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Stout-smearing, gradient flow and $c_{\rm SW}$ at one loop order

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The one-loop determination of the coefficient $c_{\rm SW}$ of the Wilson quark action has been useful to push the leading cut-off effects for on-shell quantities to $O(\alpha^2 a)$ and, in conjunction with non-perturbative determinations of $c_{\rm SW}$, to $O(a^2)$, as long as no link-smearing is employed. These days it is common practice to include some overall link-smearing into the definition of the fermion action. Unfortunately, in this situation only the tree-level value $c_{\rm SW}^{(0)} = 1$ is known, and cut-off effects start at $O(\alpha a)$. We present some general techniques for calculating one loop quantities in lattice perturbation theory which continue to be useful for smeared-link fermion actions. Specifically, we discuss the application to the 1-loop improvement coefficient $c_{\rm SW}^{(1)}$ for overall stout-smeared Wilson fermions.

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1. Introduction

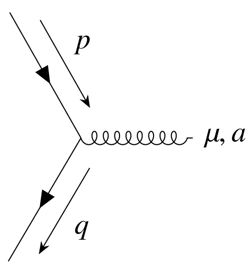
In the original work Sheikholeslami and Wohlert [1] noted that the improvement coefficient $c_{\rm SW}$ has to be equal to one at tree level. Wohlert later calculated its value at one loop order using twisted boundary conditions as a regulator [2]. More one-loop calculations of $c_{\rm SW}$ have been performed by Aoki and Kuramashi [3] focusing on different improved gauge actions and by Horsley et al. [4] for SLiNC fermions (i.e. with stout-smearing in the Wilson part of the fermion action but not in the clover term). Our aim is to calculate $c_{\rm SW}$ to one loop order for the Wilson-clover action including overall stout smearing. Furthermore because of the close relation of stout-smearing to the gradient flow of the Wilson action we are able to directly extend our results to the case of flowed fields.

2. The perturbative determination of $c_{\rm SW}$

The Sheikholeslami-Wohlert-coefficient $c_{\rm SW}$ of the $O(a)$-improved action:

$$\mathcal{S}_I = \mathcal{S}_{\rm Wilson} + c_{\rm SW} \cdot \sum_x \sum_{\mu<\nu} \bar{\psi}(x) \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x) \tag{1}$$

has a perturbative expansion $c_{\rm SW} = c_{\rm SW}^{(0)} + g_0^2 c_{\rm SW}^{(1)} + O(g_0^4)$ in powers of the bare coupling $g_0^2 = 2N_c/\beta$. It can be calculated via the quark-quark-gluon-vertex function



$$\Lambda_\mu^a(p,q) = \sum_{L=0}^{\infty} g_0^{2L+1} \Lambda_\mu^{a(L)}(p,q). \tag{2}$$

The number of loops L corresponds to the numbering of the expansion coefficients $c_{\rm SW}^{(L)}$. At tree level it is given by the lattice version of the qqg-vertex:

$$\Lambda_\mu^{a(0)}(p,q) = (V_1^a)_\mu(p,q) \tag{3}$$

$$= -g_0 T^a \left(i\gamma_\mu + a \left(\frac{1}{2}(p_\mu + q_\mu) - \frac{i}{2} c_{\rm SW}^{(0)} \sum_\nu \sigma_{\mu\nu}(p_\nu - q_\nu) \right) + O(a^2) \right) \tag{4}$$

Sandwiching its expansion in powers of a with on-shell spinors u and $\bar{u}$:

$$\bar{u}(q)\Lambda_\mu^{a(0)}(p,q)u(p) = -\, g_0 T^a \bar{u}(q) \left(i\gamma_\mu + \frac{a}{2}\left(1 - c_{\rm SW}{}^{(0)}\right)(p_\mu + q_\mu) \right) u(p) + O(a^2). \tag{5}$$

gives the condition $c_{\rm SW}{}^{(0)} = 1$ to eliminate $O(a)$ contributions[1] .

[1]Technically $c_{\rm SW}{}^{(0)} = r$ where r is the Wilson parameter.

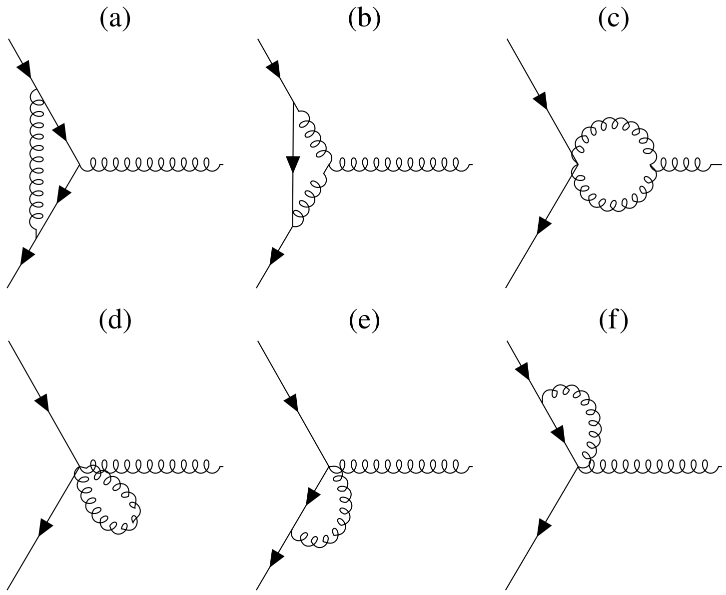


Figure 1: The six one-loop diagrams contributing to the vertex function in lattice perturbation theory.

At one-loop level the general form of the vertex function is

$$g_0^3\Lambda_\mu^{a(1)} = -g_0^3 T^a\left(\gamma_\mu F_1 + a\not{q}\gamma_\mu F_2 + a\gamma_\mu\not{p}F_3 + a(p_\mu + q_\mu)G_1 + a(p_\mu - q_\mu)H_1\right) \tag{6}$$

At $O(a)$ F_2 and F_3 do not contribute on-shell and H_1 vanishes due to symmetry arguments [3]. Sandwiching with on-shell spinors again gives

$$\begin{aligned} &g_0^3\bar{u}(q)\left(\frac{a}{2}c_{\mathrm{SW}}{}^{(1)}(p_\mu+q_\mu)T^a + \Lambda_\mu^{a(1)}(p,q)\right)u(p)\\ =&g_0^3\bar{u}(q)\left(i\gamma_\mu F_1 + \frac{a}{2}(p_\mu+q_\mu)(c_{\mathrm{SW}}{}^{(1)} - 2G_1)T^a\right)u(p) + O(p^2,q^2) + O(a^2), \end{aligned} \tag{7}$$

which results in the condition

$$c_{\mathrm{SW}}{}^{(1)} = 2G_1. \tag{8}$$

In Ref. [3] it is pointed out, that G_1 can be easily extracted from the vertex function by

$$G_1 = -\frac{1}{8}\mathrm{Tr}\left(\left(\frac{\partial}{\partial p_\mu} + \frac{\partial}{\partial q_\mu}\right)\Lambda_\mu^{a(1)} - \left(\frac{\partial}{\partial p_\nu} - \frac{\partial}{\partial q_\nu}\right)\Lambda_\mu^{a(1)}\gamma_\nu\gamma_\mu\right)_{p,q\to 0}^{\mu\neq\nu}. \tag{9}$$

The six diagrams that contribute at one loop level are shown in figure 1.

$$\Lambda_\mu^{a(1)} = \sum_{i=a,\ldots,f}\Lambda_\mu^{a(1)(i)} = \sum_{i=a,\ldots,f}\int_{-\pi}^{\pi}\frac{\mathrm{d}^4k}{(2\pi)^4}I_\mu^{a(i)}(p,q,k). \tag{10}$$

All of them except the tadpole diagram (d) lead to IR-divergent integrals. To calculate them separately their divergence needs to be analytically split off and the remaining constant part calculated numerically. One possibility is to use a small fictitious gluon mass μ (as was done in [3]). To this end an analytically solvable integrand with the same divergent behaviour J is subtracted from the original integrand I:

$$G_1^{(i)} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left[I_\mu^{(i)}(k) - \theta(\pi^2 - k^2) J_\mu^{(i)}(k, 0) \right] + \frac{\Omega_3}{(2\pi)^4} \int_0^{\pi} J_\mu^{(i)}(k, \mu^2) k^3 dk \tag{11}$$

The Heaviside-function makes J analytically solvable containing the non-zero gluon mass. As an example, let us consider diagram (c):

$$I_\mu^{a(c)} = \sum_{b,c} \sum_{\nu,\rho} V_{2\nu\rho}^{bc}(p, q, q-p-k)\, G(k)\, G(p-q+k) V_{g3\,\nu\rho\mu}^{bca}(p-q+k, -k, q-p), \tag{12}$$

where $V_{2\mu\nu}^{ab}$, $V_{g3\,\mu\nu\rho}^{abc}$ and $G(k)$ are the lattice versions of the qqgg-vertex, the ggg-vertex and the gluon propagator respectively. A possible analytically solvable integral can be obtained by expanding these to $O(a)$ which gives:

$$\mathrm{J}^{(c)} = \frac{3}{4} N_c c_{\mathrm{SW}}{}^{(0)} \frac{2\pi^2}{(2\pi)^4} \int_0^{\pi} \frac{k^3}{(k^2 + \mu^2)^2} dk \tag{13}$$

$$= \frac{3}{4} N_c c_{\mathrm{SW}}{}^{(0)} \frac{1}{16\pi^2} \left(\ln\left(\frac{\pi^2}{\mu^2}\right) - 1 \right) + O(\mu^2) \tag{14}$$

Denoting the logarithmic divergence by $L = \frac{1}{16\pi^2} \ln\left(\frac{\pi^2}{\mu^2}\right)$ and adding the finite result from the $I - J$ integral results in the following contribution from diagram (c):

$$2G_1^{(c)} = \frac{9}{2} L - 0.0813095 \tag{15}$$

The following table shows the contributions from all diagrams and how their divergent parts add up to zero. The far right column shows the values from [3] for comparison. It is worth pointing out

Diagram	Divergent part	Constant part	Aoki, Kuramashi
(a)	$-L/3$	0.00457196	0.004572(2)
(b)	$-9L/2$	0.0830768	0.08311(3)
(c)	$9L/2$	−0.0813095	−0.08133(3)
(d)	0	0.297394537	0.29739454(1)
(e)	$L/6$	−0.0175746	−0.017574(1)
(f)	$L/6$	−0.0175746	−0.017574(1)
Sum	0	0.268588292	0.26858825(1)

that calculating the finite sum of all diagrams directly instead of diagram by diagram can lead to a numerically more accurate result, as the discontinuity introduced through the Heaviside function in (12) causes a slower convergence.

3. Perturbative stout smearing and Wilson flow

Stout smearing with smearing parameter ρ of the link variable $U_\mu(x)$ is defined through

$$U_\mu^{(1)}(x) = e^{i\rho Q_\mu(x)} U_\mu(x) \tag{16}$$

with

$$Q_\mu(x) = \frac{1}{2i}\left(W_\mu(x) - \frac{1}{3}\mathrm{Tr}[W_\mu(x)]\right) \tag{17}$$

$$W_\mu(x) = \sum_{\nu\neq\mu}(P_{\mu\nu}^\dagger(x) - P_{\mu\nu}(x)), \tag{18}$$

where $P_{\mu\nu}(x)$ is the plaquette in the μ,ν-plane at lattice position x. As $Q_\mu(x)$ is anti-hermitian, $U_\mu^{(1)}(x)$ is automatically in $SU(3)$ again and the smearing can be iterated leading to $U_\mu^{(n)}(x)$ after n smearing steps.

Wilson flow is the gradient flow of the Wilson action $\mathcal{S}_W$ and defined through the following differential equation:

$$\partial_t U_\mu(x,t) = -g_0^2\left\{\partial_{x,\mu}\mathcal{S}_W[U(x,t)]\right\} U_\mu(x,t) = iQ_\mu(x,t)U_\mu(x,t) \tag{19}$$

$$U_\mu(x,0) = U_\mu(x). \tag{20}$$

Therefore it is easily seen, that the Wilson flow is generated by infinitesimal stout-smearings. This means calculations done in the stout-smearing formalism with parameters ρ and n can be transformed into the Wilson flow formalism with flow time t by performing the limit $n \to \infty$, $\rho \to 0$ with $n\rho = t =$ const. (t in lattice units, i.e. t/a^2).

Our goal is to perform perturbative calculations of c_{SW} as sketched in section 2 including stout smearing and eventually Wilson flow. Therefore we need a perturbative expansion of the smearing.

3.1 Leading order

The un-smeared original link variable $U_\mu(x)$ has an expansion in terms of the gluon field $A_\mu(x)$:

$$U_\mu(x) = 1 + ig_0 T^a A_\mu^a(x) - \frac{g_0^2}{2}\frac{1}{2}\{T^a, T^b\}A_\mu^a(x)A_\mu^b(x) + O(g_0^3) \tag{21}$$

At leading order the smeared link variable $U_\mu^{(1)}(x)$ has a similar expansion with a modified gluon field $A_\mu^{a(1)}(x)$:

$$U_\mu^{(1)}(x) = 1 + ig_0 T^a A_\mu^{a(1)}(x) + O(g_0^2) \tag{22}$$

$$A_\mu^{a(1)}(x) = A_\mu^a(x) + \rho(Q_\mu^a(x))^{\mathrm{LO}} \tag{23}$$

with

$$(Q_\mu^a(x))^{\mathrm{LO}} = \sum_{\nu=1}^{4}\left(A_\nu^a(x) + A_\mu^a(x+\hat\nu) - A_\nu^a(x+\hat\mu)\right. \tag{24}$$

$$\left. - A_\nu^a(x-\hat\nu) + A_\mu^a(x-\hat\nu) + A_\nu^a(x+\hat\mu-\hat\nu) - 2A_\mu^a(x)\right). \tag{25}$$

After a Fourier transform $A_\mu^{a(1)}(k)$ can be expressed as [5]:

$$A_\mu^{a(1)}(k) = \sum_\nu \left(f(k)\delta_{\mu\nu} - (f(k)-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k) \tag{26}$$

with $\hat{k}_\mu = 2\sin(\frac{1}{2}k_\mu)$ and $f(k) = 1 - \rho\hat{k}^2$. After n smearing steps the overall structure remains the same except for the powers of n

$$A_\mu^{a(n)}(k) = \sum_\nu \left(f(k)^n\delta_{\mu\nu} - (f(k)^n-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k), \tag{27}$$

which in turn makes it easy to perform the aforementioned limit to the Wilson flow formalism:

$$A_\mu^{a}(k,t) = \sum_\nu \left(e^{-t\hat{k}^2}\delta_{\mu\nu} - (e^{-t\hat{k}^2}-1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2} \right) A_\nu^a(k,0). \tag{28}$$

3.2 Next-to-leading order

At next-to-leading order in addition to the quadratic term involving the already known modified gluon field $A_\mu^{a(1)}(x)$ there is also an anti-symmetric part $A_\mu^{ab(1)}(x)$:

$$\begin{aligned} U_\mu^{(1)}(x) &= 1 + ig_0 T^a A_\mu^{a(1)}(x) \\ &- \frac{g_0^2}{2}\left(\frac{1}{2}\{T^a,T^b\}A_\mu^{a(1)}(x)A_\mu^{b(1)}(x) + \frac{1}{2}[T^a,T^b]A_\mu^{ab(1)}(x)\right) \\ &+ O(g_0^3) \end{aligned} \tag{29}$$

with

$$A_\mu^{ab(1)}(x) = 4\rho\cdot\left[\frac{1}{2}(Q_\mu^a(x))^{\mathrm{LO}}A_\mu^b(x) + (Q_\mu^{ab}(x))^{\mathrm{NLO}}\right]. \tag{30}$$

Fourier transforming gives:

$$A_\mu^{ab(1)}(k_1,k_2) = 4\rho\sum_{\nu,\rho} g_{\mu\nu\rho}(k_1,k_2)A_\nu^a(k_1)A_\rho^b(k_2), \tag{31}$$

where

$$\begin{aligned} g_{\mu\nu\rho}(k_1,k_2) =& \delta_{\mu\nu}\sin(\tfrac{1}{2}(2k_{1\rho}+k_{2\rho}))\cos(\tfrac{1}{2}k_{2\mu}) \\ &-\delta_{\mu\rho}\sin(\tfrac{1}{2}(2k_{2\nu}+k_{1\nu}))\cos(\tfrac{1}{2}k_{1\mu}) \\ &-\delta_{\nu\rho}\sin(\tfrac{1}{2}(k_{1\mu}-k_{2\mu}))\cos(\tfrac{1}{2}(k_{1\nu}+k_{2\nu})). \end{aligned} \tag{32}$$

After n iterations we get a sum over the $A_\mu^{(m)}(x)$ from all previous smearing steps:

$$A_\mu^{ab(n)}(k_1,k_2) = 4\rho\sum_{m=0}^{n-1}\left[\sum_{\nu,\rho} g_{\mu\nu\rho}(k_1,k_2)A_\nu^{a(m)}(k_1)A_\rho^{b(m)}(k_2)\right]. \tag{33}$$

And finally in the Wilson flow limit we end up with the simple result of $g_{\mu\nu\rho}$ contracted with the differences of the flowed and the un-flowed fields:

$$
\begin{aligned}
A^{ab}_\mu(k_1, k_2, t) = {} & 4 \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) \\
& \times \sum_{\alpha\beta} \left[(e^{-t\hat{k}_1^2} - 1)\delta_{\nu\alpha} - (e^{-t\hat{k}_1^2} - 1)\frac{\hat{k}_{1\nu}\hat{k}_{1\alpha}}{\hat{k}_1^2} \right] \\
& \times \left[(e^{-t\hat{k}_2^2} - 1)\delta_{\rho\beta} - (e^{-t\hat{k}_2^2} - 1)\frac{\hat{k}_{2\rho}\hat{k}_{2\beta}}{\hat{k}_2^2} \right] A^a_\alpha(k_1, 0) A^b_\beta(k_2, 0) \qquad (34) \\
= {} & 4 \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2)[A^a_\nu(k_1, t) - A^a_\nu(k_1, 0)][A^b_\rho(k_2, t) - A^b_\rho(k_2, 0)] \qquad (35)
\end{aligned}
$$

4. Outlook

In order to calculate diagram (d), which involves the qqggg-vertex, the smearing relation at next-to-next-to-leading order is also needed. There the expressions become rather lengthy. The next step will then be to insert the smearing relations into the expanded action to obtain the Feynman rules, which can only be partly compared to [4], as the ones coming from the clover term are not included there.

Another question we would like to investigate concerns the structure of the smearing expansion. Because the smeared link variable $U^{(1)}_\mu(x)$ is again in $SU(3)$ it is expected to have an expansion

$$U^{(1)}_\mu(x) = e^{ig_1 T^a \tilde{A}^a_\mu(x)} \qquad (36)$$

with a modified (renormalised) coupling g_1. However this expansion does not coincide with the expansion in g_0 and finding a link between the two may help to simplify calculations.

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