HVP contribution to Running Coupling and Electroweak Precision Science

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We investigate the impact of the latest Mainz/CLS collaboration's result for the hadronic vacuum polarization (HVP) on the electroweak (EW) precision science. The subject is closely related to the muon g-2 via the HVP. Both precision tests come under scrutiny with respect to physics Beyond the Standard Model. Our HVP calculation is used for the running electromagnetic coupling at low energy and linked at various matching energies to the higher energy running evaluated by phenomenological approaches. We predict the electromagnetic coupling at the Z-pole ($\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$), providing a lattice-driven input to EW-global fits. Our preliminary $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ is stable for a wide range of matching energies and comes with various systematics taken into account and consistent with phenomenological estimates.

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1. Introduction

Hadronic vacuum polarization (HVP) is a key quantity in both of the muon anomalous magnetic moment (muon g-2, a_μ) and the electromagnetic (QED) running coupling [1]. In precision science of a_μ, there exists 4.2σ tension between the Standard Model (SM) prediction [2] and the combined experimental results [3, 4]. Near future, the HVP will be directly measured in MUonE experiment [5].

Quark/hadronic contributions to QED coupling at Z-pole, $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$, is a fit parameter in the electroweak (EW) global fits. If a prior is given for $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ and Higgs boson mass ($M_{\rm higgs}$) is treated as a fit parameter with no prior, the output $M_{\rm higgs}$ shows 1.7σ tension [6] to the physical value. The EW global fits and $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ will become more important in the future collider experiments, such as international linear collider, with respect to exploring physics beyond the Standard Model (BSM).

In both of a_μ and $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$, HVP contributions have relied on R-ratio (normalized cross-section of $e^+e^- \to$ hadrons) integrals (dispersive method) and led to a major source of uncertainty. Lattice QCD has made significant progress in HVP calculations and will become competitive to the dispersive method near future [2].

In Refs. [7] and [8, 9], Mainz/CLS collaboration has investigated the leading-order (LO) HVP contribution to muon g-2 ($a_\mu^{\rm LO\text{-}HVP}$) and QED running coupling, respectively. In this proceedings, we examine $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ by using the Mainz/CLS data.

In the following section, we describe our strategy to connect Mainz/CLS data to the EW quantity $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$. Next in Sec. 3, we will show our result for $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ and compare it with the known results. And finally in Sec. 4, we will provide concluding remarks and future perspective.

2. Setup

The QED running coupling ($\alpha(s)$) in the *on-shell scheme* is parameterized as [10]

$$\alpha(s) = \frac{\alpha_0}{1 - \Delta\alpha_{\rm lep}(s) - \Delta\alpha_{\rm had}(s)} , \tag{1}$$

where $\alpha_0 \simeq 1/137.036$ denotes the fine-structure constant. In this proceedings, we focus on the hadronic correction $\Delta\alpha_{\rm had}(s)$. In particular, we are interested in the five quark-flavour hadronic contribution at Z-pole ($s = M_Z^2$), which we denote as $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$, which is proportional to the R-ratio integral and corresponds to a fit parameter in EW global fits.

In Lattice QCD, the output is a vector-current correlator and its Fourier transformation gives HVP in on-shell scheme ($\Pi^{\gamma\gamma}(-Q^2)$). Here, Q^2 represents *spacelike* momentum at hadronic scale. Thus, lattice QCD can provide

$$\Delta\alpha_{\rm had}^{\rm lat}(-Q^2) = 4\pi\alpha_0(\Pi^{\gamma\gamma}(-Q^2) - \Pi^{\gamma\gamma}(0)) . \tag{2}$$

Our gauge configurations contain the effects of full ($u\,d$, s) dynamical quarks. We measured ($u\,d\,s$)-quark-line connected and disconnected contributions to the vector current correlators. We also measured charm connected contributions, which is dominant comparing to the missing effects, the charm disconnected and charm sea-quarks, which we will discuss lator. Our data ensembles cover various lattice spacings, volumes, and pion/kaon masses. See Table 1 in Ref. [8] for our gauge ensemble collections. Our gauge configurations do not include isospin breaking (IB) effects (up/down quark mass difference and QED interaction). We have perturbatively estimated them based on QED$_L$ approach [11] using some of our ensembles [12] and taken account the result into one of systematic errors.

We have adopted the time-momentum representation (TMR) kernel to convert the vector-current correlators to the HVP. The TMR may be regarded as an interpolation scheme of discrete lattice momenta and allows us to approach arbitrary spacelike momenta Q^2. For selected momenta, ranging from $Q^2 = 0.01$ GeV^2 to 10 GeV^2, we have successfully taken the chiral and continuum extrapolations after correcting the finite volume effects at each ensemble. A detailed setup, definitions, and results are reported in Ref. [8].

The connection between $\Delta\alpha_{\rm had}^{\rm lat}(-Q^2)$ and $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ can be established via the so-called Euclidean split technique [13], i.e.

$$\begin{aligned}\Delta\alpha_{\rm had}^{(5)}(M_Z^2) &= \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\\ &+ \left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right] + \left[\Delta\alpha_{\rm had}^{(5)}(M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-M_Z^2)\right]_{\rm pQCD}\,,\end{aligned} \tag{3}$$

where the threshold energy Q_0^2 is selected around 5 GeV^2. In the literature [13, 14], $\Delta\alpha_{\rm had}^{(5)}(-Q_0^2)$ has been evaluated by employing the dispersive approach, while we utilize Mainz/CLS lattice estimate $\Delta\alpha_{\rm had}^{\rm lat}(-Q_0^2)$ [8] for that. The second line in Eq. (3) will be calculated by phenomenological methods with perturbative QCD (pQCD).

2.1 Heavy quark corrections to $\Delta\alpha_{\rm had}^{\rm lat}(-Q_0^2)$

For the central value of $\Delta\alpha_{\rm had}^{(5)}(-Q_0^2)$, we simply use our lattice estimate $\Delta\alpha_{\rm had}^{\rm lat}(-Q_0^2)$. We note that charm disconnected, charm sea-quark, and bottom quark contributions have not been included in our data but necessary for $\Delta\alpha_{\rm had}^{(5)}(-Q_0^2)$. These missing effects will be considered as additional systematic uncertainty, which we explain in the following.

In $a_\mu^{\rm LO\text{-}HVP}$ [15], the charm disconnected contribution was smaller than 1% of up/down/strange disconnected contributions. In HVP, we assume similar amounts of charm disconnected effects, which are at most $O(0.01)$% in the total HVP.

For the charm sea-quark effects, we adopt a phenomenological estimate. In the R-ratio, the leading charm contributions appear as D-meson pair creations. For D^+D^-,

$$R_{D^+D^-}(s) = \frac{1}{4}\left(1 - \frac{4m_{D^+}^2}{s}\right)^{3/2} |F_{D^+}(s)|^2\,, \tag{4}$$

and similar expression follows for the $D^0\bar{D}^0$ and $D_s^+D_s^-$ channels. Since the form factor F_{D^+} is not known precisely, we will approximate it with values at $s = 0$, which amounts to treating D-mesons in the scalar QED framework and replacing with electromagnetic charges: $\{F_{D^0}(s), F_{D^+}(s), F_{D_s^+}\} \to \{2/3, -1/3, -1/3\}$. Note that up, down, or strange quarks are assigned as valence quarks responsible for charges, so that D-mesons contain charm sea-quarks. The corresponding contributions to QED

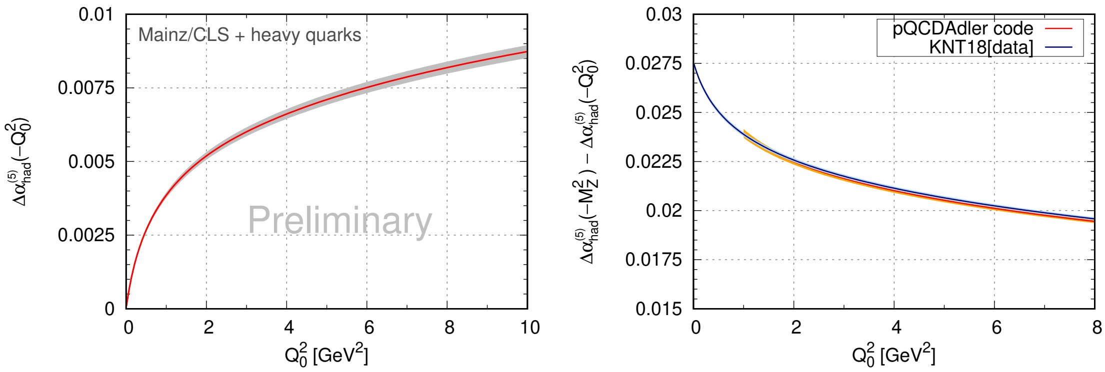

Figure 1: Left: $\Delta\alpha^{(5)}_{\rm had}(-Q_0^2)$ with our lattice data and heavy quark corrections. The gray band shows a preliminary and conservative estimate of uncertainty. Right: $[\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-Q_0^2)]$ in the Adler function (red) [19] and dispersive methods (blue) [20].

running coupling reads

$$\begin{aligned}\Delta\alpha^{\rm D}_{\rm had}(-Q_0^2) &= \frac{\alpha_0}{3\pi}\int ds \frac{(R_{D^0 D^0} + R_{D^+D^-} + R_{D_s^+ D_s^-})(s)}{s(s+Q_0^2)}\ ,\\ &= \frac{4}{9} f(Q^2/m_{D^0}^2) + \frac{1}{9} f(Q^2/m_{D_s}^2) + \frac{1}{9} f(Q^2/m_{D^+}^2)\ ,\end{aligned} \tag{5}$$

where

$$f(z) \equiv \frac{1}{144\pi^2}\left[-8(1+3/z) + 3(1+4/z)^{3/2}\log\left[\frac{2+z+\sqrt{z(4+z)}}{2}\right]\right]\ . \tag{6}$$

We find 2.6 permil contributions of $\Delta\alpha^{\rm D}_{\rm had}(-Q_0^2)$ over $\Delta\alpha_{\rm had}(-Q_0^2)$ at $Q_0^2 = 5.0$ GeV^2.

Finally, we estimate the bottom quark contributions. To this end, we utilize the result by the HPQCD collaboration for the lowest four time-moments of the HVP [16]. We construct Padé approximants from the moments, which result in a few-permil effect on the total hadronic running of the coupling. This effect is larger than the 0.04 percent effect reported in $a_\mu^{\rm LO\text{-}HVP}$ [17]. This is due to the fact that the running coupling scale Q_0^2 is not well separated from the bottom quark mass, in contrast to the muon mass case.

All contributions from heavy quarks discussed here are included in the estimate of $\Delta\alpha^{(5)}_{\rm had}(-Q_0^2)$ as additional systematic errors, added in quadrature in our lattice results [8]. The resulting increase in the total error is only a few permil and thus tiny, but comparable to some of the lattice-originated systematics.

In the left panel of Fig. 1, we show $\Delta\alpha^{(5)}_{\rm had}(-Q_0^2)$ evaluated with our lattice data and heavy quark corrections. The gray band shows a preliminary and conservative error, which contains both of lattice- and heavy-quark-originated uncertainty and amounts to a few percent.

2.2 Higher energy corrections

We shall now investigate the remaining part, the second line in Eq. (3). If threshold energy Q_0^2 is sufficiently large, the first square bracket in Eq. (3) is dominated by perturbative QCD. In practice, however, we take $Q_0^2 \sim 5$ GeV^2 for which minor non-perturbative (NP) effects are non-negligible. We adopt two approaches: Adler function method and dispersive method.

The Adler function is defined as the derivative of the timelike QED running coupling as

$$D(Q_0^2) = \frac{3\pi}{\alpha_0}\left[s\frac{d\Delta\alpha_{\rm had}^{(5)}(s)}{ds}\right]_{s=-Q_0^2} . \tag{7}$$

This quantity may be evaluated by the dispersion integral of the R-ratio, $Q_0^2 \int dsR(s)/(s+Q_0^2)^2$. In our analyses, we adopt a different strategy; for large Q_0^2, the Adler function is calculable with pQCD plus minor NP corrections [14, 18]. We have taken account of the three-loop pQCD with some of four- and five-loop corrections. For NP part, the operator product expansion and Padé approximations are considered. Thus, we do not have to use R-ratio data. This approach has been implemented in the public code [19], which we have utilized. Once $D(Q^2)$ is obtained, we can calculate

$$\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right]_{\rm pQCD'} = \frac{\alpha_0}{3\pi}\int_{Q_0^2}^{M_Z^2}\frac{dQ^2}{Q^2}D(Q^2) , \tag{8}$$

where pQCD′indicates pQCD with minor NP corrections considered in the estimate of $D(Q^2)$. For $Q_0^2 = 5$ GeV^2, uncertainty is about 1% originating to input values of the strong coupling at Z-pole and heavy quark pole-masses.

We shall move on to the second approach, dispersive method,

$$\left[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)\right] = \frac{\alpha_0}{3\pi}(M_Z^2 - Q_0^2)\int_{m_{\pi^0}^2}^{\infty} ds\frac{R(s)}{(s+Q_0^2)(s+M_Z^2)} . \tag{9}$$

The appearance of Q_0^2 in the denominator of the integrand implies that contributions from the R-ratio at low energies are suppressed along with any experimental uncertainties in their determination. For R-ratio integral, we have used KNT18 data [20] with a full covariance matrix for error estimates, which is about 1% or less at $Q_0^2 = 5$ GeV^2.

In the right panel of Fig. 1, we show $[\Delta\alpha_{\rm had}^{(5)}(-M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-Q_0^2)]$ as a function of Q_0^2. The red and blue lines are the results with the Adler function and dispersive methods, respectively. Both methods are consistent within the errors. Since the Adler function method does not rely on R-ratio data, the result would be reliable at large Q_0^2 without suffering from experimental systematics. For a small Q_0^2, however, larger uncertainty emerges from NP corrections in $D(Q_0^2)$, and we cannot extract a result at very small Q_0^2 due to the Landau pole artifact of the strong running coupling.

Finally, the second combination in square brackets in Eq. (3) provides the link between the spacelike and timelike regions at the Z boson mass, which has been evaluated in pQCD by Jegerlehner [21, 22],

$$\left[\Delta\alpha_{\rm had}^{(5)}(M_Z^2) - \Delta\alpha_{\rm had}^{(5)}(-M_Z^2)\right] = 0.000045(2) . \tag{10}$$

3. Result

We shall now extract our results for $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$ using Eq. (3). The threshold Q_0^2 should be as large as possible so that our lattice result accounts for low energy contributions in $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$.

However, $Q_0^2 = 7\ \mathrm{GeV}^2$ is the border line bellow which the continuum extrapolation is well controlled. Conservatively, we have chosen $Q_0^2 = 5\ \mathrm{GeV}^2$. We find

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)\big|_{\mathrm{Lattice+KNT18[data]}} = 0.0278(1\%)\ ,$$
$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)\big|_{\mathrm{Lattice+pQCD'[Adler]}} = 0.0277(1\%)\ . \tag{11}$$

The error estimate is still under debate and we tentatively put 1% uncertainty, which is conservative enough. We stress that the above values/errors are *preliminary*, and the official results from the Mainz/CLS collaboration are under preparation. In "Lattice + KNT18[data]" approach, our $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ is stable for $Q_0^2 \in [2,7]\ \mathrm{GeV}^2$. In "pQCD′[Adler]", the reliable region is limited to avoid Landau pole artifact: $Q_0^2 \in [4,7]\ \mathrm{GeV}^2$. The assigned 1% error covers enough the fluctuations associated with varying Q_0^2 in those ranges.

In Fig. 2, we compile results for $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ obtained using our lattice estimate of HVP, the standard dispersive approach, as well as EW global fits. Our results (11) are plotted as the first and second top symbols (red square and diamond), respectively. The gray color error-bars show 1% preliminary estimates for uncertainty.

The central values of our results are somewhat larger than the green circles which display results based on the standard dispersive approach, where the R-ratio integration is performed over the entire momentum range [22–24]. This trend originates to the lattice data $\Delta\alpha^{\mathrm{lat}}_{\mathrm{had}}(-Q_0^2)$, which tends to be larger than the results from the dispersive method beyond uncertainty. At Z-pole, however, the tension becomes much milder because of additional uncertainties from the high energy part.

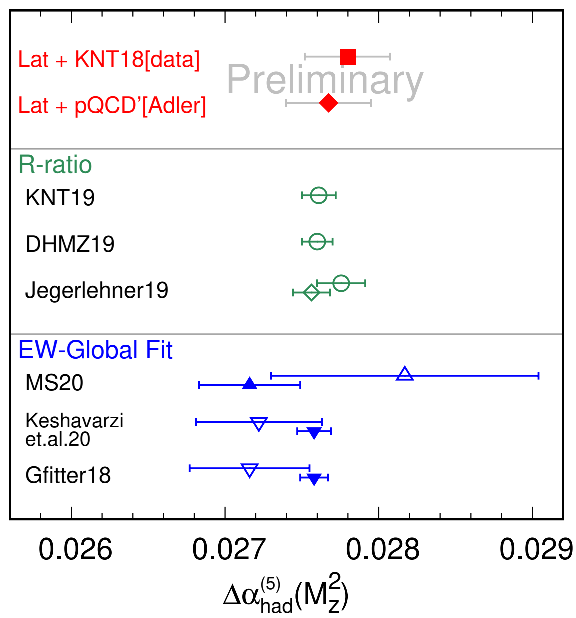

Figure 2: Compilation of $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$. See text for details. The first two data points (filled-red square/diamond symbols) represent the results from the Euclidean split technique using our lattice estimate for $\Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)$. 1% uncertainty is put as a preliminary estimate as indicated by gray color error-bars. Green circles denote results based on the standard dispersive approach. From top to bottom, Refs. [23], [24], and [22]. The green-diamond has used the Euclidean split technique. Blue upper/lower triangles represent the results from the EW global fits, published in Refs. [26], [27], and [6].

In the figure, blue upper/lower triangles represent the results from the EW global fits. For the upper open triangle, both of M_{higgs} and $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ are treated as fit parameters without priors. For the upper filled triangle, M_{higgs} is fixed at the physical value but $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ is treated as a fit parameter without priors. In the lower triangles, M_{higgs} is a fit parameter with the prior at the physical value. For the open/filled symbols, $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ is treated as a fit parameter without/with priors from R-ratio results. Our results are consistent with all of EW-global fits results within the errors.

In our data, more stringent estimates for uncertainty will be soon available. The error-bars will be significantly smaller and the soft-tension may emerge against some of EW-global fits results.

4. Summary

We have investigated Quark/hadronic contributions to QED running coupling at Z-pole, $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$, where we have utilized our lattice QCD data for the running at hadronic energy [8, 9]. In order to connect $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ defined in the timelike region and the lattice output $\Delta\alpha^{\rm lat}_{\rm had}(-Q_0^2)$ in the spacelike region at hadronic scale ($Q_0^2 \sim 5$ GeV^2), we have adopted the Euclidean split technique [13] shown in Eq. (3).

Our lattice data $\Delta\alpha^{\rm lat}_{\rm had}(-Q_0^2)$ have included full (u, d, s)-quark and charm-connected valence-quark contributions with finite volume corrections. Isospin breaking effects are taken account into the systematic errors. The missing effects (charm disconnected, charm sea-quark, and bottom quark contributions) have been considered as additional systematic uncertainty of a few permil, which is comparable with some of lattice-origin systematic errors. The total error is a few percent level, which is still preliminary.

For the energy region of $[Q_0^2, M_Z^2]$, we have adopted Adler function or dispersive method. In the former, pQCD plus minor NP expressions have been known [14, 18] and implemented in the public code [19], which we have utilized. In the dispersive method, we have used KNT18 data [20] with a full covariance matrix for error estimates. Both methods have resulted about 1% uncertainty for $Q_0^2 = 5$ GeV^2 in the first solid bracket in Eq. (3).

For the second combination in square brackets in Eq. (3), we have quoted a pQCD estimate by Jegerlehner [21, 22], which is very small.

By adding all terms in Eq. (3), we have found Eq. (11) and compared with the standard dispersive approach as well as EW global fits in Fig. 2. Our results have been *preliminary* with very conservative 1% uncertainty and consistent with the other estimates.

As a future perspective, more stringent estimates for uncertainty will be soon available. The error-bars will be significantly smaller and a soft-tension may emerge against some of phenomenological results. Our lattice-based $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ will provide an important input for EW global fits and BSM searches.

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