Pion electric polarizabilities from lattice QCD

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We report a first principle lattice calculation of the pion electric polarizability α_π at the physical pion mass. First, we derive the master formula, which relates the pion polarizabilities with the position space hadronic Compton tensor, $\langle \pi | J_\mu(x) J_\nu(0) | \pi \rangle$. The finite volume error of the master formula is exponentially suppressed by the spatial extent of the lattice. Then, the hadronic tensor is calculated using domain wall fermions (DWF) directly at physical pion mass. The gauge ensembles are generated by the RBC-UKQCD collaborations.

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1. Introduction

Polarizabilities describe the leading energy shift for a neutral particle in a constant electric and magnetic field. In Minkowski space-time convention, we have [1]:

$$H_{\text{eff}} = -\frac{4\pi}{2}\alpha E^2 - \frac{4\pi}{2}\beta B^2 \quad \text{(Minkowski)}, \tag{1}$$

where α is the electric and β is the magnetic polarizability. In Euclidean space-time, the convention for the electric field is different. Therefore, we have

$$H_{\text{eff}} = \frac{4\pi}{2}\alpha E^2 - \frac{4\pi}{2}\beta B^2 \quad \text{(Euclidean)}. \tag{2}$$

In the following discussion, we will use the Euclidean space-time convention by default. Based on this definition, the polarizabilities of neutral particles can be related to the low energy behavior of the hadronic Compton tensor, $\langle N|J_\mu(x)J_\nu(0)|N\rangle$. For a charged particle, polarizability can also be defined via the low energy behavior of the hadronic Compton tensor after subtracting the Born term contribution. The electric current operator is defined as follow

$$J_\mu(x) = J_\mu(t_x, \vec{x}) = e\left(e_u\bar{u}\gamma_\mu u + e_d\bar{d}\gamma_\mu d + e_s\bar{s}\gamma_\mu s\right), \tag{3}$$

where $e_u = 2/3$, $e_d = e_s = -1/3$, and $\alpha_{\text{QED}} = e^2/(4\pi) \approx 1/137$. The γ_μ matrices satisfy the anti-commutation relation: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$.

Two-loop calculations of the pion polarizabilities using Chiral perturbation theory have been done for both the charged pion [2–4] and the neutral pion [5, 6]. There are also some lattice calculations from first principle using the background field method [7–9]. There are also attempts to use hadronic Compton tensor with small momentum transfer to extract polarizablities [10, 11]. Realistic lattice calculations along this direction are also underway.

In this work, we derive different position space formulas using the hadronic tensor to obtain the pion electric polarizabilities. We demonstrate these formulas allow efficient lattice calculations and will show some numerical results. In particular, we emphasis the finite volume errors of the master formula obtained are exponentially suppressed by the spatial lattice size L.

2. Formulation

2.1 Neutral pion

We start the derivation in finite volume to avoid the infinities from the infinite volume. However, we assume a periodic boundary box with very large volume (much larger than the real lattice size) so the finite volume effects can be neglected. We will analyze the finite volume effects of possible lattice calculations after we obtained the final expression.

Consider the zero momentum neutral pion correlation function ($t_{\text{snk}} \gg 0 \gg t_{\text{src}}$) in the presence of very smooth and slow varying external vector potential $A_\mu(x) = A_\mu(t_x, \vec{x})$, which is only non-zero in the middle region of the pion correlation function. We expect the neutral pion correlation function takes the following form according to the definition of the pion polarizabilities.

$$\langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle_{A_\mu} = \langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle \exp\left(-\frac{4\pi}{2}\int_x \frac{\alpha_{\pi^0}E^2(x) - \beta_{\pi^0}B^2(x)}{L^3}\right), \quad (4)$$

where $\pi^0(t)$ is a pion field with vanishing spatial momentum. We can also calculate the correlation function as a perturbative expansion in the vector potential A_μ:

$$\langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle_{A_\mu} = \langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle + \langle T\pi^0(t_{\text{snk}})\int_x iA_\mu(x)J_\mu(x)\pi^0(t_{\text{src}})\rangle \quad (5)$$

$$+\frac{1}{2}\langle T\pi^0(t_{\text{snk}})\int_x iA_\mu(x)J_\mu(x)\int_y iA_\nu(y)J_\nu(y)\pi^0(t_{\text{src}})\rangle .$$

For the neutral pion,

$$\langle \pi^0(t_{\text{snk}})J_\mu(x)\pi^0(t_{\text{src}})\rangle = 0 . \quad (6)$$

Combine the above equations and use translational invariance, we obtain:

$$\frac{4\pi}{2}\int_x \frac{\alpha_{\pi^0}E^2(x) - \beta_{\pi^0}B^2(x)}{L^3}$$

$$= \frac{1}{2}\int_x \left(\int_y A_\mu(x+y)A_\nu(y)\right)\frac{\langle T\pi^0(t_{\text{snk}})J_\mu(x)J_\nu(0)\pi^0(t_{\text{src}})\rangle}{\langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle} \quad (7)$$

$$= \frac{1}{2}\int_x \left(\int_y A_\mu(x+y)A_\nu(y)\right)\frac{1}{2M_\pi}\langle \pi^0|TJ_\mu(x)J_\nu(0)|\pi^0\rangle . \quad (8)$$

In the second step, we have rewrite the correlation function in terms of the matrix elements of the zero momentum pion state. Actually, we need to subtract the vacuum contribution in the above formula (note the above derivation is valid without the pion as well, in which case we would obtain the vacuum fermion sea energy shift due to the E&M field):

$$\frac{\langle T\pi^0(t_{\text{snk}})J_\mu(x)J_\nu(0)\pi^0(t_{\text{src}})\rangle}{\langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle} \quad (9)$$

$$\rightarrow \frac{\langle T\pi^0(t_{\text{snk}})J_\mu(x)J_\nu(0)\pi^0(t_{\text{src}})\rangle}{\langle T\pi^0(t_{\text{snk}})\pi^0(t_{\text{src}})\rangle} - \langle TJ_\mu(x)J_\nu(0)\rangle .$$

We will assume this subtraction in later discussion without explicitly writing it down.

At this point, we can consider some specific choice of vector potential. Focusing on the electric polarizability, we can choose:

$$\vec{A}(t_x) = \vec{A}(t_x, \vec{x}) = \vec{A}(x) , \quad (10)$$

while taking the time component of the vector potential equal to zero. Only the electric field is non-zero with this choice. Apply the general formula Eq. (8), we obtain:

$$I_{\text{eff}} = \int_{t_x} \frac{4\pi}{2}\alpha_{\pi^0}E^2(t_x) \quad (11)$$

$$= \int_{t_x,\vec{x}} \left(\int_{t_y} A_i(t_x+t_y)A_j(t_y)\right)\frac{1}{2M_\pi}\frac{1}{2}\langle \pi^0|TJ_i(t_x,\vec{x})J_j(0,\vec{0})|\pi^0\rangle . \quad (12)$$

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Since we choose the vector potential to vary very slowly, we can expand the time dependence of $\vec{A}$:

$$A_i(t_x + t_y) \approx A_i(t_y) + t_x \partial_t A_i(t_y) + \frac{1}{2} t_x^2 \partial_t^2 A_i(t_y) + \dots . \tag{13}$$

Due to the current conservation ($\partial_\mu^x J_\mu(t_x, \vec{x}) = 0$):

$$\int_{t_x, \vec{x}} \langle \pi^0 | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi^0 \rangle = \int_{t_x, \vec{x}} \partial_\mu^x \langle \pi^0 | T x_i J_\mu(t_x, \vec{x}) J_j(0, \vec{0}) | \pi^0 \rangle = 0 , \tag{14}$$

and time reflection symmetry:

$$\int_{t_x, \vec{x}} t_x \langle \pi^0 | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi^0 \rangle = 0 , \tag{15}$$

the first two terms in Eq. (13) do not contribute to I_{eff} in Eq. (12). Only the third term remains. Therefore

$$I_{\text{eff}} = \frac{1}{2} \int_{t_x, \vec{x}, t_y} A_j(t_y) \frac{1}{2} t_x^2 \partial_t^2 A_i(t_y) \frac{1}{2M_\pi} \langle \pi^0 | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi^0 \rangle . \tag{16}$$

Integrating by part and noticing the matrix element is only non-zero (after integration) when $i = j$, we obtain:

$$I_{\text{eff}} = -\frac{1}{2} \int_{t_y} \frac{1}{2} \partial_t \vec{A}(t_y) \cdot \partial_t \vec{A}(t_y) \int_{t_x, \vec{x}} \frac{1}{3} t_x^2 \frac{1}{2M_\pi} \langle \pi^0 | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi^0 \rangle . \tag{17}$$

Comparing the above equation with Eq. (11), we obtain for α_{π^0}:

$$\alpha_{\pi^0} = -\int_{t_x, \vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_\pi} \langle \pi^0 | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi^0 \rangle . \tag{18}$$

This is our master formula in this paper to obtain α_{π^0}. While the above equation seems positive definite, it does not imply $\alpha_{\pi^0} > 0$. The reason is that the vacuum contribution needs to be subtracted as indicated in Eq. (9).

It is possible to use the general Eq. (8) with a different choice of $A_\mu(x)$. For example, we can choose a mostly time independent field $A_\mu(x) = A_\mu(\vec{x})$ within a very long time interval with length t_{int}. The fields smoothly vanishes outside the time interval. We have the following relations:

$$\int_{t_x, \vec{x}} E^2(t_x, \vec{x}) \approx t_{\text{int}} \int_{\vec{x}} E^2(\vec{x}) , \tag{19}$$

$$\int_{t_x, \vec{x}} B^2(t_x, \vec{x}) \approx t_{\text{int}} \int_{\vec{x}} B^2(\vec{x}) . \tag{20}$$

With this choice of vector potential, we can similarly derive the the following formulas for α_{π^0} and β_{π^0}:

$$\alpha_{\pi^0} = -\int_{t_x, \vec{x}} \frac{\vec{x}^2}{24\pi} \frac{1}{2M_\pi} \langle \pi^0 | T J_t(t_x, \vec{x}) J_t(0, \vec{0}) | \pi^0 \rangle , \tag{21}$$

$$\beta_{\pi^0} = \int_{t_x, \vec{x}} \frac{\vec{x}^2}{48\pi} \frac{1}{2M_\pi} \langle \pi^0 | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi^0 \rangle \tag{22}$$

$$= -\int_{t_x, \vec{x}} \frac{1}{24\pi} \frac{1}{2M_\pi} \langle \pi^0 | T \vec{x} \cdot \vec{J}(t_x, \vec{x}) \vec{x} \cdot \vec{J}(0, \vec{0}) | \pi^0 \rangle .$$

Since the matrix elements satisfy the current conservation condition, we can obtain different but equivalent formulas for both α_{π^0} and β_{π^0}.

2.2 Charged pion

The charged pion polarizabilities are more conveniently defined via low-energy expansion coefficients of the (virtual) Compton scattering processes after the leading generalized Born terms being subtracted.

$$\langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle_S = \langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle - \langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle_{\text{Born}} . \quad (23)$$

The Born term can be defined as Eq. (115) in Ref. [1]. For the Compton scattering processes, in case where the photons are on-shell, the Born term is equal to the scalar QED contribution. If the photons are off-shell, the (generalized) Born term is defined to include the electromagnetic form factor of the pions. The Euclidean space-time expression for the Born term is:

$$T_{\mu,\nu}^{\text{Born}}(q_t,\vec{q}) = \int_{t_x,\vec{x}} e^{iq_t t_x - i\vec{q}\cdot\vec{x}}\langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle \quad (24)$$

$$= e^2 F^2(q_t^2+\vec{q}^2)\left(2\delta_{\mu,\nu} - \frac{(2p+q)_\mu(2p+q)_\nu}{(p+q)^2+M_\pi^2} - \frac{(2p-q)_\mu(2p-q)_\nu}{(p-q)^2+M_\pi^2}\right) , \quad (25)$$

where $p = (iM_\pi,\vec{0})$.

After the Born term subtraction, the polarizabilities for both neutral and charged pions can be defined in a uniform way as the low-energy expansion coefficients. Therefore, the derived formulas Eqs. (18,21,22,23) is valid for the charged pions if we use the Born term subtracted Compton tensor defined in Eq. (23). [1]

In particular, for our master equation, Eq. (18), we can explicitly calculate the Born term contribution, which we need to subtract. Based on the Born term definition, we have:

$$\left.\frac{\partial^2}{\partial q_t^2}T_{k,k}^{\text{Born}}(q_t,\vec{0})\right|_{q_t=0} = -\int_{t_x,\vec{x}} t_x^2\langle\pi|TJ_k(t_x,\vec{x})J_k(0,\vec{0})|\pi\rangle_{\text{Born}} \quad (26)$$

$$= \frac{\partial^2}{\partial q_t^2}\left(e^2 F^2(q_t^2)2\delta_{k,k}\right) . \quad (27)$$

For charged pion, we have $F_{\pi^\pm}(q^2) \approx 1 - r_\pi^2 q^2/6$, where $r_\pi = 0.659(4)$ fm [13], is the $\pi^\pm$ charge radius. Combining Eqs. (18,23,26,27), we obtain the expression for $\alpha_{\pi^\pm}$:

$$\alpha_{\pi^\pm} = -\int_{t_x,\vec{x}} \frac{t_x^2}{24\pi}\frac{1}{2M_\pi}\langle\pi^\pm|T\vec{J}(t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi^\pm\rangle - \alpha_{\pi^\pm}^{\text{Born}} , \quad (28)$$

where $\alpha_{\pi^\pm}^{\text{Born}} = -\alpha_{\text{QED}}\frac{r_\pi^2}{3M_\pi} = -14.94(18)\times 10^{-4}$ fm^3.

3. Lattice calculation

On the lattice, we calculate the Euclidean space-time hadronic Compton tensor from the four-point function:

$$\frac{1}{2M_\pi}\langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle = L^3\frac{\langle\pi(t_x+t_{\text{sep}})J_\mu(t_x,\vec{x})J_\nu(0,\vec{0})\pi^\dagger(-t_{\text{sep}})\rangle_L}{\langle\pi(t_x+t_{\text{sep}})\pi^\dagger(-t_{\text{sep}})\rangle_L} , \quad (29)$$

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[1]For neutral pions, the formulas are valid with or without the Born term subtraction, i.e. $\alpha_{\pi^0}^{\text{Born}} = 0$.

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where we use Coulomb gauge fixed wall source for the pion operator $\pi(t)$, and t_{sep} is the time separation between the current operator and the pion interpolating operator. The separation t_{sep} is fixed for each ensemble and set to be large enough (~ 1.5 fm) to ensure projection to the pion state. Calculation of the hadronic Compton tensor is performed on four ensembles generated by the RBC and UKQCD collaborations. The names and attributes of the ensembles are shown in Table 1. The properties of these ensembles are studied in detail in Ref. [12].

To study the size of the excited states contamination in our calculation, we define the following ratio:

$$R(t) = \frac{\langle \pi(t)\pi^{\dagger}(-t)\rangle_L}{\cosh\left(M_{\pi}(2t - T/2)\right)} \tag{30}$$

The ratio should reach constant for large enough t, and its deviation from that constant at small t represent the norm of the excited states. The lattice results of the above ratio is plotted in Figure 1. It can be seen that the norms of the excited states are below 1 % and are statistically insignificant.

	Volume	a^{-1} (GeV)	L (fm)	M_{π} (MeV)	t_{sep} (a)
48I	$48^3 \times 96$	1.730(4)	5.5	135	12
64I	$64^3 \times 128$	2.359(7)	5.4	135	18
24D	$24^3 \times 64$	1.0158(40)	4.7	142	8
32D	$32^3 \times 64$	1.0158(40)	6.2	142	8

Table 1: List of the ensembles used in the calculations and their properties. They are generated by the RBC and UKQCD collaborations. [12] Note we use a partially quenched quark mass for 48I and 64I ensembles. The unitary pion mass for both 48I and 64I ensembles is 139 MeV. We use unitary quark masses for all the other ensembles.
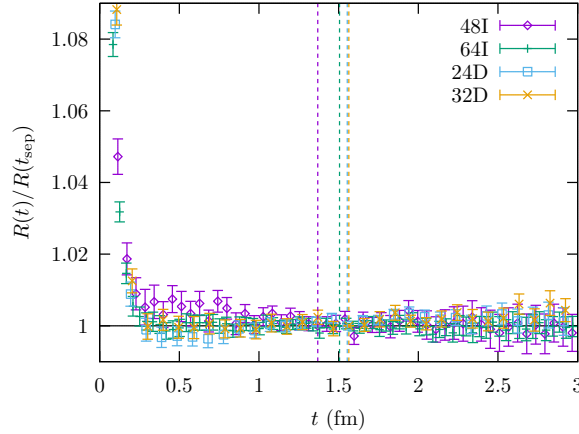

Figure 1: Ratio of the pion correlation function and its pion state contribution normalized at t_{sep}. $R(t)$ is defined in Eq. (30). Vertical lines correspond to t_{sep} used for each ensemble.

In this calculation of the pion polarizabilities, we only included the contribution from the quark connected diagrams, which are shown in Figure 2. We plot the results in Figure 3 as a function of t:

$$\alpha_{\pi}(t) = -\int_{-t < t_x < t} \int_{\vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_{\pi}} \langle \pi | T\vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi \rangle - \alpha_{\pi}^{\text{Born}} \; . \tag{31}$$

Figure 2: Quark connected diagrams of the four-point hadronic function (hadronic Compton tensor) used to calculate the pion polarizabilities. The dot represent the vector current operator insertion.
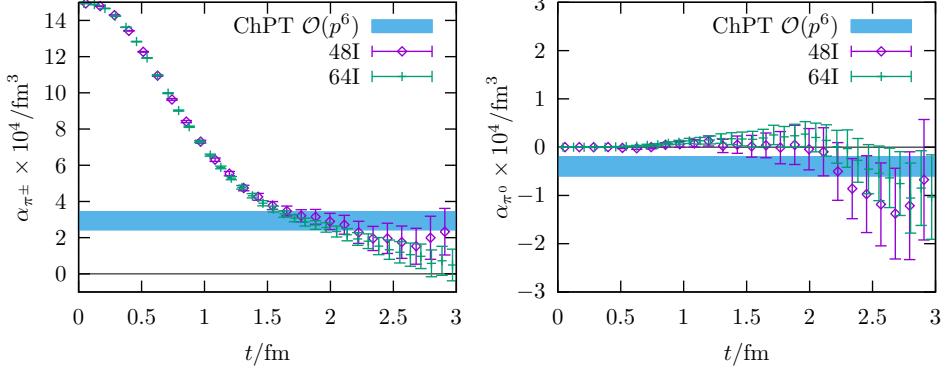
Figure 3: Electric polarizabilities of charged pion (left) and neutral pion (right). The t dependence is given by Eq. (31), $\alpha_\pi = \alpha_\pi(t \to +\infty)$. ChPT results are from Ref. [2–6].
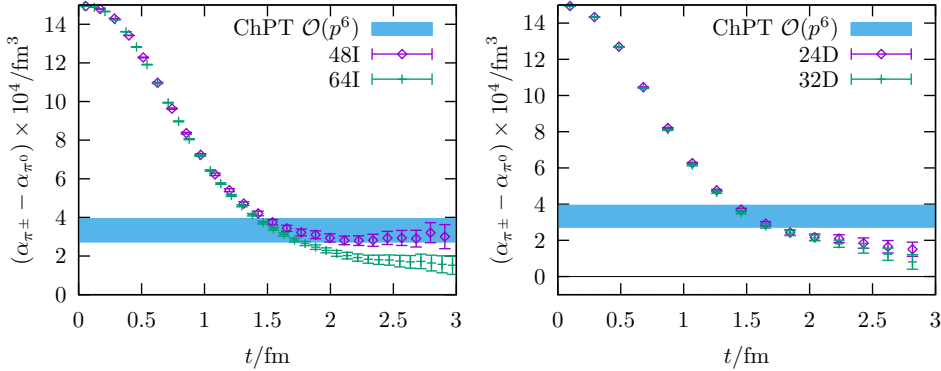
Figure 4: Difference of the electric polarizabilities between charged pion and neutral pion $\alpha_{\pi^\pm} - \alpha_{\pi^0}$. The t dependence is given by Eq. (31), $\alpha_\pi = \alpha_\pi(t \to +\infty)$. ChPT results are from Ref. [2–6].

This is the partial sum of the t integral of our master formula Eq. (18,28), and $\alpha_\pi = \alpha_\pi(t \to +\infty)$.

We notice that the difference between the electric polarizabilities of the charged pion and the neutral pion $\alpha_{\pi^\pm} - \alpha_{\pi^0}$ only depends on the left diagram in Figure 2 (and a disconnected diagram which we ignore). The numerical lattice result for this diagram is more precise than the other diagram. Therefore, we plot the difference in Figure 4. In the right plot, we find the finite volume effect is very small since the results from the 24D and 32D ensembles agree well.

4. Conclusion

We have derived several position space formulas for calculating the polarizabilities of both neutral and charged hadrons on the lattice, using pions as example. The finite volume effect of

all these type of formulas is exponentially suppressed by the lattice size and it is found to be numerically small in our lattice calculation as well. We also find the finite volume effects to be about 1% in one-loop ChPT, for the charged minus neutral difference. We have calculated the electric polarizabilities for both charged and neutral pion. At present, the lattice results tend to be lower than the ChPT predictions in the charged minus neutral difference. Further investigation is needed to reach firm conclusions. Improvement of the precision of the lattice calculation in the future is possible.

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