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Inhomogeneous Phases in the Chiral Gross-Neveu Model on the Lattice

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We discuss possible existence of inhomogeneous phase in low temperature and high density region in the 1+1 dimensional chiral Gross-Neveu model on the lattice.

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1. Introduction

Understanding the quantum chromodynamics (QCD) phase diagram is one of important issues in elemental particle physics, hadron physics and nuclear physics. In high temperature and low density region, intensive studies from both of theoretical side and experimental side clarify the QCD equation of state and even bulk properties of the QCD matter [1]. However, in low temperature and high density region, the QCD phase transition and possible phases in the QCD phase diagram are in the middle of being discussed. For example, though lattice QCD calculation is a powerful tool for investigation of non-perturbative aspects, usual Monte Carlo simulation is not applicable because of a notorious problem, the sign problem. High-energy heavy-ion collisions which achieve success to obtain information of Quark-Gluon Plasma in low density region can not reach the high density region at present. On other hand, the recent observation of neutron stars and gravity waves sheds light on exploration of the QCD equation of state at finite density which is revealed from relation between mass and radius of neutron star [2].

In low temperature and high density region possible interesting phases are discussed from effective theories which exhibit the same symmetries as QCD, chiral symmetry. They are pion condensation, color super conducting phases and inhomogeneous chiral condensation. For spatial dependence of inhomogeneous chiral condensation, various kinds of structures are discussed; chiral density wave, kinks as solitonic solutions [3]. Usually the investigations are limited to specific ansatz such as a selected set of Fourier modes. One of pioneering works is carried out without using ansatz for spacial structure of chiral condensation [4]. Also, lattice calculation of the 1+1 Gross-Neveu (GN_2) model is performed [5]. They focus on characteristic features of GN_2 model with finite number of flavor from comparison with those with infinite number of flavor [5]. Here, to obtain the insight of the QCD phase diagram at high density, we apply lattice simulation to the 1+1 dimensional chiral GN ($\chi\mathrm{GN}_2$) model which does not have the sign problem. In the $\chi\mathrm{GN}_2$ model inhomogeneous chiral condensate exists in $N_f \to \infty$ limit [6].

This paper is organized as follows. We begin in Section 2 by showing brief explanation of the $\chi\mathrm{GN}_2$ model. In Section 3 we explain the lattice simulation setup for the $\chi\mathrm{GN}_2$ model. In Section 4, we show our numerical results of the $\chi\mathrm{GN}_2$ model, temperature dependence of vacuum and inhomogeneous phase. We end in Section 5 with our conclusions.

2. The chiral Gross-Neveu Model

The 1+1 dimensional chiral Gross-Neveu ($\chi\mathrm{GN}_2$) model is a relativistic quantum field theory describing N_f flavors of Dirac fermion with both scalar and pseudscalar four-fermion interaction terms. The fermions described by a field $\psi = (\psi_1, \ldots, \psi_{N_f})$ have two-component Dirac spinor indices. The lagrangian is given by

$$\mathcal{L} = \bar{\psi} i \gamma^\nu \partial_\nu \psi + \frac{g^2}{2N_f} \left[(\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2 \right], \tag{1}$$

where g^2 is a coupling constant. From point of view of QCD, important features of the $\chi\mathrm{GN}_2$ model are asymptotic freedom and spontaneous symmetry breaking of continuum chiral symmetry in large N_f limit [6]. In particular, because it has no sign problem, Monte Carlo simulation is applicable.

To perform the fermion integration one follows Hubbard and Stratonovich by introducing an auxiliary scalar fields $\sigma(x_1, x_2)$ and $\pi(x_1, x_2)$ which are represented by the operators $\bar{\psi}\psi$ and $\bar{\psi}i\gamma_5\psi$ respectively in the integration terms,

$$S_{\sigma,\pi} = \int dx_1 dx_2 \left[\bar{\psi}M\psi + \frac{N_f}{2g^2}\left(\sigma^2 + \pi^2\right)\right], \qquad Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\pi\, e^{-S_{\sigma,\pi}}, \tag{2}$$

where

$$M = \gamma_\nu\partial_\nu + \sigma + i\gamma_5\pi - \gamma_2\mu \tag{3}$$

is the Dirac operator. In Eqs. (2) and (3) a chemical potential μ is included for study on the system at finite fermion density. Expectation values of operators $O(\psi, \bar{\psi}, \sigma, \pi)$ in the grand canonical ensemble are given by

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\pi\, O(\psi, \bar{\psi}, \sigma, \pi)e^{-S_{\sigma,\pi}}. \tag{4}$$

Note that the integration is over fermion fields, which are anti-periodic in the Euclidean time direction, with period $\beta = 1/T$, while the auxiliary scalar fields are periodic.

Integrating over the fermion fields leads to

$$S_{\text{eff}} = \frac{1}{2g^2}\int dx_1 dx_2 \left(\sigma^2 + \pi^2\right) - \log\det M, \qquad Z = \int \mathcal{D}\sigma\mathcal{D}\pi\, e^{-N_f S_{\text{eff}}} \tag{5}$$

with expectation values of operators $O(\sigma, \pi)$ given by

$$\langle O\rangle = \frac{1}{Z}\int \mathcal{D}\sigma\mathcal{D}\pi\, O(\sigma, \pi)e^{-N_f S_{\text{eff}}}. \tag{6}$$

3. Simulation setup

We determine lattice simulation setup of the χGN$_2$ model by reference to that of the GN$_2$ model [5]. To explore the existence of an inhomogeneous phase and the μ-T phase diagram in the χGN$_2$ model, we generate a large number of ensembles of field configurations $\sigma(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ (Tab. 1). We employ naive fermion for analyses. In the previous study on the GN$_2$ model [5], they found that there is no lattice fermion dependence to the phase structure of the GN$_2$ model, using naive fermion and SLAC fermion. It suggests that one can use both kinds of fermion actions. We fix the spatial extents N_s to $N_s = 32$ in the calculation, though $N_s = 64$ is used in Ref. [5]. Furthermore we also find that in the homogeneous phase $N_s = 32$ is enough large to give the same phase structure as that in continuum theory with large N_f [7]. We carry out the calculation on $N_t = 4, 6, \ldots 32$ due to the limitation of evaluation of fermion determinant in naive fermion.

4. Numerical results

4.1 Finite temperature at $\mu = 0$

First we investigate the phase structure of the χGN$_2$ model at vanishing chemical potential, changing temperature. We calculate the expectation values of $\Sigma^2 = \langle\bar{\sigma}^2\rangle$ and $\Pi^2 = \langle\bar{\pi}^2\rangle$ as a

Table 1: Ensembles of field configurations for the χGN$_2$ model.

fermion	N_f	$N_s = L/a$	$N_t = 1/Ta$	g^2	$a\Delta_0$	μ/Δ_0
				1.9332	0.4153(3)	0.0, 0.8886
naive	8	32	4, 6, . . ., 32	1.8132	0.3791(2)	0.0, 0.8886
				1.7132	0.3436(2)	0.0, 0.8886

function of Monte Carlo time τ (Fig. 1 (a)). Here we measure the observables on thermalized configurations separated by 10 Monte Carlo units. We observe that amplitudes of Σ^2 and Π^2 are not constant: they are fluctuating to Monte Carlo time with negative correlation to the other. This behavior suggests that $\Delta^2 = \Sigma^2 + \Pi^2$ is constant. In Fig. 1 (b) we show Δ^2 as a function of Monte Carlo time τ. The values of Δ^2 are constant, which means that Δ^2 is a candidate of an order parameter for chiral symmetry in the χGN$_2$ model.

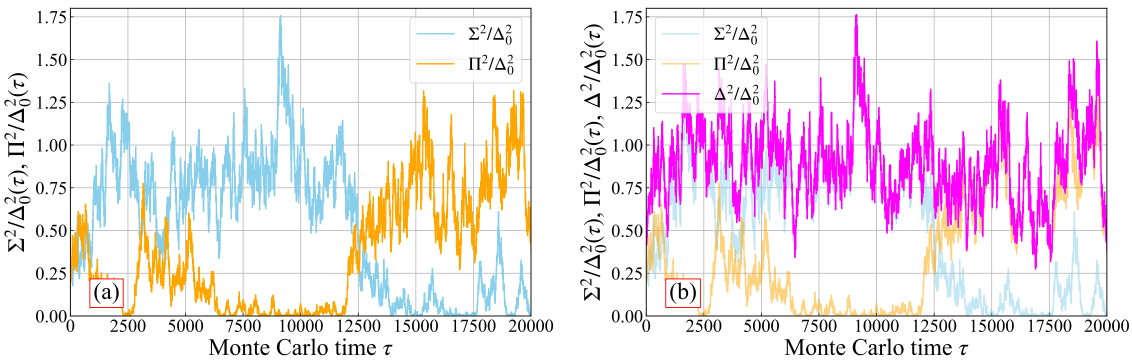

Figure 1: (a) Σ^2 (light blue solid line) and Π^2 (orange solid line) as a function of Monte Carlo time. (b) Δ^2 (magenta solid line) as a function of Monte Carlo time (b), together with Σ^2 and Π^2.

In Fig. 2, Δ^2 as a function of temperature is shown at vanishing chemical potential. Here we perform the calculation with coupling constants g^2=1.7132, 1.8132 and 1.9332 to check lattice spacing dependence. We normalize values Δ^2 by Δ_0^2 which is measured at $N_s = 64$ corresponds to $T = 0$. The behavior of Δ^2 as a function of T is consistent among the three couplings. The lattice spacing dependence to Δ^2 is negligible within the couplings, though further investigation with more variation of coupling is needed [8, 9]. The value of Δ^2 at low temperature is around 1 and starts to decrease at $T/\Delta_0 \sim 0.09$ and approaches 0. Even in $N_f = 8$ the chiral symmetry is broken at vanishing temperature and is restored at finite temperature. This is the same as that in large N_f limit [6]. In large N_f limit behavior of the χGN$_2$ model is the same as that of the GN$_2$ model [6].

In Fig. 3 we compare the temperature dependence of Δ^2 and that of Σ^2. Here we adjust the value of the coupling so that lattice spacing in both cases is the same (Tab. 2). In the GN$_2$ model Σ_0 is fixed at $N_s = 32$ which corresponds to $T = 0$. The result also implies that in $N_f = 8$ the behavior of physical observables is already same as that in large N_f limit.

However, there are several caveats to existence of homogeneous phase in the calculation. Most serious issue is the lattice size. If the lattice size is not enough large to include the entire structure of fluctuation of Δ, only part of it is observed and the behavior of it is not distinguished from that in

Table 2: Ensembles of field configurations for the GN_2 model.

fermion	N_f	$N_s = L/a$	$N_t = 1/Ta$	g^2	$a\sigma_0$	μ/σ_0
naive	8	32	2, 4, 6, . . ., 64	1.9132	0.4190(1)	0.0

inhomogeneous phase. Therefore further investigation of behavior of Δ with small lattice spacing and large lattice size is indispensable.

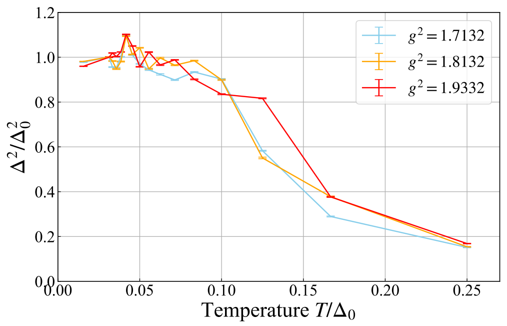

Figure 2: Δ^2 as a function of temperature T/Δ_0 in the χGN_2 model, in the case of $g^2 = 1.7132$ (light blue solid line), 1.8132 (orange solid line) and 1.9332 (red solid line).

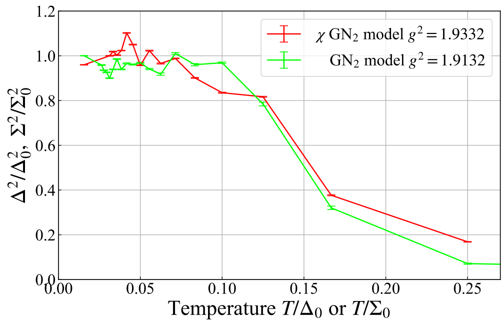

Figure 3: Comparison between temperature T/Σ_0 dependence of Σ^2 in the GN_2 model and temperature T/Δ_0 dependence of Δ^2 in the χGN_2 model.

4.2 Analysis on inhomogeneous phase

We investigate existence of inhomogeneous phase in low temperature and high chemical potential region. We focus on spatial correlation functions of σ and π,

$$C_{\sigma\sigma}(x) = \frac{1}{N_t N_x} \sum_{t,y} \langle \sigma(t, y + x)\sigma(t, y)\rangle, \tag{7}$$

$$C_{\pi\pi}(x) = \frac{1}{N_t N_x} \sum_{t,y} \langle \pi(t, y + x)\pi(t, y)\rangle. \tag{8}$$

In Fig. 4 we show correlation functions of $C_{\sigma\sigma}$ and $C_{\pi\pi}$ as a function of spatial lattice n_x at (a) $(T/\Delta_0, \mu/\Delta_0) = (0.1204, 0.8886)$ and (b) $(T/\Delta_0, \mu/\Delta_0) = (0.1003, 0.8886)$. We find that at both temperatures correlators of $C_{\sigma\sigma}$ and $C_{\pi\pi}$ are fluctuating as a function of n_x, which originates from the existence of inhomogeneous phase. In the case of $T/\Delta_0 = 0.1204$ the behavior of $C_{\sigma\sigma}$ is almost the same as that of $C_{\pi\pi}$, though small deviation of amplitude between them is observed. In the case of lower temperature (b), the period of fluctuation of correlators does not change, but amplitude of them becomes larger. In effective theory the amplitude of fluctuation is related with the order parameter of chiral symmetry [6]. Again, the deviation between amplitude of $C_{\sigma\sigma}$ and that of $C_{\pi\pi}$ becomes large. It may imply consequence of phase shift between σ and π. However, it may also suggest too small lattice size and rough lattice spacing where the maximum value of the amplitudes and entire structure of correlators are not caught correctly. More investigation with smaller lattice spacing and large lattice size calculation is needed. In our lattice size, only one cycle of fluctuation of $C_{\sigma\sigma}$ and $C_{\pi\pi}$ is included. In spite of $N_f = 2$ calculation, very clear spiral structure is observed on fine lattice spacing like $a\Delta_0 = 0.19$ and 0.08 [8]. They also investigate density dependence of the spiral structure.

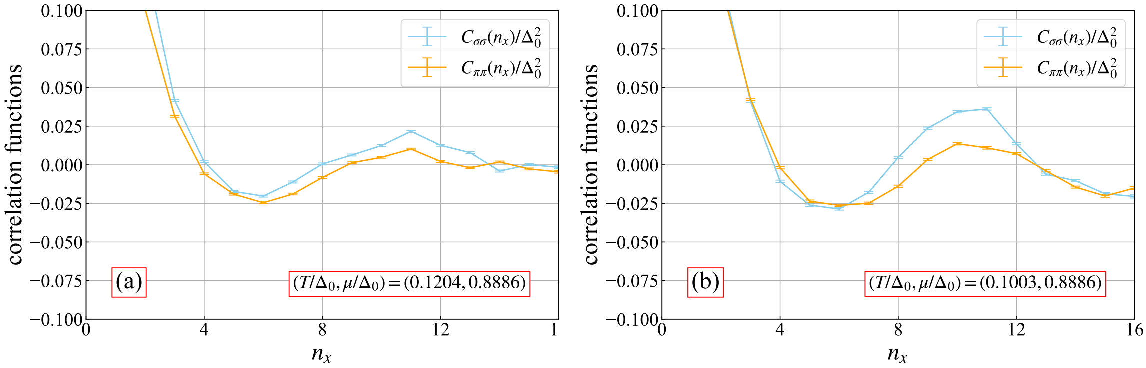

Figure 4: Correlation functions $C_{\sigma\sigma}$ and $C_{\pi\pi}$ as a function of spatial lattice n_x at (a) $(T/\Delta_0, \mu/\Delta_0) = (0.1204, 0.8886)$ and (b) $(T/\Delta_0, \mu/\Delta_0) = (0.1003, 0.8886)$.

In Fig. 5 we show the correlation function of Δ. The analyses on homogeneous phase suggest that the order parameter of chiral symmetry in the χGN_2 model is not Σ or Π itself, but Δ. It is consistent with large N_f limit [6]. Because Δ is not fluctuating, the signal of correlation function of Δ is more clearly than that of Σ or Π. The amplitude of the correlation function of Δ is larger, as temperature is lower. The same tendency is observed in Refs. [8, 9]. The amplitude of Δ is around twice as large as those of Σ and Π. Though detailed comparison between them is too early, because of small lattice size and rough lattice spacing in the current calculation. However, because Δ consists of Σ and Π, we will be able to extract detailed information from Δ even in small lattice. Here, we can not obtain clear signal of the cross terms of σ and π. We need to analyze them at lower temperature in larger lattice. More detailed analyses such as correlators of Δ, cross term of σ and π and phase difference between σ and π are needed.

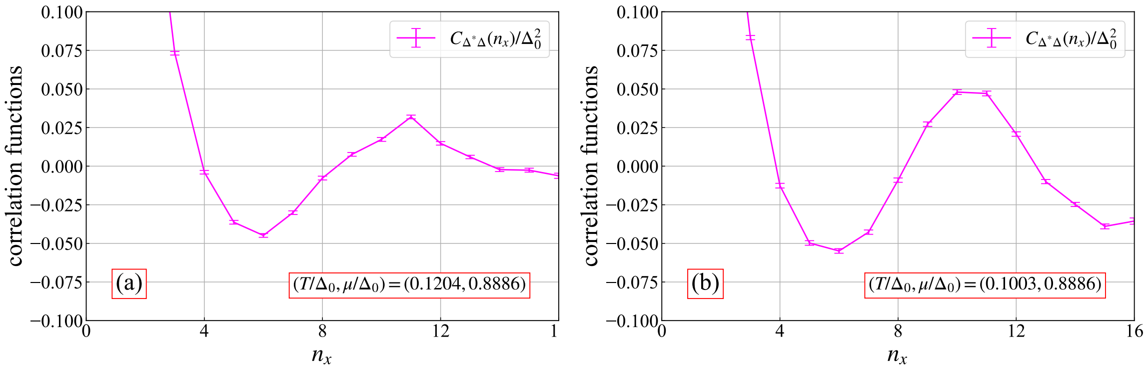

Figure 5: Correlation function $C_{\Delta^*\Delta}$ as a function of spatial lattice n_x at (a) $(T/\Delta_0, \mu/\Delta_0) = (0.1204, 0.8886)$ and (b) $(T/\Delta_0, \mu/\Delta_0) = (0.1003, 0.8886)$.

5. Summary

We discussed the possible existence of inhomogeneous phase in 1+1 dimensional chiral Gross-Neveu model on the lattice. First we investigated the phase structure of the χGN_2 model at vanishing chemical potential, changing temperature. From behavior of $\Delta^2 = \Sigma^2 + \Pi^2$ as a function of Monte Carlo time, a candidate of an order parameter for chiral symmetry in the χGN_2 model is Δ. At vanishing chemical potential, we observed restoration of chiral symmetry at high T. In low temperature and high chemical potential region, we observed the existence of inhomogeneous phase in spatial correlation functions of σ and π. However, to conclusive results further investigation of behavior of Δ and its correlation function with small lattice spacing and large lattice size is indispensable.

We leave for future work the phase diagram in larger N_f and at larger density. For example, the period of the spiral structure is smaller at larger density. Furthermore baryon and thermodynamic quantities in 1+1 dimensional chiral Gross-Neveu model on the lattice and addition of superconducting term to the model can extract more interesting information of the QCD phase diagram.

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