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Continuum limit of two-dimensional multiflavor scalar gauge theories

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We address the interplay between local and global symmetries by analyzing the continuum limit of two-dimensional multicomponent scalar lattice gauge theories, endowed by non-Abelian local and global invariance. These theories are asymptotically free. By exploiting Monte Carlo simulations and finite-size scaling techniques, we provide numerical results concerning the universal behavior of such models in the critical regime. Our results support the conjecture that two-dimensional multiflavor scalar models have the same continuum limit as the σ-models associated with symmetric spaces that have the same global symmetry.

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1. Introduction

Local and global symmetries play a crucial role in determining the main features of a many-body physical system. Non-Abelian gauge invariance is often associated with color confinement and asymptotic freedom [1], while global invariance and the corresponding symmetry-breaking pattern is related to phase transitions and universality. We focus on the intertwining of these two symmetries in the presence of a critical phenomenon, to better understand the role played by local (color) and global (flavor) degrees of freedom in determining the modes that become critical and therefore the universality class observed approaching the continuum limit. Due to the Mermin-Wagner theorem [2], two-dimensional systems cannot undergo a spontaneous breaking of a continuous symmetry at finite temperature. However, a critical behavior may be expected in the asymptotic zero-temperature limit, as it happens in 2D non-linear σ models (NLSMs) [3]. Indeed, these theories develop an exponentially diverging correlation length in the low-temperature regime

$$\xi \underset{T\to 0^+}{\sim} T^{-p} e^{c/T} \tag{1}$$

similarly to QCD, the theory of strong interactions. Generalizations of NLSMs, e.g. gauged σ-models defined on a symmetric space G/H, are expected to be asymptotically free as well [4].

In this proceeding, we review some results on Multiflavor Scalar Gauge (MSG) theories on a two-dimensional lattice, characterized by a non-Abelian SO(N_c) gauge symmetry and O(N_f) global invariance, as paradigmatic examples to discuss the critical properties of MSG models [5]. We only consider fields transforming according to the fundamental representation of the gauge group. To investigate the critical behavior of these lattice systems, we performed finite-temperature Monte Carlo simulations and a detailed analysis of the minimum-energy configurations. All results support the following conjecture: a two-dimensional MSG model has the same continuum limit as the NLSM defined on a symmetric space, possessing the same global symmetry as the gauge model. This is confirmed by the study of the simplest formulation, i.e., in the absence of any local potential in the Hamiltonian. In this case $\mathbb{RP}^{N_f-1}$ critical behavior is observed for any $N_c \geq 3$, N_c and N_f being the number of colors and flavors, respectively.

However, different symmetry-breaking patterns and critical behaviors may be driven by the introduction of a quartic potential that does not break explicitly any symmetry of the lattice model. Indeed, by varying the potential parameters, one can vary the structure of the scalar-field vacuum, which crucially determines the continuum limit realized for $\beta \to \infty$. For multiflavor scalar theories in the unit-length limit, we considered the most general gauge-invariant quartic potential that preserves the O(N_f) global invariance of the model, to understand the role that the potential plays in determining the universality classes and continuum limits of MSG models. We present here some numerical results for $N_c = 3$ with a nonvanishing quartic coupling [they are taken from [6], that studied MSGs with fields in the adjoint representation of SU(2), which is equivalent to the fundamental representation of SO(3)]. These results can be extended to more general models undergoing transitions belonging to more exotic universality classes, associated with a NLSM defined on a symmetric space. In particular, for $N_f > N_c$ and appropriate quartic couplings, the continuum limit is described by a NLSM defined on the quotient group $\mathrm{SO}(N_f)/(\mathrm{SO}(N_c) \otimes \mathrm{SO}(N_f - N_c))$ [4, 8], corroborating our general conjecture.

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2. Lattice model and observables

We define the model on a square lattice with periodic boundary conditions. On each site x of the lattice, we define a real matrix ϕ_x^{if} (where $i = 1, .., N_c$ and $f = 1, .., N_f$ are the color and flavor indices, respectively) satisfying the unit-length constraint

$$\mathrm{Tr}\, \phi_x^t \phi_x = 1\,. \tag{2}$$

To implement the SO(N_c) gauge symmetry, we introduce link variables $U_{x,\mu} \in \mathrm{SO}(N_c)$ on each link connecting two nearest-neighbor sites, according to the Wilson prescription. The Hamiltonian and the partition function of the lattice model read as follows

$$H_{\mathrm{MSG}} = -N_f \sum_{x,\mu} \mathrm{Tr}\, \phi_x^t U_{x,\mu} \phi_{x+\mu} - \frac{\gamma}{N_c} \sum_x \mathrm{Tr}\, U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu}^t U_{x,\nu}^t + V(\phi_x) \tag{3}$$

$$Z = \sum_{\{\phi,U\}} e^{-\beta H}\,, \quad \beta = 1/T\,, \tag{4}$$

where V(ϕ_x), as discussed in the introduction, is a gauge-invariant potential term which preserves the symmetries of the lattice model. We consider the most general quartic potential compatible with this requirement, i.e.,

$$V(\phi_x) = w \sum_x \mathrm{Tr}\, \phi_x^t \phi_x \phi_x^t \phi_x\,. \tag{5}$$

The system is characterized by SO(N_c) gauge invariance

$$\phi_x \mapsto \phi_x' = W_x \phi_x, \qquad U_{x,\mu} \mapsto U_{x,\mu}' = W_x U_{x,\mu} W_{x+\mu}^t, \qquad W_x \in \mathrm{SO}(N_c)\,, \tag{6}$$

and O(N_f) global invariance

$$\phi_x \mapsto \phi_x' = \phi_x M, \qquad U_{x,\mu} \mapsto U_{x,\mu}' = U_{x,\mu}, \qquad M \in \mathrm{O}(N_f). \tag{7}$$

To determine the critical behavior associated with the breaking of the SO(N_f) global symmetry, we study the condensation of a traceless spin-2 symmetric operator Q_x

$$B_x^{fg} = \sum_{i=1}^{N_c} \phi_x^{if} \phi_x^{ig} \qquad Q_x^{fg} = B_x^{fg} - \frac{\delta^{fg}}{N_f}\,, \tag{8}$$

which is the simplest local gauge-invariant operator that can be defined on the lattice. In particular, to classify the models at criticality, we focus on two Renormalization Group (RG) invariant quantities such as the quartic Binder cumulant U and the ratio $R_\xi = \xi/L$,

$$\xi^2 = \frac{1}{4\sin^2\frac{\pi}{L}} \frac{\tilde{G}(0) - \tilde{G}(p_m)}{\tilde{G}(p_m)}, \quad U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 = \frac{1}{V^2} \sum_{x,y} \mathrm{Tr}\, Q_x Q_y\,, \tag{9}$$

ξ being the second-moment correlation length, L the lattice size, p_m the minimum value of the momentum consistent with periodic boundary conditions, and $\tilde{G}$ the Fourier transform of the two-point function $G(x-y) = \langle \mathrm{Tr}\, Q_x Q_y \rangle$. See [5] for additional technical details. Given two RG invariant quantities, for instance U and R_ξ, we determine $U(R_\xi)$, i.e., how the Binder parameter U

(N_c, N_f)	$\langle \mathrm{Tr}\, B_x^2 \rangle_{w=0}$	$\langle \mathrm{Tr}\, B_x^2 \rangle_{w=1}$
(3, 2)	0.9998(4)	0.49981(13)
(3, 3)	0.9999(5)	0.3331(2)
(3, 4)	1.0000(4)	0.3332(2)
(4, 3)	1.00000(1)	-
(4, 4)	1.00001(2)	-

Table 1: Estimates of $\langle \mathrm{Tr}\, B_x^2 \rangle$ on minimum-energy configurations for square lattices of size $L = 4$ and several color-flavor combinations.

depends on R_ξ, for several lattice sizes. According to the Finite-Size Scaling (FSS) theory, if two models belong to the same universality class, in the FSS limit $U(R_\xi)$ is the same, i.e., the function $F(R_\xi)$ defined by

$$U(R_\xi) \underset{L\to+\infty}{\approx} F(R_\xi) \tag{10}$$

is the same. For finite-size systems Eq. (10) holds apart from scaling corrections, whose leading behavior is controlled by the RG scaling dimension of the lowest-dimensional irrelevant operator appearing in the theory. In the case of asymptotically free models, such as two-dimensional vector models, corrections decrease as L^{-2}, multiplied by powers of $\ln L$ [10]. This strategy allows us to compare different models at criticality without tuning any non-universal parameter, as the function $F(R_\xi)$ only depends on the boundary conditions and the universality class associated with the system.

3. Minimum-energy configurations

The study of the minimum-energy configurations is a very useful and handy way to predict the critical behavior of two-dimensional multiflavor scalar systems. Practically, we perform simulations on very small systems for very large values of β, and then we extrapolate the expectation values, to obtain the $\beta \to +\infty$ limit. Here, we restrict the discussion to $\mathrm{Tr}\, B_x^2$, whose average value $\langle \mathrm{Tr}\, B_x^2 \rangle$ allows us to determine the structure of the scalar-field vacuum. In agreement with general considerations, different results, corresponding the different vacuum structures, are obtained for $w > 0$ and $w \leq 0$, see, for instance, Table 1.

These results can be easily understood. Applying the singular value decomposition to ϕ_x^{ia}, one can prove that two minima exists and, correspondingly, $\mathrm{Tr}\, B^2$ is either equal to 1 or to $1/q$, where $q = \min[N_c, N_f]$. More precisely, in the limit $\beta \to +\infty$, the rectangular matrix ϕ_x can be casted in one of the two following forms, depending on the quartic coupling sign:

$$\begin{aligned} &w \leq 0 : \phi_x^{ia} = s_x^i z_x^a, \qquad \text{where} \quad s_x^i, z_x^a \text{ are unit-length vectors} \\ &w > 0 : \phi_x^{ia} = \sqrt{\frac{1}{q}} \sum_{k=1}^{q} C_x^{ik} O_x^{ak}, \qquad \text{where} \quad C \in \mathrm{O}(N_c), O \in \mathrm{O}(N_f). \end{aligned} \tag{11}$$

For $w \leq 0$, $\mathrm{Tr}\, B_x^2$ is 1 (see App. A of [6] for a discussion of the case $w = 0$) and the bilinear operator B_x^2 reduces to a projector P_x^{fg} onto a one-dimensional space

$$B_x^{fg} = z_x^f z_x^g = P_x^{fg}, \quad \text{with} \quad P_x^2 = P_x . \tag{12}$$

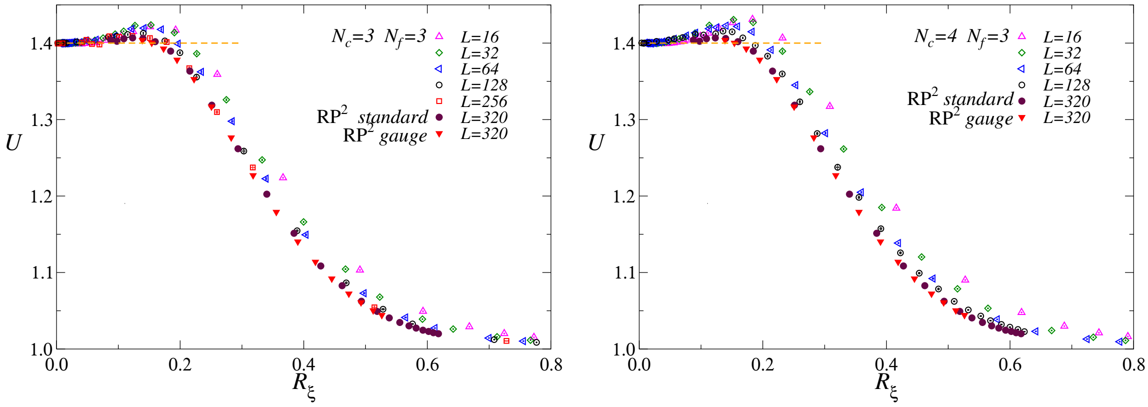

Figure 1: Left: Plot of U versus R_ξ for $N_f = 3$ and $w = 0$. Left: results for $N_c = 3$; right: results for $N_c = 4$. In both plots MSG data are compared with the scaling curve of two different formulation of the $\mathbb{R}\mathrm{P}^2$ model (see [7] for more details).

If we assume that the dynamics in the gauge model is completely determined by the fluctuations of the order parameter B_x, by substituting Eq. (11) in the Hamiltonian, we identify the effective scalar model as the $\mathbb{R}\mathrm{P}^{N_f-1}$ model [5, 6]. Indeed, the standard nearest-neighbor $\mathbb{R}\mathrm{P}^{N_f-1}$ action is given by

$$\mathrm{H}_{RP^{N-1}} = -J \sum_{x,\mu} \mathrm{Tr}\, P_x P_{x+\mu}, \quad P_x^{fg} = \phi_x^f \phi_x^g \tag{13}$$

This is fully confirmed by the results of Table 1 and by the FSS analyses we will provide in the next section.

For $w > 0$, the scalar vacuum is different and $\langle \mathrm{Tr}\, B_x^2 \rangle$ is equal to $1/q$, as verified in Table 1. We will not proceed further discussing this case, as the topic is more technical and out of the scope of this proceeding. We only mention that preliminary results support a critical behavior associated with a NLSM defined on the symmetric space $\mathrm{SO}(N_f)/(\mathrm{SO}(N_c) \otimes \mathrm{SO}(N_f - N_c))$, if $N_f > N_c$ [8]. Note that these universality classes depend on the number of colors N_c—at variance with the case $w \leq 0$—and have a peculiar symmetry under $N_c \mapsto N_f - N_c$.

4. Numerical results

We present some FSS analyses taken from [5, 6]. We will make extensive use of the relation Eq. (10), comparing plots of the Binder cumulant U as a functions of R_ξ: if two theories have the same critical behavior, the asymptotic curve $F_U(R_\xi)$ is the same in the FSS limit. We first show data in the absence of a potential term ($w = 0$) for $N_f = 3$ and $N_c = 3, 4$, see the left and right panel of Fig. 1. In both cases, increasing the lattice size, MSG data approach monotonically the asymptotic curve of the $\mathbb{R}\mathrm{P}^2$ model: tiny finite-size deviations are interpreted as scaling corrections. The figure provides numerical evidence that MSG theories with $N_f = 3$ and non-Abelian gauge symmetry (without any potential term in the Hamiltonian) are in the same universality class as the $\mathbb{R}\mathrm{P}^2$ vector model.

We then present results for $N_c = 3, N_f = 4$ (note that $N_f > N_c$) and different values of the quartic coupling, $w = 0$ and 10, see Fig. 2. In the absence of a potential term (left panel), we still

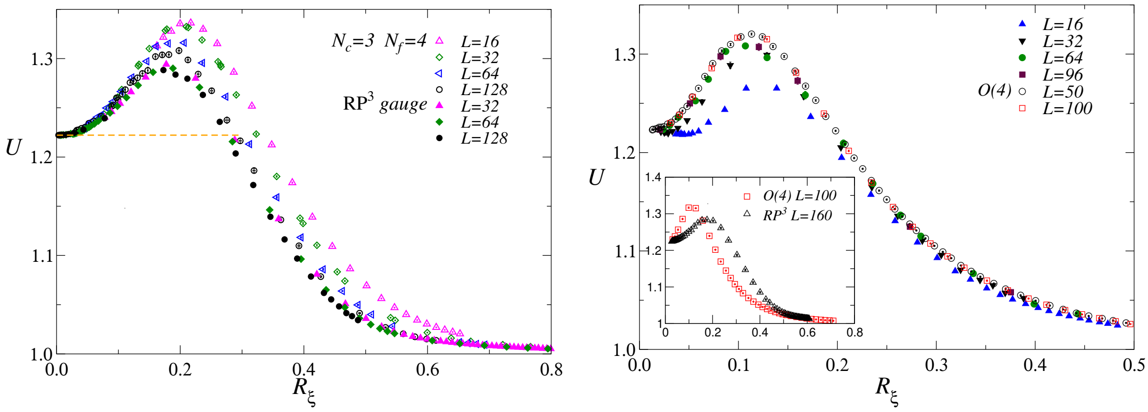

Figure 2: Plot of U versus R_ξ for $N_c = 3, N_f = 4$. In the left panel we show results for $w = 0$, in the right panel results for $w = 10$. MSG data are compared with the scaling curve of the $\mathbb{R}\mathrm{P}^3$ model (left) and of the O(4) vector model (right). The scaling curves of these two models are also reported in the inset (right panel).

observe a $\mathbb{R}\mathrm{P}^{N_f-1}$ critical behavior: the four-flavor scaling curve is consistent with $\mathbb{R}\mathrm{P}^3$ data, within scaling corrections. This is in agreement with the analysis of the minimum energy configurations: for $w = 0$ one obtains $\langle \mathrm{Tr}\, B_x^2 \rangle = 1$ as in the $\mathbb{R}\mathrm{P}^3$ model, see Table 1. A different continuum limit is observed for $w > 0$. In the right panel of Fig. 2, we present results for $w = 10$, which are in complete agreement with an O(4) critical behavior. As shown in the inset, $\mathbb{R}\mathrm{P}^3$ and O(4) data for the largest sizes at our disposal (they provide very good approximations of the universal curves $F_U(R_\xi)$ associated with each of the two universality classes) are clearly different, so the two different continuum limits can be easily distinguished.

5. Conclusions

We presented numerical studies of multiflavor scalar models with a non-Abelian SO(N_c) gauge symmetry, an O(N_f) global invariance and fields transforming in the fundamental representation of the gauge group. The continuum limit that is realized in the $\beta \to +\infty$ limit crucially depends on the structure of the scalar field vacuum. In the absence of a quartic potential term (actually if $w \leq 0$, see [6]), the critical behavior is expected to be associated with a projective-space σ model. This conjecture is confirmed by the analyses of the minimum-energy configurations. We obtained $\langle \mathrm{Tr}\, B_x^2 \rangle = 1$, which implies that the lattice model is effectively described by a theory of projectors. We also performed Monte Carlo simulations for ($N_c = 3, 4$ and $N_f = 3$) and ($N_c = 3, N_f = 4$). All results are in agreement with a critical behavior belonging to the $\mathbb{R}\mathrm{P}^{N_f-1}$ universality class. These findings can be extended in several directions, some of which have been already explored in the literature. One can consider complex scalar fields transforming in the fundamental representation of the SU(N_c) gauge group with a U(N_f) global symmetry [9]. Analogous arguments to the ones presented in this proceeding support in this case a $\mathbb{C}\mathrm{P}^{N_f-1}$ critical behavior. Alternatively, one may consider other group representations. Specifically in [6], for instance, the authors considered scalar fields transforming in the adjoint representation of the SU(N_c) gauge group. A different possibility is to consider the effects of a positive quartic coupling in driving more complex critical behaviors, whenever $N_f > N_c$. As we already mentioned, in that case we expect the universality class to be associated with NLSMs defined on the symmetric spaces $\mathrm{SO}(N_f)/(\mathrm{SO}(N_c)\otimes\mathrm{SO}(N_f - N_c))$ [4, 8]. Preliminary results, as well as all numerical data we gathered, seem to strengthen the conjecture according to which MSG theories have the same continuum limit as NLSMs defined on a symmetric spaces possessing the same global symmetries as the lattice model.

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