 1$ when $D > 2$.)

The result of the coplanar double-winding Wilson loop average up to the leading contribution is given by

$$\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \begin{cases} c_1(\beta)^{|S_1| - |S_2|} & (N = 2) \\ c_1(\beta)^{|S_1|} & (N = 3) \\ c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1| - |S_2|} & (N \geqslant 4) \end{cases} . \tag{11}$$

Then we obtain the (non-zero) string tension from this result:

$$\sigma(\beta) \simeq \ln \frac{1}{c_1(\beta)} > 0 . \tag{12}$$

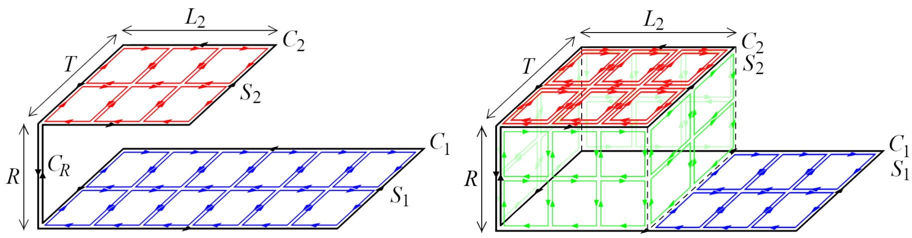


(a) $N = 2$ **(b)** $N \geqslant 3$

Figure 3: A coplanar double-winding Wilson loop, (a) $N = 2$, (b) $N \geqslant 3$

In the strong coupling region, this result reproduces the area law falloff in the $SU(N)$ lattice gauge model obtained in [3]. Moreover, by taking the continuous group limit $N \to \infty$,we find that the area law for $N \geqslant 4$ persists in the $U(1)$ lattice gauge model.

Furthermore, we also evaluate the expectation value of a shifted double-winding Wilson loop in the lattice Z_N pure gauge model. The leading contribution to a shifted double-winding Wilson loop average can be given by the 2 types of tiling by a set of plaquettes, as shown in the Fig.4.



(a) R-independent contribution **(b)** R-dependent contribution

Figure 4: A shifted double-winding Wilson loop, (a) R-independent contribution, (b) R-dependent contribution

The result of the shifted double-winding Wilson loop average up to the leading contribution is given by

$$\langle W(C_1 \cup C_2) \rangle_{R \neq 0} \simeq \begin{cases} c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_1(\beta)^{|S_1|-|S_2|} & (N=2) \\ c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_1(\beta)^{|S_1|} & (N=3) \\ c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases} . \tag{13}$$

This result reproduces the R-dependent behavior of the shifted double-winding Wilson loop average in [3]. In particular, we obtain the (non-zero) mass gap from the case of $S_1 = S_2 = 1$ and $R \gg 1$ in the above result:

$$\Delta(\beta) = 4 \ln \frac{1}{c_1(\beta)} > 0 \, . \tag{14}$$

3. Lattice Z_N gauge-scalar theory

Next, we consider the lattice Z_N gauge-scalar model with the frozen scalar field norm R for simplicity. The action of this model with the coupling constants defined by $\beta := 1/g^2$ and $K := R^2$ on a D-dimensional lattice Λ with unit lattice spacing is given by

$$S[U,\varphi] = \beta \sum_{p\in\Lambda} \mathrm{Re}\, U_p + K \sum_{\ell\in\Lambda} \mathrm{Re}\left(\varphi_x U_\ell \varphi^*_{x+\ell}\right) , \tag{15}$$

where ℓ labels a link, and p labels an elementary plaquette. U_ℓ is a Z_N link variable on link ℓ and φ_x is a Z_N scalar field at site x which transforms according to the fundamental representation of the gauge group Z_N.

In this model, the expectation value of an operator $\mathscr{F}$ has the form

$$\begin{aligned} \langle \mathscr{F} \rangle_\Lambda &:= Z_\Lambda^{-1} \int \prod_{\ell\in\Lambda} dU_\ell \prod_{x\in\Lambda} d\varphi_x\, e^{S[U,\varphi]} \mathscr{F} = Z_\Lambda^{-1} \int \prod_{\ell\in\Lambda} dU_\ell\, h[U]\, e^{\beta \sum_{p\in\Lambda} \mathrm{Re}\, U_p} \mathscr{F} , \\ Z_\Lambda &:= \int \prod_{\ell\in\Lambda} dU_\ell \prod_{x\in\Lambda} d\varphi_x\, e^{S[U,\varphi]} , \quad h[U] := \int \prod_{x\in\Lambda} d\varphi_x\, e^{K \sum_{\ell\in\Lambda} \mathrm{Re}(\varphi_x U_\ell \varphi^*_{x+\ell})} . \end{aligned} \tag{16}$$

According to [6], we can perform the *cluster expansion* by introducing the new variable ρ_p and the new measure $d\mu_\Lambda$ which absorbs the scalar part $h[U]$:

$$\langle \mathscr{F} \rangle_\Lambda = \frac{\int d\mu_\Lambda \prod_{p\in\Lambda} (1+\rho_p)\, \mathscr{F}}{\int d\mu_\Lambda \prod_{p\in\Lambda} (1+\rho_p)} = \sum_{Q(Q_0)\subset\Lambda} \int d\mu_\Lambda\, \mathscr{F} \prod_{p\in Q(Q_0)} \rho_p \cdot \frac{Z_{[Q(Q_0)\cup Q_0]^c}}{Z_\Lambda} , \tag{17}$$

$$d\mu_\Lambda := \frac{\prod_{\ell\in\Lambda} dU_\ell\, h[U]}{\int \prod_{\ell\in\Lambda} dU_\ell\, h[U]} , \quad \rho_p := e^{\beta \mathrm{Re}\, U_p} - 1 , \tag{18}$$

where Q_0 is the set of plaquettes which is the support of $\mathscr{F}$ and $Q(Q_0)$ is the set of plaquettes which is connected to Q_0. For the general set of plaquettes Q, Q^c represents the complement of Q. Here, Z_Q is defined by

$$Z_Q := \sum_{Q'\subset Q} \int d\mu_\Lambda \prod_{p\in Q'} \rho_p . \tag{19}$$

Note that $\rho_p \sim O(\beta)$ for $\beta \ll 1$. It has been showed in [7] that the confinement region ($0 \leqslant \beta \ll 1, K \ll 1$) and the Higgs region ($\beta \gg 1, K_c \leqslant K < \infty$) are analytically continued in a single " analytic region ", where the cluster expansion converges uniformly. See Fig.5.

To evaluate $h[U]$, we apply the character expansion and perform the group integration. Ignoring the contributions from multiple plaquettes, then we obtain the expression which is valid up to the lowest plaquettes order:

$$\begin{aligned} h[U] &= \int \prod_{x\in\Lambda} d\varphi_x \prod_{\ell\in\Lambda} \left[b_0(K) + b_1(K)\varphi_x U_\ell \varphi^*_{x+\ell} + \cdots + b_{N-1}(K)\left(\varphi_x U_\ell \varphi^*_{x+\ell}\right)^{N-1} \right] \\ &= N^{|\Lambda|} b_0(K)^{D|\Lambda|} \prod_{p\in\Lambda} \sum_{n=0}^{N-1} c_n(K)^4 U_p^n + \cdots . \end{aligned} \tag{20}$$

We estimate the leading contribution to the double-winding Wilson loop average with the above $h[U]$, we also apply the character expansion for ρ_p and evaluate the upper bound of the cluster

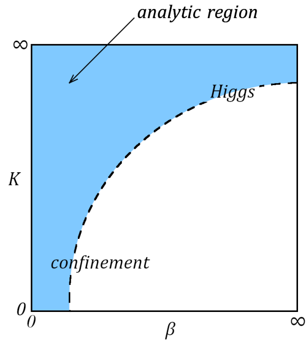


Figure 5: The analytic region on the β-K plane

expansion by using the binominal expansion. We find that there is an correspondence between the evaluation for the Z_N lattice gauge model and for the estimated upper bound for the Z_N lattice gauge-scalar model:

$$c_n(\beta) \mapsto a_n(\beta, K) := \frac{\left[b_0(\beta) - e^{\beta}\right] c_n(K)^4 + b_1(\beta) c_{n+1}(K)^4 + \cdots + b_{N-1}(\beta) c_{N+n-1}(K)^4}{b_0(\beta) + b_1(\beta) c_1(K)^4 + \cdots + b_{N-1}(\beta) c_{N-1}(K)^4} + c_n(K)^4 \ . \quad (\ \mathrm{mod}\ N\ ,\ n = 1, \cdots, N-1) \tag{21}$$

Note that $a_n(\beta, 0) = c_n(\beta)$ and $a_n(\beta, \infty) = 1$. The above estimation is valid only for the values of parameter β and K on the analytic region in the range where the string breaking does not occur.

By applying the same method as the above, we obtain the estimation for the coplanar double-winding Wilson loop average:

$$\langle W(C_1 \cup C_2) \rangle_{R=0} \lesssim \begin{cases} a_1(\beta, K)^{|S_1|-|S_2|} & (N = 2) \\ a_1(\beta, K)^{|S_1|} & (N = 3) \\ a_2(\beta, K)^{|S_2|} a_1(\beta, K)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases} \tag{22}$$

and we obtain the (non-zero) string tension from the above result:

$$\sigma(\beta, K) \gtrsim \ln \frac{1}{a_1(\beta, K)} > 0 \ . \tag{23}$$

This estimation suggests that the area law falloff in the Z_N lattice gauge model persists in the Z_N lattice gauge-scalar model and the $K \to 0$ limit agrees with the pure gauge case. Moreover, for $\sigma(\beta, K)$, the $K \to 0$ limit agrees with $\sigma(\beta)$ in the Z_N lattice gauge model, and $K \to \infty$ limit converges to 0 uniformly in β. In other words, the string tension is non-zero on the analytic region.

Additionally, we also estimate the shifted double-winding Wilson loop average:

$$\langle W(C_1 \cup C_2) \rangle_{R \neq 0} \lesssim \begin{cases} a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_1(\beta, K)^{|S_1|-|S_2|} & (N = 2) \\ a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_1(\beta, K)^{|S_1|} & (N = 3) \\ a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_2(\beta, K)^{|S_2|} a_1(\beta, K)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases} \tag{24}$$

and we obtain the (non-zero) mass gap from the case of $S_1 = S_2 = 1$ and $R \gg 1$ in the above result:

$$\Delta(\beta, K) \gtrsim 4 \ln \frac{1}{a_1(\beta, K)} > 0 \ . \tag{25}$$

For $\Delta(\beta, K)$, the $K \to 0$ limit agrees with $\Delta(\beta)$ in the Z_N lattice gauge model, and $K \to \infty$ limit converges to 0 uniformly in β. In other words, the mass gap is non-zero on the analytic region.

4. Conclusion

We investigated the area law falloff of the double-winding Wilson loops in the Z_N lattice gauge model and Z_N lattice gauge-scalar model, where the gauge group is the center group of the original $SU(N)$. First, we evaluated the N-dependent area law falloff for the coplanar double-winding Wilson loop average up to the leading contribution. We found the N-dependence of the area law falloff in the Z_N lattice gauge model, which reproduces the area law falloff in the $SU(N)$ lattice gauge model obtained in [3]. Secondly, we also checked the limit $N \to \infty$, the area law falloff for $N \geqslant 4$ persists in the $U(1)$ lattice gauge model. This result implies that the coplanar double-winding Wilson loop average in the $U(N)$ lattice gauge model and the $SU(N)(N \geqslant 4)$ lattice gauge model obeys the same area law up to the leading contribution. Furthermore, we also considered the shifted double-winding Wilson loop average up to the leading contributions. This result reproduces the R-dependent behavior in the $SU(N)$ lattice gauge model obtained in [3]. We obtained the (non-zero) mass gap $\Delta(\beta)$ from this result. Finally, we extended the above study for the Z_N lattice gauge-scalar model on the analytic region. We found that the area law falloff in the Z_N lattice gauge model persists in the Z_N lattice gauge-scalar model. We discovered that the string tension and the mass gap are non-zero on the analytic region from this estimation.

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