PROCEEDINGS OF SCIENCE

Recent work on tessellations of hyperbolic geometries

Muhammad Asaduzzaman,[a] **Simon Catterall,**[a] **Jay Hubisz,**[a] **Roice Nelson and Judah Unmuth-Yockey**[b,*]

[a] *Department of Physics, Syracuse University, Syracuse NY*

[b] *Theory division, Fermilab, Batavia, IL*

E-mail: masaduzz@syr.edu, smcatter@syr.edu, jhubisz@syr.edu, roice3@gmail.com, jfunmuthyockey@gmail.com

We review the construction and definition of lattice curvature, and present progress on calculations of the two-point correlation function of scalar field theory on hyperbolic lattices. We find the boundary-to-boundary correlation function possesses power-law dependence on the boundary distance in both the free, and interacting theories in both two and three dimensions. Moreover, the power-law dependence follows the continuum Klebanov-Witten formula closely.

The 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021 Zoom/Gather@Massachusetts Institute of Technology

*Speaker

PoS(LATTICE2021)016

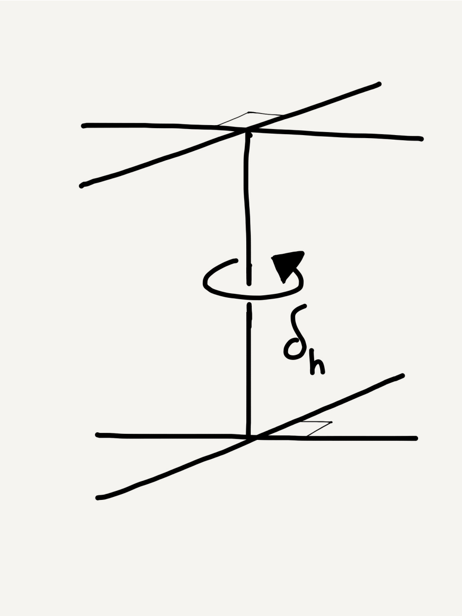

Figure 1: An illustration of the deficit angle around an edge. Here the lattice is a three-dimensional cubic lattice, whose hinges are edges.

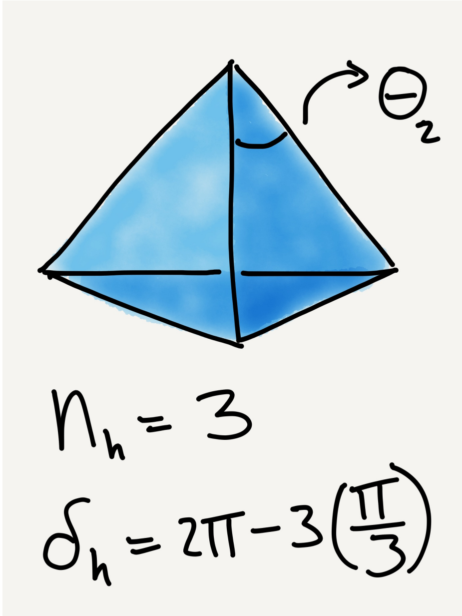

Figure 2: A tetrahedron. This is a closed surface with positive curvature. The hinges are the vertices, and there are three equilateral triangles around each hinge.

1. Introduction

To begin discussing hyperbolic spaces we must first define what curvature means on a lattice. We will think of the lattice as being composed of zero-dimensional objects (vertices), one-dimensional objects (edges), etc. up to D-dimensional objects. For constant curvature spaces there are three cases to point out. Those are: flat, positive, and negative curvature. These three cases can be distinguished by their deficit angles,

$$\delta_h = 2\pi - n_h \times \theta_D, \tag{1}$$

where δ_h measures a deviation from flat. Here n_h is the number of D-dimensional gons around a $(D-2)$-dimensional "hinge", h, and θ_D is the angle between $(D-1)$-dimensional faces attached to h. We can see that if the total amount of space contained in the gons surrounding a hinge is greater than 2π we have a negatively curved lattice. In contrast, if the number of D-gons around a hinge is less than 2π the curvature is positive. And of course if the deficit angle is zero the lattice is flat. Figure 1 shows an example of flat space in three dimensions, with four cubes around the "hinge"—an edge—and how the deficit angle is measured around that edge. To solidify the concept we will demonstrate three situations where each lattice is constructed from the same elements, but each possesses a different deficit angle, and hence different curvatures.

The first example is that of a tetrahedron. Figure 2 shows an illustration of the lattice. This is a positively, closed surface built entirely of equilateral triangles. In this case, $\theta_2 = \pi/3$, and the hinge is a vertex. Each vertex possesses three 2-gons around it, *i.e.* $n_h = 3$. Using these values in

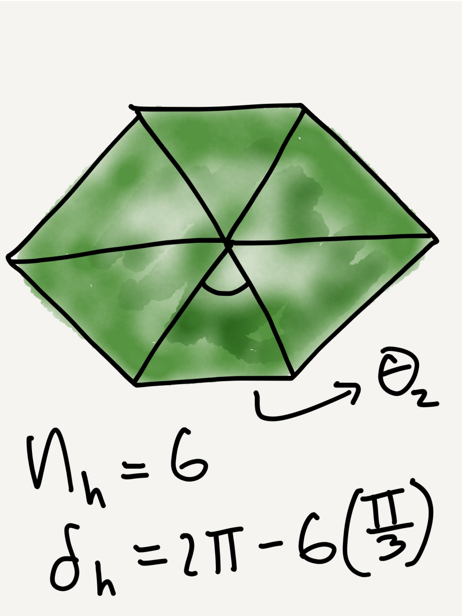

Figure 3: A section of a flat triangular lattice. This surface is open. The hinges are at the vertices, and there are six equilateral triangles around each hinge.

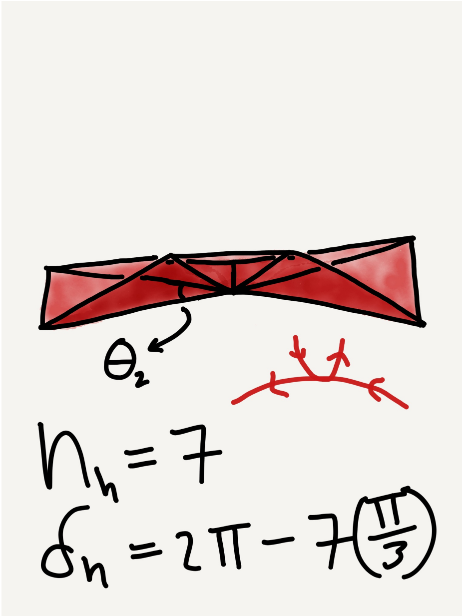

Figure 4: A section of a negatively curved triangular lattice. This surface is open. The hinges are at the vertices, and there are seven equilateral triangles around each hinge.

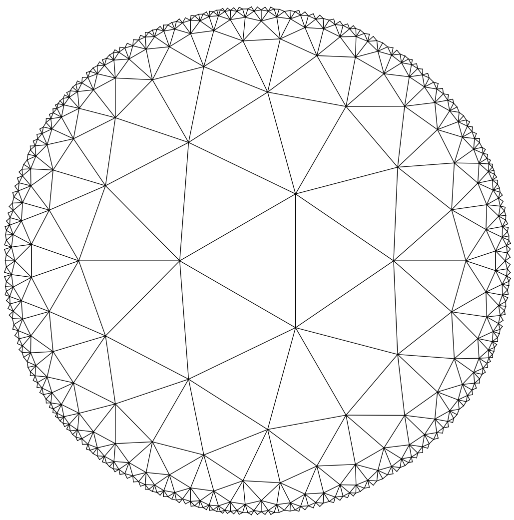

Figure 5: The Poincaré disk projection of the $\{3,7\}$ lattice. While in this view the edge lengths appear different as one moves to the boundary, this is simply a consequence of the projection.

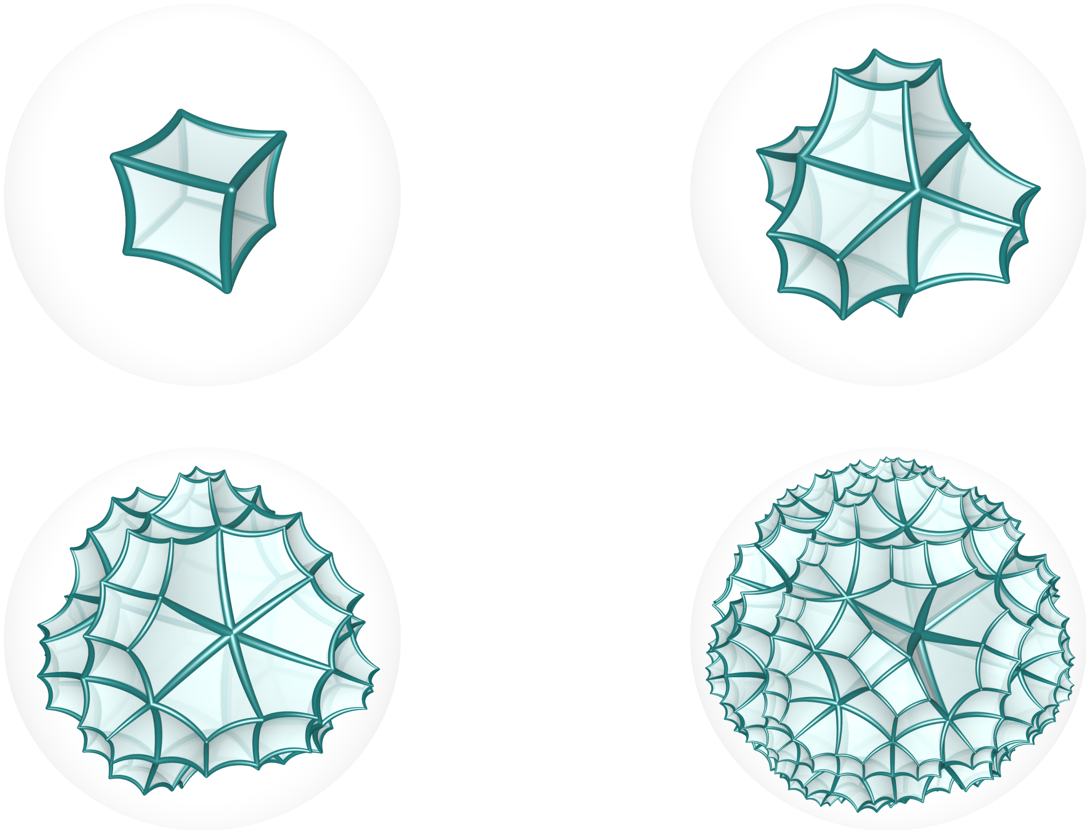

Figure 6: Several figures showing the addition of layers on the order-5 cubic honeycomb lattice. Because of the excess of cubes around an edge, the number of boundary vertices grows exponentially. This view is the Poincaré ball projection, where the lattice is mapped inside the unit ball.

the deficit angle gives $\delta_h = 2\pi - 3(\pi/3) = \pi$ which is a positive value. Notice this is the same for every vertex on the lattice.

The second example is shown in Fig. 3, which is an excerpt from a lattice. Here again the lattice is built from the same objects—equilateral triangles. $\theta_2 = \pi/3$ as before, however in this case $n_h = 6$. Using these values for the deficit angle we find $\delta_h = 2\pi - 6(\pi/6) = 0$. This is a flat (triangular) lattice. The deficit angle here is the same for every vertex.

Finally we give an example of a negatively curved lattice. This can be seen in Fig. 4, where we again consider just an excerpt of a greater lattice. The lattice is comprised of equilateral triangles with $\theta_2 = \pi/3$ again. Here $n_h = 7$ which, based on the previous example is the minimal increase which will take the deficit angle negative. We find $\delta_h = 2\pi - 7(\pi/3) = -\pi/3$ indicating negative curvature. Since this deficit angle is found at every lattice vertex, the lattice takes on a hyperbolic saddle shape. A larger image of the negatively curved lattice shown here can be seen in Fig. 5. The particular view in Fig. 5 is a projection of the hyperbolic lattice into the unit disk. This projection shortens the edge lengths as one approaches the boundary of the space. This is a consequence of the projection. The original lattice's edges are all the same length, and it is made up of equilateral triangles.

Lattices can be specified compactly with a specific notation called a Schläfli symbol. The Schläfli symbol is specified by a collection of positive integers in curly brackets, $\{p, q, r, \ldots\}$. This is read in english as "p-sided polygons, q of them around each vertex, r of those around each edge, etc.". For the case in Fig. 5 the lattice is given by $\{3, 7\}$ because there are 3-sided polygons, seven of them around each vertex.

The fact that there are excess D-gons (in the sense that there are more than the flat case) around a hinge provides, perhaps, unexpected behavior for the lattice. Because of this the size of the boundary grows exponentially as more layers of D-gons are added. This implies that the boundary of the lattice possesses a number of points which is a constant fraction of the the total number of points in the lattice. In this way the boundary never becomes "negligible", since the contribution of volume from the boundary does not vanish as the total volume increases, rather it approaches a constant fraction of the total. Figure 6 shows a hyperbolic cubic lattice as successively more layers are added onto it. The Fig. 6 lattice is a $\{4, 3, 5\}$ lattice, with three squares around each vertex, and five cubes around each edge. We will examine the effect the $\{3, 7\}$ and $\{4, 3, 5\}$ lattices have on matter fields in the following sections.

2. Including fields

In this section we will consider scalar fields—both the free and interacting cases—on hyperbolic lattices. In the continuum the free, massive scalar field action is written,

$$S_{\text{cont}} = \int d^2x\sqrt{g}\frac{1}{2}(\partial_\mu\phi\partial_\mu\phi + m_0^2\phi^2). \tag{2}$$

This action gets straight-forwardly transcribed to the lattice in the form,

$$S_{\text{lat}} = \sum_{x,y} \phi_x L_{xy} \phi_y \tag{3}$$

where L_{xy} is a matrix, and the lattice version of the Klein-Gordan operator, which couples nearest-neighbor, and on-site fields. The explicit form of this matrix is,

$$L_{xy} = q_x \delta_{xy} m_0^2 - A_{xy} \tag{4}$$

with A_{xy} the adjacency matrix for the lattice, q_x the coordination number at site x, and m_0 the bare mass. The simple lattice form of the action gives the lattice propagator immediately. The two-point correlator is given as

$$C(|x - y|) = L^{-1}_{xy}, \tag{5}$$

the matrix inverse of L.

For the case of the interacting theory we consider a truncated scalar field in the form of the Ising model. The lattice action has the form,

$$S_{\text{lat}} = -\beta \sum_{\langle xy\rangle} \sigma_x\sigma_y - h\sum_x \sigma_x \tag{6}$$

PoS(LATTICE2021)016

where $\beta = 1/T$ and h are the nearest-neighbor coupling (inverse temperature) and external field, and the field σ takes on only two values, ± 1. The two-point correlation function for this theory is given in the usual way,

$$C(|x - y|) = \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle, \tag{7}$$

where the angled brackets are considered an ensemble average.

There has been work on including scalar fields on hyperbolic lattices in the past. Initial work focused on the existence of a phase transition and the critical temperature, and in fact evidence was presented that there are multiple phase transitions [1, 2]. Later work considered the critical exponents for the Ising model in both two and three dimensional hyperbolic lattices [3–7], and it was argued that the critical exponents are mean-field.

In addition a very thorough study of correlation functions and a continuum limit was carried out in Ref. [8]. The authors focused on ϕ^4 theory and considered two- and four-point correlation functions. They looked at the large- and small-mass limits, and considered a refinement of the hyperbolic lattice in order to recover a continuum hyperbolic manifold.

Here we draw attention to Ref. [9], which is the basis for this plenary talk. In this paper the authors considered a free massive scalar field on both two- and three-dimensional hyperbolic lattices. In the case of two dimensions, they considered multiple different tessellations of $\mathbb{H}_2$. They found power-law correlations in the two-point function on the $d = 1$ and $d = 2$ boundaries of the two- and three-dimensional spaces, respectively. Moreover, the power-law correlations obeyed the Klebanov-Witten formula [10] precisely, despite the finite lattice spacing, and finite volume of the lattice.

3. Expectations

Having defined the field theories let us discuss limiting cases, and what is known generally to be expected in these cases. For the free field, continuum, boundary two-point propagator it is predicted that the propagator should decay like a power,

$$C(r) \propto r^{-2\Delta_\pm}. \tag{8}$$

In fact the power-law behavior is known explicitly,

$$2\Delta_\pm = d \pm \sqrt{d^2 + 4L^2 m_0^2} \tag{9}$$

where d is the boundary dimension, L is the radius of curvature, and $\Delta_\pm$ are two different powers that depend on the boundary conditions at infinity. From this we see the rate of decay is only dependent on the dimension of spacetime, the bare mass, and the radius of curvature of the hyperbolic space.

Looking at the interacting case, consider the high-temperature expansion for the Ising model two-point correlation function,

$$Z \propto \sum_{\{\Gamma\}} \tanh^{\Gamma}(\beta), \tag{10}$$

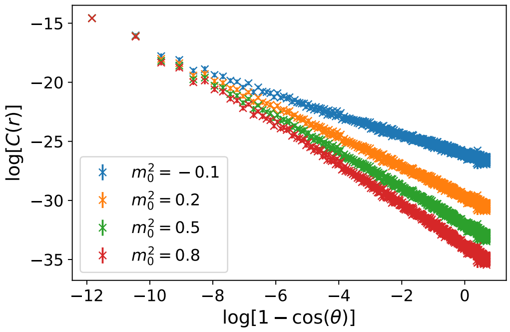

Figure 7: Boundary-to-boundary correlation functions as a function of angular distance around the disk. Correlators for several bare masses are plotted. The apparent linear form is indicative of a power-law in the log-log scale used here.

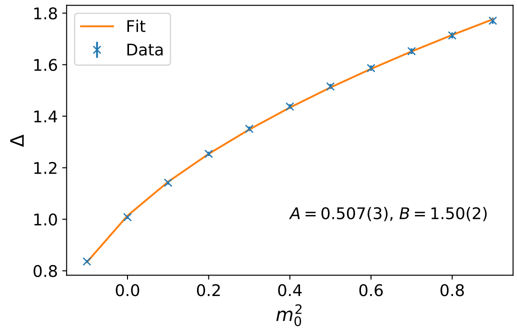

Figure 8: The powers from power-law fits to the correlators in Fig. 7 versus the bare mass. A fit is plotted along with the data which agrees well with the Klebanov-Witten formula.

which is a weighted sum over all possible closed, intersecting loops, Γ. If we consider the case of the correlation between two boundary points, we find the leading-order contribution to the correlation function goes like,

$$C(R) \propto \tanh^R(\beta) = e^{-\log(\coth\beta)R} \tag{11}$$

where R is the geodesic through the bulk. This makes it clear that at least at high-temperature, the correlation function is expected to be exponential through the bulk. However, if cast in terms of the boundary distance, we see $R \propto \log r$,[1] where r is the distance between the points along the boundary. Using this relation between bulk and boundary distance we find, in terms of r, the correlator obeys,

$$C(r) \propto r^{-\log(\coth\beta)}. \tag{12}$$

We find power-law dependence for the boundary correlator, and moreover we see the limiting dependence on β for the power.

4. Results

Having considered some general results and limiting cases for the scalar field on a hyperbolic lattice, we now present results. In Fig. 7 we see actual two-point correlator data for several bare masses in the case of the free scalar field on a $\{3,7\}$ lattice. The data is plotted on a log-log scale, and displays linear behavior indicative of a power-law. The x-axis has been recast in terms of the angular distance around the boundary. This is because this form fully takes into account the finite lattice boundary volume. The data can just as well be plotted in term of the boundary distance r, with finite-volume effects evident at long-distances.

[1]Note that the proportionality factor is dependent on the type of tessellation of hyperbolic space used. Here we will take it to be 1 for simplicity. The overall functional form of the correlator is unchanged by this choice.

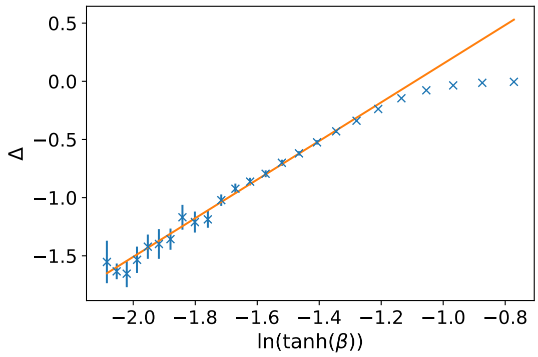

Figure 9: The powers from a power-law fit to the boundary spin-spin correlation function for the Ising model. The high-temperature (small β) data obeys a linear relationship w.r.t. $\log(\tanh\beta)$, which is consitent with the high-temperature expansion results.

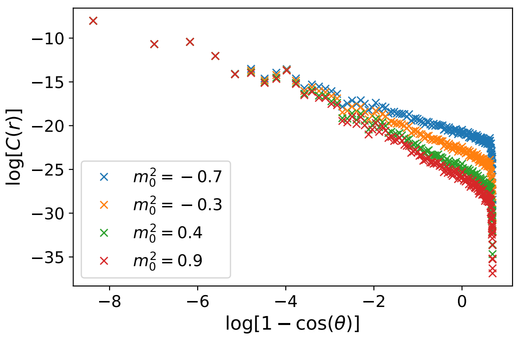

Figure 10: Boundary-to-boundary correlation functions as a function of angular distance on the sphere. Correlators for several bare masses are plotted. The apparent linear form is indicative of a power-law in the log-log scale used here.

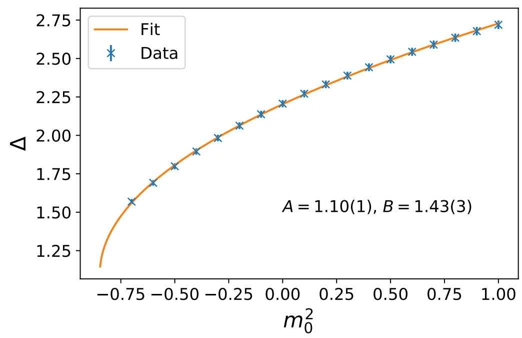

Figure 11: The powers from power-law fits to the correlators in Fig. 10 versus the bare mass. A fit is plotted along with the data which agrees well with the Klebanov-Witten formula.

From the two-point correlation function data we extract a power as a function of the bare mass. A plot of the power, Δ, as a function of the bare mass is shown in Fig. 8. We fit the data to the predicted form from the Klebanov-Witten formula,

$$\Delta = A + \sqrt{A^2 + Bm_0^2} \tag{13}$$

where A and B are fit parameters. We find A, and B match the analytic formula well.

For the Ising model, we extract the power-law behavior from the boundary spin-spin correlation function. We expect at small β to find the correlator,

$$C(r) \sim r^{\log(\tanh\beta)}. \tag{14}$$

Figure 9 shows the power Δ as a function of $\log(\tanh\beta)$. We see for small β the expected linear relationship between the two, which is emphasized with a linear fit.

The previous results were in two dimensions. We now consider the free massive scalar field on a three-dimensional hyperbolic lattice. The lattice is the order-5 cubic honeycomb lattice, or a

$\{4, 3, 5\}$ lattice. In Fig. 10 the boundary two-point correlation function is shown for several bare masses. Again, the correlator is plotted versus the angular distance along the boundary, in log-log coordinates. We find linear behavior indicative of a power-law. The power laws extracted from the correlator fits at different masses can be seen in Fig. 11. Again we see the data obeys the same functional form as the Klebanov-Witten formula, with relatively good agreement between the fit parameters and their theoretically predicted values.

5. Conclusions & Future work

Because of the observed power-law behavior of the boundary two-point correlation function on a hyperbolic lattice, the holographic idea that the boundary of a hyperbolic space supports a conformal field theory appears to survive a lattice discretization. In this case, we see from the high-temperature expansion that this power-law dependence can be traced back to exponential growth of the boundary of the lattice in comparison to the bulk. In addition to the power-law behavior, the specific nature of the power-law is one that follows the predicted Kelbanov-Witten formula closely.

Possible new directions include further studies in both two and three dimensions where the boundary theories would be conformal in one, and two dimensions respectively. It would be interesting to include other matter fields, like fermions, or gauge fields. A more difficult task, but an admirable one, is to attempt to identify the specific effective action for the conformal boundary theory from the bulk-boundary Monte Carlo data. We leave these questions to be addressed in future work.

References

[1] C.C. Wu, *Ising models on hyperbolic graphs*, *Journal of Statistical Physics* **85** (1996) 251.

[2] Y. Jiang, I. Dumer, A.A. Kovalev and L.P. Pryadko, *Duality and free energy analyticity bounds for few-body ising models with extensive homology rank*, *Journal of Mathematical Physics* **60** (2019) 083302 [https://doi.org/10.1063/1.5039735].

[3] K. Ueda, R. Krcmar, A. Gendiar and T. Nishino, *Corner transfer matrix renormalization group method applied to the ising model on the hyperbolic plane*, *Journal of the Physical Society of Japan* **76** (2007) 084004 [https://doi.org/10.1143/JPSJ.76.084004].

[4] R. Krcmar, A. Gendiar, K. Ueda and T. Nishino, *Ising model on a hyperbolic lattice studied by the corner transfer matrix renormalization group method*, *Journal of Physics A: Mathematical and Theoretical* **41** (2008) 125001.

[5] S.K. Baek, H. Mäkelä, P. Minnhagen and B.J. Kim, *Ising model on a hyperbolic plane with a boundary*, *Phys. Rev. E* **84** (2011) 032103.

[6] D. Benedetti, *Critical behavior in spherical and hyperbolic spaces*, *Journal of Statistical Mechanics: Theory and Experiment* **2015** (2015) P01002.

[7] N.P. Breuckmann, B. Placke and A. Roy, *Critical properties of the ising model in hyperbolic space*, *Phys. Rev. E* **101** (2020) 022124.

PoS(LATTICE2021)016

[8] R.C. Brower, C.V. Cogburn, A.L. Fitzpatrick, D. Howarth and C.-I. Tan, *Lattice setup for quantum field theory in* ads_2, *Phys. Rev. D* **103** (2021) 094507.

[9] M. Asaduzzaman, S. Catterall, J. Hubisz, R. Nelson and J. Unmuth-Yockey, *Holography on tessellations of hyperbolic space*, *Phys. Rev. D* **102** (2020) 034511.

[10] I.R. Klebanov and E. Witten, *Ads/cft correspondence and symmetry breaking*, *Nuclear Physics B* **556** (1999) 89.