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Spin-1 fields and RG flows in 4 dimensions

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The most general local, classically scale invariant, perturbatively renormalizable, globally $SU(N)$ invariant Lagrangian is constructed for spin-1 fields in 4 dimensions. The total number of independent couplings is 7 and the 1-loop β-functions are computed in the $\overline{\text{MS}}$ scheme. A number of asymptotically free RG flows are identified corresponding to non-trivial QFTs. None of these are gauge theories. The details of the large-N limit are also worked out and it is shown that the RG phase space is qualitatively similar for all $N > 5$ including the $N \to \infty$ limit.

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1. Introduction

In this work a straightforward QFT question is asked: what type of QFT can describe interacting, asymptotically free spin-1 (vector) fields in 4 dimensions? If gauge invariance is imposed Yang-Mills theory is unique and well-known, hence we do not require gauge invariance here only a *global* $SU(N)$ invariance, beyond locality, perturbative renormalizability and classical scale invariance. The latter requirement is not essential it simply limits the number of allowed couplings to those which are dimensionless.

At first one might think that gauge theory is the only option for having asymptotic freedom with spin-1 fields but it turns out this is not the case, at least in Euclidean signature. The explicit computation of the 1-loop β-functions in the space of 7 couplings (corresponding to the 7 allowed operators in the most general Lagrangian) shows that for any N a finite number of asymptotically free RG flows exist, more precisely 4 of these for $N > 5$. These RG flows correspond to non-trivial perturbative and asymptotically free quantum field theories which are not gauge theories. Straightforward large-N scaling works as expected, and the qualitative features of the $N \to \infty$ model is the same as with finite $N > 5$.

Similar questions as the one addressed in this work were discussed in the abelian case in [1] and rather qualitatively for the non-abelian case in [2].

The most general Lagrangian for the study of spin-1 fields is given in section 2. The β-functions are computed in section 3 to 1-loop and the resulting RG flows are studied as well. Asymptotically free RG flows are identified and the large-N limit is also spelled out. Finally, section 4 contains our conclusions and outlook to possible refinements and further research.

2. Lagrangian

The spin-1 fields will be labelled by A_μ^a in the adjoint representation of $SU(N)$. We seek the most general 4-dimensional, globally $SU(N)$ and Euclidean invariant Lagrangian with at most two derivatives, dimensionless couplings and perturbatively renormalizable interactions. It is straightforward to show that up to total derivatives a possible parametrization in terms of 7 couplings,

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$(z, g_1, g_2, g_3, g_4, h_1, h_2)$ is,

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}\partial_\mu A^a_\nu \partial_\mu A^a_\nu - \frac{1}{2}\left(1 - \frac{1}{z}\right)(\partial_\mu A^a_\mu)^2 + h_1 \tilde{\mathcal{O}}_1 + h_2 \tilde{\mathcal{O}}_2 + \mathscr{V} \\
\tilde{\mathcal{O}}_1 &= A^a_\mu A^b_\nu \partial_\mu A^c_\nu d_{abc} \\
\tilde{\mathcal{O}}_2 &= A^a_\mu A^b_\nu \partial_\mu A^c_\nu f_{abc} \\
\mathscr{V} &= \sum_{i=1}^{4} g_i \mathcal{O}_i \\
\mathcal{O}_1 &= \frac{1}{8} A^a_\mu A^b_\mu A^c_\nu A^g_\nu d_{abe} d_{cge} \geq 0 \\
\mathcal{O}_2 &= \frac{1}{8N} (A^a_\mu A^a_\mu)^2 \geq 0 \\
\mathcal{O}_3 &= \frac{1}{8N} A^a_\mu A^b_\mu A^a_\nu A^b_\nu \geq 0 \\
\mathcal{O}_4 &= \frac{1}{4} A^a_\mu A^b_\mu A^c_\nu A^g_\nu f_{ace} f_{bge} \geq 0 \,,
\end{aligned} \tag{1}$$

where f_{abc} is the totally anti-symmetric and d_{abc} is the totally symmetric tensor of $SU(N)$. For a well-defined path integral representation $z \geq 0$ is required as well as a non-negative potential $\mathscr{V}$. The requirement on (g_1, g_2, g_3, g_4) for the latter to hold is non-trivial, one of the following two conditions is necessary,

$$\begin{aligned}
g_1 &\geq 0 \,, \qquad g_2 + g_3 \geq -g_1(N-2) \\
g_1 &\leq 0 \,, \qquad g_2 + g_3 \geq -g_1 \frac{2(N-2)^2}{N-1} \,,
\end{aligned} \tag{2}$$

and any one of the following is sufficient,

$$\begin{array}{llll}
g_1 \geq 0 \,, & 4g_2 + g_3 \geq 0 \,, & g_3 \geq 0 \,, & g_4 \geq 0 \\
g_1 \geq 0 \,, & 4g_2 + g_3 \geq 8g_1 \,, & g_3 \geq 0 \,, & 3g_4 \geq -2g_1 \\
g_1 \geq 0 \,, & g_2 + g_3 \geq 0 \,, & g_3 \leq 0 \,, & g_4 \geq 0 \\
g_1 \leq 0 \,, & g_2 + 2(N-1)g_1 \geq 0 \,, & g_3 \geq 0 \,, & g_4 \geq 0 \,.
\end{array} \tag{3}$$

A complete set of minimal necessary *and* sufficient conditions is presently not known.

3. β-functions and RG flows

Since all possible terms allowed by symmetry are included in (1), all terms are perturbatively renormalizable and a well-defined path integral can be defined in Euclidean signature, the β-functions of the 7 couplings can be computed in a straightforward manner. The diagrams contributing in $\overline{\text{MS}}$ at 1-loop are listed in figure 1. For simplicity let us introduce $g_5 = h_1^2$ and $g_6 = h_2^2$. Schematically, the 1-loop β-functions are,

$$\begin{aligned}
\mu \frac{dz}{d\mu} &= \beta_z = z L_z(g_5, g_6) \\
\mu \frac{dg_i}{d\mu} &= \beta_i = Q_i(g_1, g_2, g_3, g_4, g_5, g_6) \qquad i = 1, 2, 3, 4 \\
\mu \frac{dg_i}{d\mu} &= \beta_i = g_i L_i(g_1, g_2, g_3, g_4, g_5, g_6) \qquad i = 5, 6 \,,
\end{aligned} \tag{4}$$

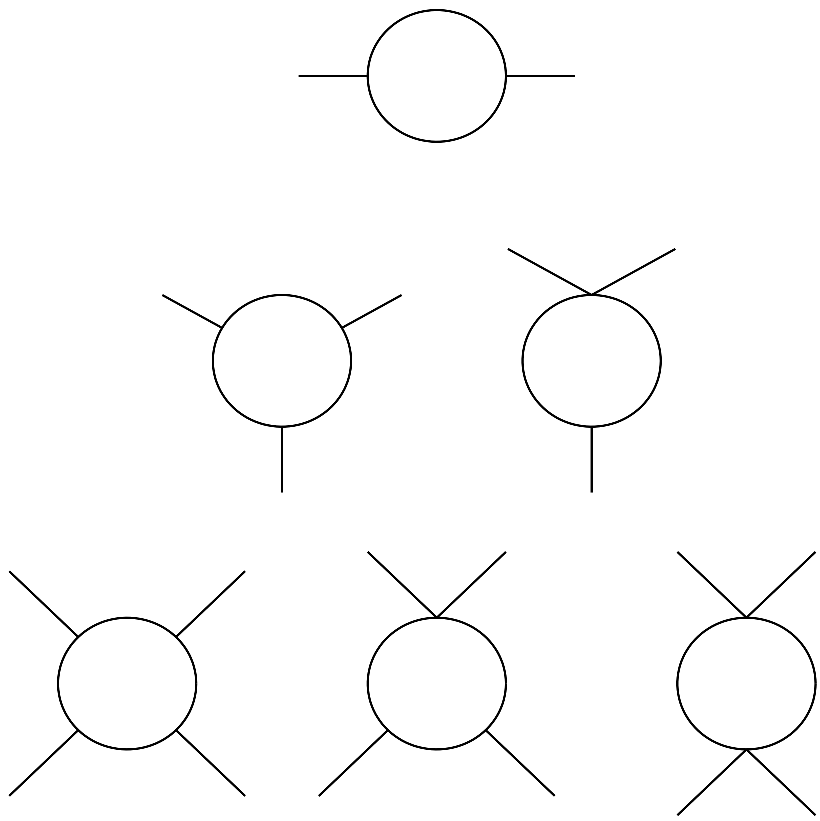

Figure 1: Diagrams contributing at 1-loop order in dimensional regularization. Rows from top to bottom: propagator, renormalization of z; 3-vertex, renormalization of (h_1, h_2); 4-vertex, renormalization of (g_1, g_2, g_3, g_4).

where $Q_{1,2,3,4}$ are quadratic monomials in the couplings with coefficients which are themselves polynomial in z and $L_{z,5,6}$ are linear in the couplings and also polynomial in z. All expressions depend on N as well. Clearly, z, g_5, g_6 renormalize multiplicatively. The precise form of the β-functions can be found in [3], which were computed with the extensive help of FORM [7–9].

There is a line of Gaussian fixed points in the space of couplings given by an arbitrary z and $g_i = 0$ for all $i = 1, \ldots, 6$. Clearly, $\beta_z = \beta_i = 0$ everywhere on this line. We will be looking for RG flows which in the UV end up on this line asymptotically. Such an RG trajectory will define a non-trivial perturbative quantum field theory. Both g_5 and g_6 can not be identically zero, but for the sake of simplicity let's assume $g_5 = 0$. The situation with $g_5 \neq 0$ is spelled out in detail in [3]. Now we are dealing with 6 couplings $(z, g_1, g_2, g_3, g_4, g_6)$ and look for RG flows which for $\mu \to \infty$ behave as,

$$\begin{aligned} g_i(\mu) &\sim 16\pi^2 \frac{C_i}{\log \frac{\mu}{\Lambda}}, \qquad i = 1, \ldots, 4, 6 \\ z(\mu) &\sim const\ , \end{aligned} \tag{5}$$

with some scale Λ and constants C_i which are subject to the non-trivial positivity constraint mentioned in section 2. Assuming an asymptotically free RG flow as in (5), clearly the ratios $r_i = g_i/g_6$ for $i = 1, \ldots, 4$ are constant towards the UV, $r_i \to C_i/C_6$. Hence our goal is to find UV fixed points in the space (z, r_1, r_2, r_3, r_4) and asymptotically free g_6, which is a straightforward exercise once the β-functions are known explicitly. The results will be solutions of complicated polynomial equations for every N and are given in table 1.

It is clear from table 1 that for any N there is a finite number of asymptotically free RG flows. Once an RG flow is identified it may be characterized by the stability or instability of $\mathscr{V}$ and also by its stability in the RG sense.

The point $z = 0$ is always a fixed point and the only fixed point for $N > 5$. The $N \leq 5$ cases are qualitatively different from $N > 5$ also in the sense that in the latter case there is a unique fixed point in the $z = 0$ plane which is stable in the RG sense. Furthermore, all fixed points for $N > 5$ correspond to a stable potential $\mathscr{V}$. Fixed points which correspond to a stable $\mathscr{V}$ lead to perfectly well-defined perturbative quantum field theories of spin-1 fields, which are not gauge theories. Those which are stable in the RG sense as well are insensitive to small deformations, as usual. The $z = 0$ fixed points can be interpreted as having a constraint $\partial_\mu A^a_\mu = 0$ because of the appearance of the coupling $1/z$ in (1). As a result the original 4 degrees of freedom are reduced to 3. Note that the $\partial_\mu A^a_\mu = 0$ constraint has nothing to do with gauge fixing since gauge invariance is not present to begin with. The constraint arose dynamically from the nature of the particular UV fixed points.

It should be noted that we have been working in Euclidean signature and Wick rotation back to a unitary theory in Minkowski space time is not possible. This is because, as is well-known, gauge invariance is required to kill off the negative norm states which is of course not present on any of the RG flows considered here. In order to study how gauge symmetry emerges in a perturbative treatment such as ours, one must include ghost fields; for more details see [3].

Another aspect of table 1 is the smoothness of the large-N limit. Similarly to the situation in gauge theory the $N \to \infty$ limit is performed at constant Ng_i. The fixed point ratios $r_i = C_i/C_6$ have well-defined large-N limits of course and so does NC_i. Qualitatively all $N > 5$ cases are similar, the strict $N \to \infty$ limit only makes some of the ratios between different fixed points degenerate. The 4 fixed points only differ in r_2 and r_3 in this limit and one of them is stable in the RG sense.

4. Conclusion and outlook

In this work a seemingly simple QFT question was posed: what is the most general QFT describing a set of spin-1 fields with global $SU(N)$ invariance. The RG phase space was mapped out in the 1-loop approximation and a finite number of asymptotically free RG flows were found for any N. More precisely, only classically scale invariant couplings were considered, i.e. dimensionless couplings. Note that in this case scale invariance does not imply conformal invariance [4–6]. If dimensionful couplings are allowed, but global $SU(N)$ invariance is still imposed, a mass term can be added to the Lagrangian,

$$\mathscr{L}_m = \frac{m^2}{2} A^a_\mu A^a_\mu \,. \tag{6}$$

The perturbative expansion of the corresponding anomalous dimension is beyond the scope of the present work but would be interesting to work out in the future.

N	z	r_1	r_2	r_3	r_4	NC_6	$\mathscr{V}$
3	0	0.054652	0.122003	0.485317	0.970537	0.138656	stable
3	0	0.064145	0.133021	0.665179	0.964086	0.137153	stable
3	0	-0.647582	-0.580231	1.889786	1.204615	0.138656	unstable
3	0	-0.562664	-0.493787	1.918797	1.173022	0.137153	unstable
3	25/3	0.000334	0.079592	-0.251950	1.020083	0.148484	unstable
3	25/3	0.010673	0.074642	-0.144563	1.004360	0.145542	unstable
3	25/3	-0.108161	-0.028903	-0.034960	1.056248	0.148484	unstable
3	25/3	-0.080316	-0.016348	0.037417	1.034690	0.145542	unstable
4	0	0.044841	0.106784	0.351786	0.979028	0.140948	stable
4	0	0.074162	0.083060	1.368389	0.960858	0.136196	stable
4	25/3	0.004413	0.111209	-0.323177	1.013219	0.146900	unstable
4	25/3	0.016297	0.243636	-0.344606	0.995511	0.145494	unstable
4	25/3	0.017435	0.119096	-0.223217	0.997309	0.144605	unstable
4	25/3	0.017931	0.235838	-0.327356	0.993784	0.145177	unstable
5	0	0.042754	0.103223	0.327436	0.981138	0.141567	stable
5	0	0.054311	1.073479	0.536511	0.957994	0.142046	stable
5	0	0.067257	-0.066910	1.896637	0.967324	0.136857	stable
5	0	0.069027	0.516675	1.600829	0.956705	0.138188	stable
5	25/3	0.012566	0.149475	-0.375377	1.003344	0.145326	unstable
5	25/3	0.021321	0.180212	-0.347564	0.993910	0.144298	unstable
6	**0**	**0.041817**	**0.101590**	**0.316866**	**0.982127**	**0.141864**	**stable**
6	0	0.048648	1.137578	0.428569	0.966346	0.142530	stable
6	0	0.059916	-0.214070	2.277709	0.972682	0.137748	stable
6	0	0.062649	0.434621	2.043391	0.963808	0.138624	stable
7	**0**	**0.041301**	**0.100682**	**0.311136**	**0.982683**	**0.142032**	**stable**
7	0	0.045944	1.161333	0.383774	0.971232	0.142626	stable
7	0	0.054742	-0.321825	2.541816	0.976034	0.138570	stable
7	0	0.057376	0.412019	2.341096	0.968497	0.139238	stable
10	**0**	**0.040625**	**0.099483**	**0.303720**	**0.983425**	**0.142259**	**stable**
10	0	0.042691	1.184351	0.334451	0.977917	0.142606	stable
10	0	0.047136	-0.495636	2.966144	0.980468	0.140207	stable
10	0	0.048800	0.401667	2.839408	0.975942	0.140564	stable
50	**0**	**0.040047**	**0.098451**	**0.297474**	**0.984071**	**0.142458**	**stable**
50	0	0.040124	1.198242	0.298567	0.983855	0.142474	stable
50	0	0.040300	-0.680516	3.425269	0.983967	0.142360	stable
50	0	0.040376	0.410710	3.418741	0.983752	0.142375	stable
100	**0**	**0.040030**	**0.098420**	**0.297287**	**0.984091**	**0.142464**	**stable**
100	0	0.040049	1.198589	0.297559	0.984037	0.142468	stable
100	0	0.040093	-0.686738	3.440904	0.984065	0.142439	stable
100	0	0.040112	0.411281	3.439259	0.984011	0.142443	stable
∞	**0**	**0.040024**	**0.098409**	**0.297224**	**0.984097**	**0.142466**	**stable**
∞	0	0.040024	1.198704	0.297224	0.984097	0.142466	stable
∞	0	0.040024	-0.688818	3.446135	0.984097	0.142466	stable
∞	0	0.040024	0.411476	3.446135	0.984097	0.142466	stable

Table 1: Non-trivial fixed points with $g_5 = 0$ for the ratios $r_i = g_i/g_6 = C_i/C_6$, and the coefficient C_6; see (5). The last column indicate whether the potential $\mathscr{V}$ is stable or not. For $N > 5$ there is a unique fixed point for which $\mathscr{V} \geq 0$ and is stable in the RG-sense in the $z = 0$ plane, these are shown in bold.

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Similarly, a worthwhile extension of the present work would be the calculation of the β-functions to 2-loops or more. Since asymptotic freedom can be established by the 1-loop calculation alone, it is expected that the main conclusion will not change, namely that for any N well-defined, asymptotically free, perturbative Euclidean quantum field theories exist, which are not gauge theories.

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