Phases at finite winding number of an Abelian lattice gauge theory

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Pure gauge theories are rather different from theories with pure scalar and fermionic matter, especially in terms of the nature of excitations. For example, in scalar and fermionic theories, one can create ultra-local excitations. For a gauge theory, such excitations need to be closed loops that do not violate gauge invariance. In this proceedings, we present a study on the condensation phenomenon associated with the string-like excitations of an Abelian lattice gauge theory. These phenomena are studied through numerical simulations of a $U(1)$ quantum link model in 2+1 dimensions in a ladder geometry using matrix product states. In this proceedings, we show the existence of ground states characterized by the presence of such string-like excitations. These are caused due to the condensation of torelons. We also study the relationship between the properties of the plaquettes in the ground state and the presence of such condensation phenomenon.

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1. Introduction

Quantum simulators are promising concepts, which enable experimental realizations of theoretical models. They are increasingly used to study the physics of lattice quantum field theories[1]. Nevertheless, classical simulations are still the most reliable method to study lattice field theory, with results close to the thermodynamic limit. In particular, tensor networks are a great tool to study lattice field theory in situations where standard Markov chain Monte Carlo is known to fail [2]. In addition, such classical tensor network simulation methods can guide and benchmark quantum simulations since they share the same Hamiltonian. In this regard, such classical simulation methods are very well positioned to guide frontier research in this field.

In this work, we expand on exploring the phases which can be realized in a strongly interacting lattice gauge theory. It is well-known that for a pure gauge theory, it is possible to have loop operators which wind around the spatial volume, which we will denote as *winding number operator* and which are termed as torelons [3]. For the non-Abelian gauge groups $SU(2)$ or $SU(3)$ commonly used for studies of confinement, these operators are associated with a $Z(2)$ or $Z(3)$ center symmetry. In the case of the Abelian $U(1)$ lattice gauge theory, however, such operators carry a global $U(1)$ quantum number. Consequently it becomes possible to couple a chemical potential to this winding number operator. Here, we study the condensation phenomena of the string-like excitations (expectation value of the winding number operators) at large values of the chemical potential coupled to the winding number operator.

The presence of string-like excitations that can spread over the entire lattice extent in a cylindrical geometry is of great interest [4, 5]. In particular, one can ask to what extent the confining properties of the ground state are disturbed when such excitations are present. To illustrate why this can be the case, consider the physics of an interacting bosonic or fermionic model with a global symmetry. A pedagogical example is the XY-model, where one can also couple a chemical potential. In the absence of the chemical potential at weak couplings, the excitations are short ranged, and the system has a mass gap. However, upon subjecting the system to a sufficiently large chemical potential, the particle excitations condense, and long-range correlations (signalled by a non-zero value of the superfluid stiffness) appear in the system [6]. We aim to investigate such physics in the pure gauge theory context. Furthermore, these string-like excitations of a gauge theory have their own unique dynamics, which can in principle be very different from the dynamics of point-particles in scalar theories or mesons in fermionic theories.

The goal of this study is to demonstrate the presence of the string excitations in the ground state numerically. In scalar or fermionic theories, the expectation value of the particle number operator presents discrete jumps when the chemical potential is increased (e.g., [6, 7]). When we raise the system's density, a higher particle state may have lower energy than the corresponding lower particle state. We expect the same behaviour for string-like excitations in our Abelian gauge theory. This sting-like excitations divide the Hilbert space in sectors that are exponentially hard to explore by Quantum Monte Carlo simulations. For this reason, tensor network algorithms have been chosen to simulate this model.
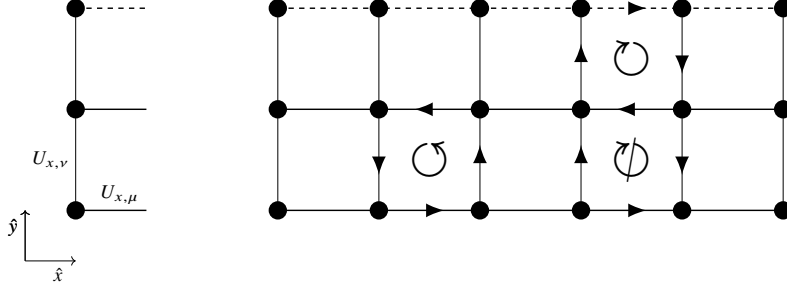
Figure 1: Ladder geometry of the lattice and its local basis used inside the DMRG algorithm. The periodicity in $\hat{y}$ is indicated by the dashed lines. Furthermore we show the two flippable plaquettes (↺, ↻) and a non flippable plaquette (↻̸).

2. Quantum link model

While a Wilson-type Abelian gauge theory has an infinite-dimensional Hilbert space for every link, the quantum link models (QLMs) regulate this infinite-dimensional Hilbert space in a completely gauge-invariant fashion [8–10]. This is achieved by replacing the quantum rotors in the Wilson theory for a quantum spin in the QLM, such that the resulting finite-dimensional Hilbert spaces have sizes $(2S + 1)$ locally on the links. The spin representations take the values $S = 1/2, 1, 3/2, \ldots$ and the Wilsonian theory is obtained when $S \to \infty$ [11]. In this work we consider the $U(1)$ quantum link model in 2+1 dimensions with the spin $S = 1/2$ representation. Due to the low finite dimensional Hilbert space of this model, quantum simulator proposals have been made [12, 13], or this could be simulated on larger NISQ devices [14].

The Hamiltonian of the system is defined as:

$$\mathcal{H} = -J \sum_{\square} (U_{\square} + U_{\square}^{\dagger}) + \sum_{\square} \lambda (U_{\square} + U_{\square}^{\dagger})^2 , \tag{1}$$

where we have defined the plaquette operator $U_{\square} = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^{\dagger} U_{x,\nu}^{\dagger}$ and the operators $U_{x,\mu}$ act on the links. The Hamiltonian has a local $U(1)$ gauge symmetry and the generator of the symmetry is:

$$G_x = \sum_{\mu} \left(E_{x-\hat{\mu},\mu} - E_{x,\mu} \right) = \sum_{\mu} \left(S_{x-\hat{\mu},\mu}^z - S_{x,\mu}^z \right) \tag{2}$$

where $E_{x,\mu}$ is the electric field operator. The gauge-field operator is canonically conjugate to the electric field operator, i.e., $[E_{x,\mu}, U_{y,\nu}] = U_{x,\mu} \delta_{\mu,\nu} \delta_{x,y}$ and $[E_{x,\mu}, U_{y,\nu}^{\dagger}] = -U_{x,\mu}^{\dagger} \delta_{\mu,\nu} \delta_{x,y}$. As explained before, we can use quantum spins as degrees of freedom, which satisfy the above commutation relations. In the S^z basis, we can represent: $U_{x,\mu} = S_{x,\mu}^+$, $U_{x,\mu}^{\dagger} = S_{x,\mu}^-$ and $E_{x,\mu} = S_{x,\mu}^z$, with the spin raising and lowering operators S^+ and S^-.

3. Symmetries of the model

We consider a 2+1 dimensional lattice with extension $L_x \times L_y$, shown in Figure 1. In addition to the $U(1)$ gauge symmetry, the Hamiltonian also has the point group symmetries (translation, (discrete) rotation, and parity). The Hamiltonian is invariant under the charge conjugation symmetry

(Z_2). Moreover, the system has a global invariance with respect to winding number transformations ($U(1) \otimes U(1)$). The generators of these symmetries are the winding number operators:

$$W_x = \frac{1}{L_y} \sum_y^{L_y} S^z_{(x,y),\hat{x}} \quad \text{and} \quad W_y = \frac{1}{L_x} \sum_x^{L_x} S^z_{(x,y),\hat{y}}, \tag{3}$$

respectively for the x direction and for the y direction. The generators of this symmetry commute with the Hamiltonian $[H, W_x] = [H, W_y] = 0$ and are thus simultaneously diagonalizable. We therefore add these operators to the Hamiltonian with the coefficient μ_x and μ_y used as chemical potential in the x and in the y direction. The resulting Hamiltonian is:

$$H = -J \sum_\square (U_\square + U_\square^\dagger) + \lambda \sum_\square (U_\square + U_\square^\dagger)^2 + \mu_x \sum_x W_x + \mu_y \sum_y W_y. \tag{4}$$

The operators W_x and W_y allow for the formation of stringy excitations that spread over the lattice. Moreover, in analogy with the particle number operators in fermionic and scalar theories, the ground state energy is given by:

$$E_{GS} = E_{\mathcal{H}} - \mu_x N^x - \mu_y N^y, \tag{5}$$

where N^x and N^y are the expectation values of $\sum_x W_x$ and $\sum_y W_y$. We now consider all the possible states one plaquette can have. Since a plaquette is formed, in our quantum link model, by four different spins 1/2, the number of states is $n_{\text{states}} = 2^4 = 16$. The two plaquette operators present in the Hamiltonian are:

$$U_\square = S^+_{x,\mu} S^+_{x+\hat{\mu},\nu} S^-_{x+\hat{\nu},\mu} S^-_{x,\nu} \quad \text{and} \quad U_\square^\dagger = S^-_{x,\mu} S^-_{x+\hat{\mu},\nu} S^+_{x+\hat{\nu},\mu} S^+_{x,\nu}. \tag{6}$$

In our representation, a plaquette can be any combination of the four different spins that compose the plaquette (e.g., $|\downarrow\downarrow\downarrow\uparrow\rangle$). Between all the possible 16 states a plaquette can have, only two are not annihilated by the action of the $U_\square$ and $U_\square^\dagger$ operators. The two possible states $|\circlearrowright\rangle$ and $|\circlearrowleft\rangle$ are visualized in Figure 1. We define $|\circlearrowright\rangle$ as the classical state in which the plaquette state has a clockwise orientation ($|\uparrow\uparrow\downarrow\downarrow\rangle$). We define $|\circlearrowleft\rangle$ as the classical state in which the plaquette state has a counter-clockwise orientation ($|\downarrow\downarrow\uparrow\uparrow\rangle$). We define the flippability operator O_{flipp} as the difference between the fippability in even and odd sites in the x direction over the lattice:

$$O_{\text{flipp}} = \sum_\square (-1)^x (U_\square + U_\square^\dagger)^2 = \sum_\square (-1)^x (U_\square U_\square^\dagger + U_\square^\dagger U_\square). \tag{7}$$

We note that, within the quantum link formulation, the operator $U_\square U_\square^\dagger$ is not the identity operator since $U_\square$ is not unitary. Hence, this operator can measure non-trivial correlations. In the numerical studies, we will demonstrate a correlation between this operator and the number of excitations present in the ground state.

4. Numerical simulations

In our numerical studies, the wave function of the system is represented by matrix product states and the ground state of the model is computed with the Density matrix renormalization group algorithm (DMRG)[15]. The DMRG algorithm performs extremely well in 1+1 dimensions. For
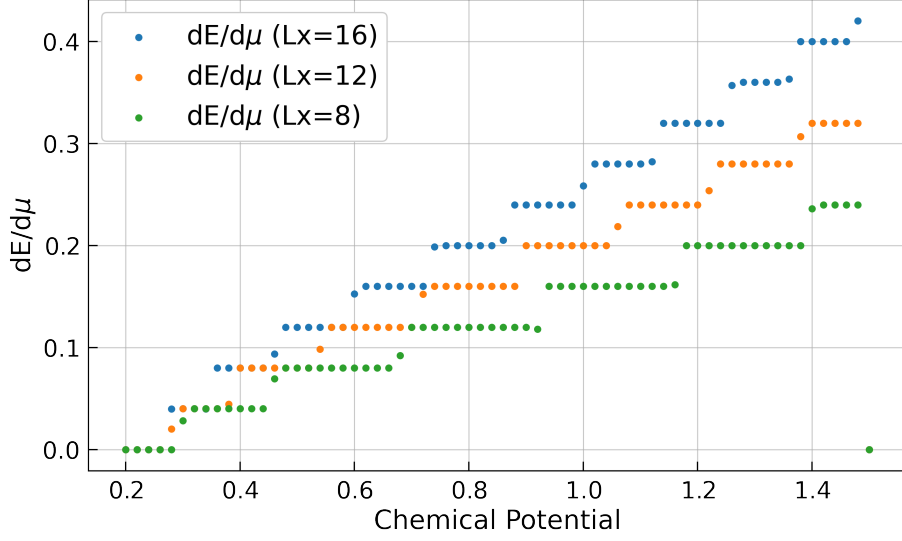
Figure 2: Derivative of the energy as a function of the chemical potential.

this reason, we have studied the system in a ladder geometry (see Figure 1) with open boundary conditions in the *x* direction and periodic boundary condition in the *y* direction (the theory lives on a cylinder). The local basis used inside the DMRG algorithm is shown in Figure 1. It is staged together as a chain to form the ladder geometry of our lattice. The dashed lines indicate the links on the periodic boundary. In the numerical studies, We keep $\mu_x = 0$ in order to prevent the system from generating a winding in the x-direction, which would otherwise kill all dynamics.

The commutator $[H, W_y]$ is no longer zero when open boundary conditions are imposed. However, the non-commuting terms only appear in the boundaries, and therefore for large enough extents in the x-direction this does not pose a problem. Moreover, note that the effect of the W_y operator can also be realized by imagining a static charge-anti-charge pair at the boundaries, which inject a background electric flux into the system. In our numerical simulations, we explore the effect of the chemical potential (μ_y) on the ground state properties of the theory. We fix the parameter in the Hamiltonian to be $J = 1$ and $\lambda = -1$. We study the properties of the ground state as a function of the chemical potential μ_y.

In Figure 2 we show the numerical derivative of the ground state energy extrapolated in the infinite bond dimension limit with respect to the chemical potential as a function of the chemical potential. The derivative of the expectation value of the energy of the ground state corresponds to the expectation value of the winding number operator on the ground state. The winding number operator can be seen as an operator that counts the number of string-like excitations in the ground state. In this case the excitations are strings that propagate through the whole lattice. Discrete jumps in the number of excitations in the ground state are clearly visible in Figure 3. When the chemical potential is increased, the system favors a lower energy state with more particles in the ground state.

In Figure 3 we plot two different quantities as a function of the chemical potential. In blue we plot the absolute value of the flippability of the state defined in Equation 7. When the values in the plot are close to zero, they are zero to numerical precision (10^{-10} in our case). We plot the absolute
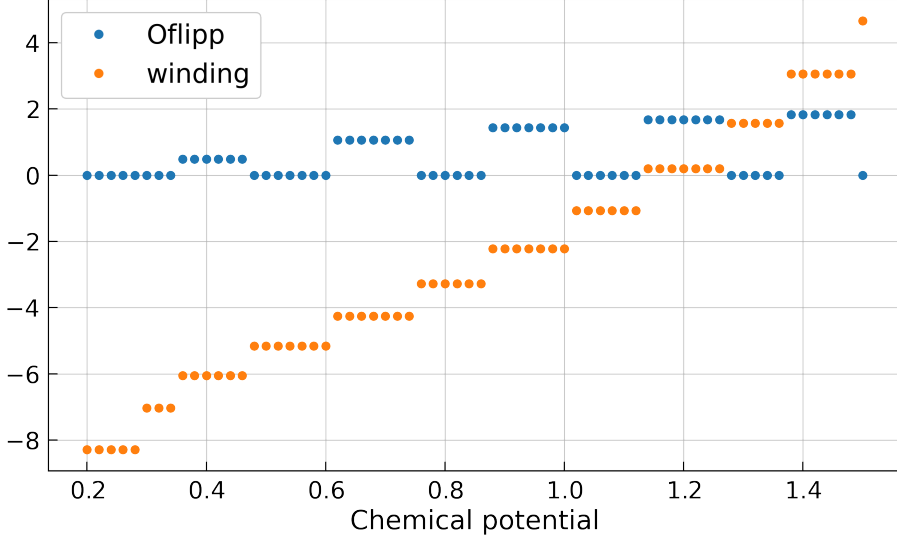
Figure 3: Correlation between flippability and winding sectors.

value of the expectation value because the operator is symmetric under charge conjugations. In orange, we plot the expectation value of the winding number operator. It is clear from this plot that the two quantities are correlated. Starting from small chemical potential, we have a low number of excitations and a zero flippability. When we increase the chemical potential, we see a transition in the winding number and in the flippability. At the next transition point for the winding number operator, the flippability goes back to zero. This pattern is repeated when the chemical potential is increased. Since the flippability operator is defined as the difference of the flippability of the single plaquettes in even and odd site, we can measure if we have an even or odd number of excitations in the ground state by computing the expectation value of the flippability.

5. Conclusions

In this proceeding, we analyzed a $U(1)$ quantum link model in 2+1 dimensions in a ladder geometry. Having motivated the importance of coupling a chemical potential to the winding number operators, we have studied the properties of the ground state of the model at finite volume and increasing values of the chemical potential. We have demonstrated how the increase of chemical potential corresponds to the condensation of string excitations in the ground state. In fact, increasing the chemical potential changes the flippability operator, as well as the number of string-like excitations present in the ground state in discrete steps. In bosonic and fermionic theories, increasing the chemical potential causes non-trivial particle numbers states to be the ground state. Similarly, in this case, when we increase the chemical potential, we find the ground state to comprise non-zero winding numbers. This causes the expectation value of the winding numbers to change in discrete steps. We have also seen a correlation between the winding number sectors and the average flippability of the plaquettes. With the finite dimensional Hilbert space at each link, this model is also a good candidate for cold atom simulations without resorting to any further truncation [16].

Acknowledgments

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