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Aspects of finite temperature QCD towards the chiral limit

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QCD under extreme conditions has been studied for a long time, and the chiral limit has been a grey area mostly. In this write-up of my talk, I review some of the recent developments made by the community to unveil various features of QCD towards the chiral limit, which includes calculation of the chiral critical temperature and determination of the order of chiral phase transition for various numbers of flavors. Acknowledging the importance of the studies regarding the effective restoration of $U_A(1)$, I try to give a comprehensive overview about the various studies done in the last few years in a comparative manner to realize the current status of the community in this regard. I also discuss very recent efforts about the relevance of various energy-like observables w.r.t. the chiral phase transition.

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1. Why this talk?

Let me start with some prelude about why it is interesting to study the Quantum Chromo-Dynamics (QCD) in/towards the chiral limit, defined by the vanishing quark masses, which can't be realized in nature.

Two fundamental features of QCD are spontaneous breaking of chiral symmetry and confinement. More than four decades ago, there were studies to understand their interplay. One set of studies claimed that the spontaneous breaking of the chiral symmetry happens in a confining environment [1, 2]. Almost at the same time other studies tried to answer the question by calculating and comparing the corresponding transition temperatures [3, 4], but this remains somewhat inconclusive w.r.t. the behavior of theories with different number of colors. Although not very sophisticated w.r.t. present day calculations, the above-mentioned works shows the importance of the chiral transition temperature, which in itself defines a fundamental scale.

On the phenomenological side also, the chiral transition temperature and the nature of the chiral transition is very important to establish the global phase structure of strongly interacting matter. To illustrate this point, I show the conjectured phase diagram of QCD in the space of temperature (T) – baryon chemical potential (μ_B) – degenerate mass of u and d quarks in fig. 1, which I borrowed from ref. [5]. Since chiral symmetry is exact in the chiral limit, spontaneous breaking of chiral symmetry must happen through a phase transition [6]. The zero temperature transition is expected to be of first order and it extends in the $T-\mu_B$ plane and bends toward the T-axis. This is shown by the thick black line in fig. 1. For vanishing chemical potential, on the other side, it was argued in a seminal work [7] that the chiral phase transition for massless u and d quarks will be of second order belonging to the $O(4)$ universality class (red dot in fig. 1). This also extends for finite μ_B and bends towards the μ_B-axis (represented by the thick red line in fig. 1) and meets the before-mentioned first order line in a tri-critical point (magenta dot in fig. 1). This completes the picture in the chiral limit and indicates $T_c > T_{\rm tri}$. When one leaves the chiral plane, the first order transition at low T and high μ_B remains first order but the transition at vanishing and small but non-vanishing values of μ_B becomes a crossover which is represented by the black dashed line in fig. 1. The first order line in the $T-\mu_B$ plane for finite quark masses ends in a (bi-)critical point which shifts to smaller T (so $T_{\rm tri} > T_{\rm cep}$) and larger μ_B [8], which defines the so-called wing line(s), represented by thick blue line in fig. 1. The critical end-point at finite masses are connected to the temperature axis through a crossover, and this crossover temperature at $\mu_B = 0$ for physical pion mass has been recently determined very precisely [9, 10]. It is not hard to realize that the QCD phase diagram at physical value of pion mass is nothing but a slice of the more general phase diagram represented in fig. 1. These interlinked pieces of information gives rise to a very important inequality: $T_{\rm cep} < T_c$, *i.e.* the chiral critical temperature puts an upper bound on the position of the

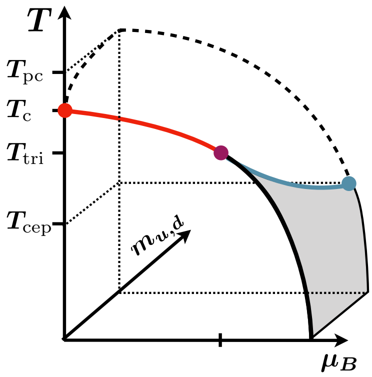

Figure 1: Schematic phase diagram of QCD in the space of temperature (T) – baryon chemical potential (μ_B) – degenerate mass of u and d quarks. Figure is taken from ref. [5].

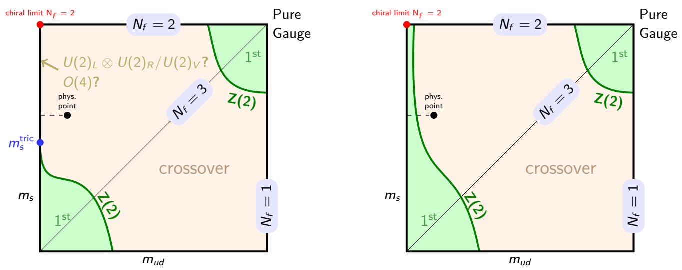

Figure 2: Two most celebrated versions of the Columbia plot. Figures are taken from ref. [12].

conjectured critical end point in the physical world QCD.

The picture presented in the last paragraph is not undisputed. In the before-mentioned work by Pisarski and Wilczek [7], it is also mentioned that if the strength of the anomaly decreases with increasing temperature and $U_A(1)$ gets effectively restored at the critical temperature then the chiral transition for two massless flavors can be of first order. Then the scenario will be quite different compared to what is shown in fig. 1. Although later it was proposed [11] that the chiral transition for two massless flavors can still be continuous even if $U_A(1)$ is restored, but then the universality class will be different.

In this context, two widely celebrated versions of a flag diagram, known as the Columbia plot [13] which presents the order of the chiral transition in the plane of light (degenerate u and d) and strange quark masses, is shown in fig. 2. In the left panel the second order scenario for 2- and $(2 + 1)$-flavor is shown, whereas the right panel shows the scenario where the chiral transition is of first order. I will come to more details of these diagrams later. Various other aspects about the Columbia plot can be found in [14].

I hope by now I convinced the reader that why/how the calculation of the chiral critical temperature and determination of the order of the chiral phase transition is of fundamental necessity. In this talk I shall try to map the progress made by the community to shed light on various aspects of QCD towards the chiral limit. My apologies if I miss any reference; it goes without saying that would be completely unintentional.

2. Results: a few mentions

In this part I shall try to mention various works that have tried and are trying to answer various fundamental questions about the chiral phase transition. First I mention the calculations regarding the chiral transition temperature or to determine the order of the chiral transition for two degenerate light flavors ($N_f = 2$), and then I mention the same for three ($N_f = 3$) and larger than three ($N_f > 3$) numbers of degenerate flavors. Finally I come back to $N_f = 2(+1)$ and mention very recent developments regarding the energy-like observables.

2.1 $N_f = 2(+1)$: the chiral transition temperature

Let me start with a well known scaling formula for the pseudo-critical temperatures, defined by the peak position of order parameter susceptibility for various quark masses:

$$T_{\rm p}(H) = T_c \left(1 + \frac{z_p}{z_0} H^{1/\beta\delta}\right) + \text{sub-leading} \,, \tag{1}$$

where $T_p(H)$ is the peak position for the symmetry breaking field H, which in QCD is proportional to the light quark masses. The chiral transition temperature T_c and the scale of the scaling variable z_0 are non-universal parameters, and z_p specifies the peak position of the scaling function of the order parameter susceptibility. For the relevant universality classes regarding the chiral transition for two massless flavors, the value of z_p is order unity [15] which gives rise to a significant drop in a non-linear fashion of the pseudo-critical temperature towards the chiral limit, w.r.t. *e.g.* physical values of the light quark masses. Authors of [15] came up with two novel estimators of pseudo-critical temperatures by the following conditions;

$$\frac{H\chi_M(T_\delta), H}{M(T_\delta, H)} = \frac{1}{\delta} \quad \text{and} \quad \chi_M(T_{60}, H) = 0.6\, \chi_M(T_p, H) \,, \tag{2}$$

where M and χ_M are the order parameter (proportional to the chiral condensate) and its susceptibility (proportional to the chiral susceptibility) for the chiral phase transition. The values of the scaling variable corresponding to the conditions in eq. 2, named as z_δ and z_{60}, are two orders of magnitude smaller compared to z_p [15], which results in way less variation of the pseudo-critical temperatures towards the chiral limit.

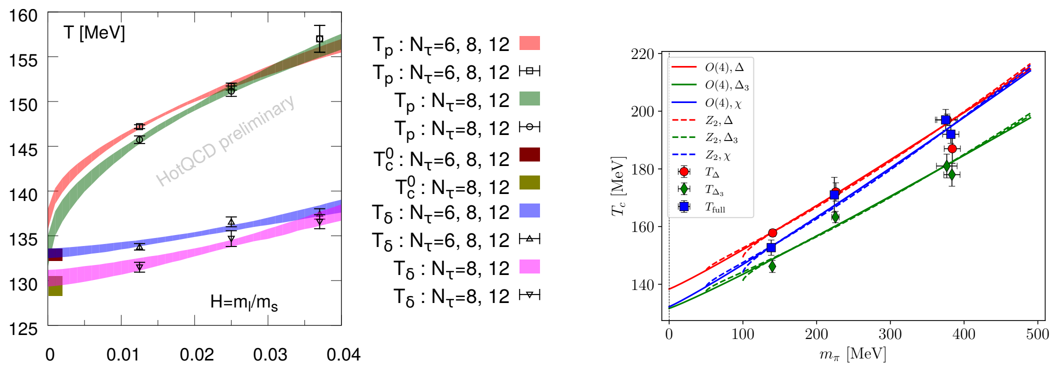

Figure 3: Chiral extrapolation of various pseudo-critical temperatures. Left: calculation with HISQ and the plot is taken from [16]. Right: calculation with twisted mass Wilson fermion and the plot is taken from [17].

In fig. 3 the chiral extrapolation of various pseudo-critical temperatures from two different groups is shown. The left panel shows the *continuum extrapolated* results of T_δ and T_{60}, and they are compared with the conventional estimator of T_p (*also continuum extrapolated*), calculated with highly improved staggered quarks (HISQ). The stability of the new estimators is vivid in the plot. In ref. [15] it is found that the behavior of various pseudo-critical estimators is consistent with $O(4)$ scaling ansätze. Finally the chiral critical temperature, *in the continuum*, is quoted as $T_c = 132^{+3}_{-6}$ MeV [15]. In the right panel the chiral extrapolation of various pseudo-critical estimators, calculated with twisted mass Wilson fermions [17], are shown. In contrary to ref. [15], this calculation is performed within a fixed scale approach. Still the quoted chiral critical temperature, $T_c = 134^{+6}_{-4}$ MeV, agrees well with the other calculation. Consistency with $O(4)$ scaling is found. Moreover, the chiral extrapolations of pseudo-critical estimators are found not quite sensitive to the difference on the universality classes.

2.2 $N_f = 2(+1)$: order of the chiral transition

Coming to the nature of the chiral transition for $N_f = 2$, there exist two major ways to look into it: either from the scaling perspective or through direct determination of the fate of $U_A(1)$. In the following I will take the tour in both ways.

2.2.1 $N_f = 2(+1)$: order of the chiral transition from scaling

Close to the critical point the ratio of the order parameter and its susceptibility is given by [16]

$$\left.\frac{M}{\chi_M}\right|_{T_X,H} = \left.\frac{f_G(z)}{f_\chi(z)}\right|_{z_X} H + \text{sub-leading}. \tag{3}$$

where X = p, δ or 60. It can be clearly seen that in absence of the sub-leading corrections the RHS is determined solely through the universal contribution. This fact has been exploited in the left panel of fig. 4. Parameter free comparison of the data to the scaling expectation clearly depicts that the data prefers an $O(N)$ scenario ($O(4)$ in continuum and $O(2)$ for staggered calculation at a finite temporal extent, N_τ) over a first order transition in the chiral limit. Little deviations of the data w.r.t. the scaling expectation towards larger quark masses can be accommodated using a regular term, which is proportional to quark mass in the order parameter. Similar results are found for z_{60}. More details about this analysis can be found in [16].

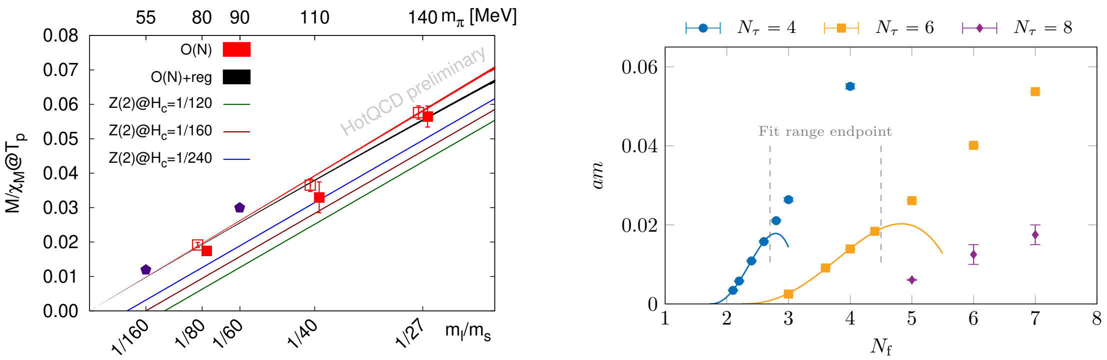

Figure 4: Left: Continuum extrapolated ratio of the chiral order parameter and its susceptibility evaluated at the peak of the chiral susceptibility T_p plotted as function of the scaled light quark masses. Plot is taken from [16]. Right: Bare critical quark mass at which a second order phase transition belonging to Z_2 universality class is found, shown as a function of number of flavors N_f for various N_τ. Plot is taken from [18].

There is another attempt to determine the order of the chiral transition using tri-critical scaling. In ref. [18] the number of degenerate quark flavors was treated as a real continuous parameter and found the phase boundary in an extended parameter space. In the right panel of fig. 4 only the projection in the plane of quark mass and number of flavors is shown. Each data point implicitly specifies a critical coupling. The chiral transition is first order below this set of points for a given N_τ and above it is crossover. Each of these critical points corresponds to second order phase transitions belonging to the Z_2 universality class. These are usually calculated on the basis of the finite size

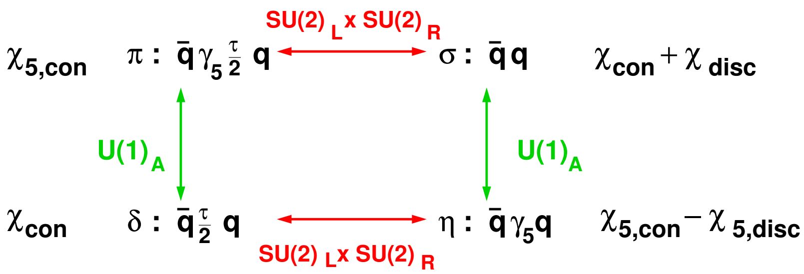

Figure 5: Various mesonic channels of QCD and their relation through either the flavor non-singlet chiral transformation or by the flavor singlet axial transformation. Figure is taken from ref. [19].

scaling arguments of the kurtosis of the chiral condensate. These critical points, creating a line, are eventually expected to go to a tri-critical point at some critical value of N_f, as can be realized from the left panel of fig. 2 comparing the situation for $N_f = 3$ and 2. In the right panel of fig. 4, one can clearly see the sizeable shift of the critical value of N_f w.r.t. N_τ, and it is quite clear that in the continuum the critical N_f will be larger than 2, implying that the chiral transition for $N_f = 2$ will be of second order.

2.2.2 $N_f = 2(+1)$: effective restoration of $U_A(1)$

The $U_A(1)$ symmetry, which is broken in nature upon quantization, is expected to get effectively restored at high temperature because of the deceasing number of non-perturbative topological configurations [7]. So investigation regarding the effective restoration of $U_A(1)$ is extremely important to establish the nature of the chiral transition for 2-flavor and (2+1)-flavor QCD. Before I go to mentioning different studies in this direction, let me start with a brief introduction how effective restoration of $U_A(1)$ is usually studied in the literature. In this context I borrowed fig. 5 from ref. [19]. In this schematic diagram various mesonic channels are connected through either the flavor non-singlet chiral transformation or by the flavor singlet axial transformation, $U_A(1)$. The well known expected degeneracy between iso-triplet pseudo-scalar meson (π) and iso-scalar scalar meson (σ or f_0) can be seen through the horizontal connection between the top two entries of fig. 5. Similarly the iso-triplet pseudo-scalar meson (π) is related to the iso-triplet scalar meson (δ or a_0) by the flavor singlet axial transformation, $U_A(1)$, which is represented by the vertical connection between the two left side entries of fig. 5. To investigate the effective restoration of $U_A(1)$, usually the degeneracy between the masses or susceptibilities is checked, both of which is actually based upon the degeneracy between the correlation functions in those channels. Specifically most of the literature calculates differences like $m_{a_0/\delta} - m_\pi$ or $\chi_\pi - \chi_{a_0/\delta}$ as a measure of the $U_A(1)$ breaking.

It is interesting to note that in the *chirally symmetric background* one can use the degeneracy between the diagonal entries of fig. 5 to check the effective restoration of $U_A(1)$. When one chooses to connect the diagonal containing δ and σ, it can be immediately realized that the corresponding susceptibilities are given by the connected part and the total chiral susceptibility. As a result of this identification one can write [20]

$$\begin{aligned} \chi_\pi - \chi_{a_0/\delta} &= (\chi_\pi - \chi_\sigma) + \chi_{\mathrm{disc}} \\ &= \chi_{\mathrm{disc}}, \quad \text{in the chirally symmetric phase,} \end{aligned} \tag{4}$$

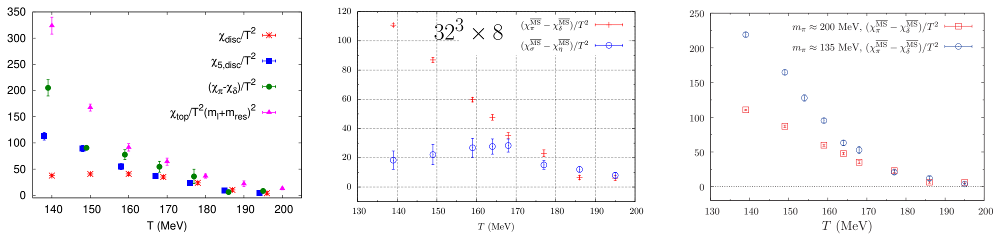

Figure 6: Temperature variation of symmetry breaking measures from calculation with (2+1)-flavor domain-wall fermions (DWF) for $N_\tau = 8$. Plot taken from - Left: ref. [19], Middle: ref. [21], Right: ref. [22].

which implies the condition for the effective restoration of $U_A(1)$ boils down to the condition whether in the chirally symmetric background $\chi_{\rm disc}$ vanishes or not.

A plethora of literature can be found about the effective restoration of $U_A(1)$ and is not possible to cover all of those here. In the following I rather list some of the recent efforts which are almost equally divided into the groups which either favors or disfavors the effective restoration of $U_A(1)$. I show one key plot from each study and mention few features of the calculations belonging to either world.

In the left panel of Fig. 6 the temperature variation of symmetry breaking measures is shown from a calculation with (2+1)-flavor domain-wall fermions (DWF) for $N_\tau = 8$ [19] with a $\sim 45\%$ heavier pion compared to the physical one. $\chi_\pi - \chi_\delta$ remains non-zero for the entire temperature interval of the study suggesting that $U_A(1)$ is effectively broken even after restoration of chiral symmetry. Moreover degeneracy of $\chi_\pi - \chi_\delta$ with the disconnected scalar/chiral susceptibility $\chi_{\rm disc}$ and disconnected pseudo-scalar susceptibility $\chi_{5,\rm disc}$ at high temperatures ensures that the non-vanishing $\chi_\pi - \chi_\delta$ originates indeed from $U_A(1)$ breaking and not from explicit breaking due to quark masses.

A follow-up study [21] with the same set up as above but with larger volumes confirms that the finite volume effects are under control. As can be seen from the middle panel of fig. 6 the $U_A(1)$ breaking measures have a clearly non-vanishing value, and the near equality between the different measures ensures that the explicit chiral symmetry breaking is tiny around the highest temperatures of this study. Further analyses of the volume dependence and the chirality of the near-zero modes agrees with the expectations from the dilute instanton gas approximation picture, again confirming breaking of $U_A(1)$ above the pseudo-critical temperature.

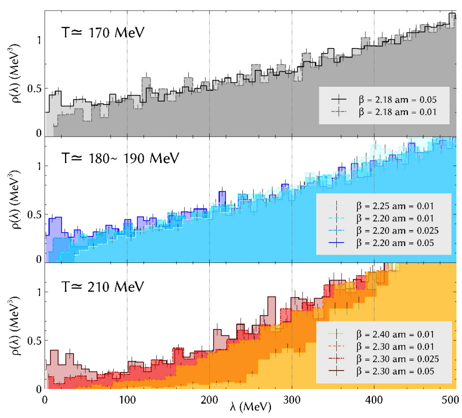

Figure 7: Eigenvalue spectrum of the massless 2-flavor overlap Dirac operator for $N_\tau = 8$ [23]. Lighter shed represents smaller masses.

In the right panel of fig. 6 the temperature variation of $\chi_\pi - \chi_\delta$ from calculation with (2+1)-flavor domain-wall fermions (DWF) for $N_\tau = 8$ [22] with a physical pion is shown. Again the $U_A(1)$ breaking measure remains non-zero over the entire temperature range of the study. At low temperature the expected increase of $\chi_\pi - \chi_\delta$ with decreasing quark mass can be seen. The

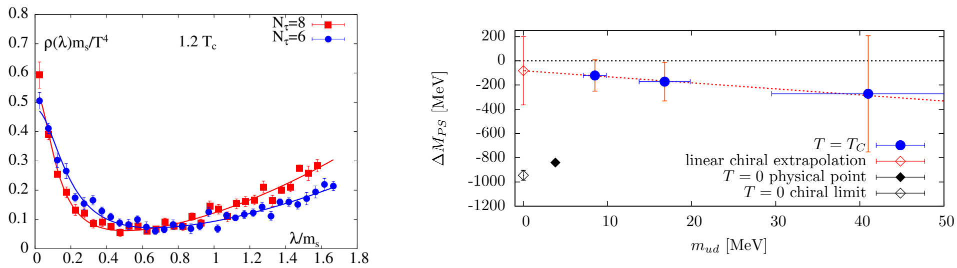

Figure 8: Left: Renormalized eigenvalue spectra calculated with overlap fermions on (2+1)-flavor HISQ ensembles [24] for $N_\tau = 6, 8$. Right: Chiral extrapolation of the difference between the pseudo-scalar and scalar masses, calculated using non-perturbatively $O(a)$-improved 2-flavor Wilson fermions [25] with $N_\tau = 16$.

apparent mass independence of $\chi_\pi - \chi_\delta$ at high temperatures confirms that the non-vanishing value comes due to the breaking of $U_A(1)$ and not due to small explicit breaking through non-vanishing quark masses.

In fig. 7 the eigenvalue spectrum of the massless 2-flavor overlap Dirac operator for $N_\tau = 8$ [23] is shown. Lighter sheds represent smaller masses. The upper panel corresponds to $T < T_{\rm pc}$, where reduction of the mass reduces the density of near zero modes $\rho(0)$, which is compatible with finite volume effects. In the infinite volume limit $\rho(0)$ is expected to be finite. The middle panel corresponds to $T \sim T_{pc}$, and here decreasing the mass resulted in suppression of eigenvalues up to ~ 40 MeV, which is consistent with a vanishing $\rho(0)$ in the chiral limit. This suppression is even more prominent for $T > T_{\rm pc}$ and indicates towards a gap in the spectrum in the chiral limit. Such a gapped spectrum eventually leads to the degeneracy between correlators in π and δ channels, implying effective restoration of $U_A(1)$ in the chiral limit.

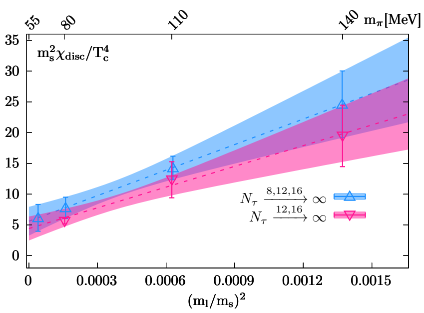

Figure 9: Chiral extrapolation of $\chi_{\rm disc}$ calculated using (2+1)-flavor HISQ action with $N_\tau = 8$, 12 and 16, for $T = 1.6\,T_c$ [26].

In the left panel of fig. 8 renormalized eigenvalue spectra calculated with overlap fermions on (2+1)-flavor HISQ ensembles [24] for $N_\tau = 6, 8$ are shown. Accumulation of near-zero modes being independent of the cutoff and being stable under smearing of the gauge fields ensures that the accumulation is not due to lattice artifacts. Since these near-zero modes will result into a non-vanishing $\chi_\pi - \chi_\delta$, apparently $U_A(1)$ seems to be broken even above the pseudo-critical temperature. Detailed study of the spatial structure and localization properties of these near-zero modes reveals agreement with the expectation from weakly interacting instantons and anti-instantons.

In the right panel of fig. 8 a chiral extrapolation of the difference between the pseudo-scalar and scalar masses at the pseudo-critical temperature of the respective masses is shown, calculated using non-perturbatively $O(a)$-improved 2-flavor Wilson fermions [25] with $N_\tau = 16$. Extrapolations with linear and square-root ansätze w.r.t. quark masses lead to a vanishing value (within uncertainties) of the mass difference shown in the plot, implying an effective restoration of $U_A(1)$

in the chiral limit.

In fig. 9 chiral extrapolation of $\chi_{\rm disc}$ calculated using the (2+1)-flavor HISQ action with $N_\tau = 8$, 12 and 16 is shown for $T = 1.6\ T_c$, where T_c is the chiral critical temperature in the continuum [26]. The blue band represents a joint chiral and continuum extrapolation using an ansätz that is quadratic in quark mass, and the coefficients have a quartic correction in lattice spacing. The magenta band, on the other hand, shows the result from the 'proper' order of limits for staggered calculations where the continuum limit is taken first with a quadratic correction in lattice spacing using the two largest N_τ and then mass extrapolation using a quadratic ansätz. Both extrapolations lead to a non-vanishing value of $\chi_{\rm disc}$ with more than 95% confidence, implying that $U_A(1)$ is broken in continuum even at $1.6\ T_c$. Similar results are obtained from the analysis of $\chi_\pi - \chi_\delta$. Analyses of the correlation function of the eigenvalue density reveals that the microscopic origin of the axial anomaly at high temperature can be described within the weakly interacting (quasi)instanton gas picture. A similar analysis at somewhat lower temperature, closer to $T_{\rm pc}$, on the other hand shows that the eigenvalue spectrum is quite different from that at higher temperatures and is not consistent with the dilute instanton gas picture [27].

In fig. 10 eigenvalue histograms of domain-wall (top panel), partially quenched overlap (middle panel) and reweighted overlap (bottom panel) Dirac operators on ensembles generated with 2-flavor domain-wall sea quarks [28] are shown. The apparent suppression of near-zero modes for domain-wall and reweighted overlap spectra towards the chiral limit seems to be stable under changing lattice spacing and spatial volume implying that the observed effect is not a lattice artifact. Eigenvalues in the lowest bin monotonically decrease w.r.t. decreasing quark mass and become consistent with zero in the chiral limit. A highly contrasting behavior can be observed for the partially quenched overlap spectrum - a sharp peak is present in the lowest bin and persists against the decrease of quark mass. Since this peak does not appear in the domain-wall and reweighted overlap spectra, this was considered as an artifact of partial quenching. After removing this artifact $\chi_\pi - \chi_\delta$ is found to be consistent with zero in the chiral limit, implying effective restoration of $U_A(1)$ in the chiral limit.

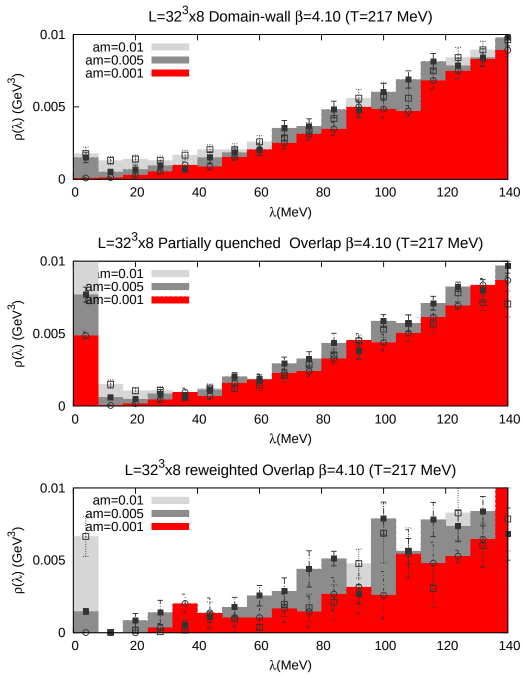

Figure 10: Eigenvalue histograms of domain-wall (top), partially quenched overlap (middle) and reweighted overlap (bottom) Dirac operators on ensembles generated with 2-flavor domain-wall sea quarks [28].

In the left panel of fig. 11 renormalized eigenvalue densities of appropriately mass-tuned valence overlap operator on (2+1)-flavor HISQ ensembles for three different light quark masses [29] are shown. The occurrence of a near-zero peak seems to be stable under a change of quark mass. The calculation of the renormalized $U_A(1)$ breaking measure using these eigenvalues confirms that

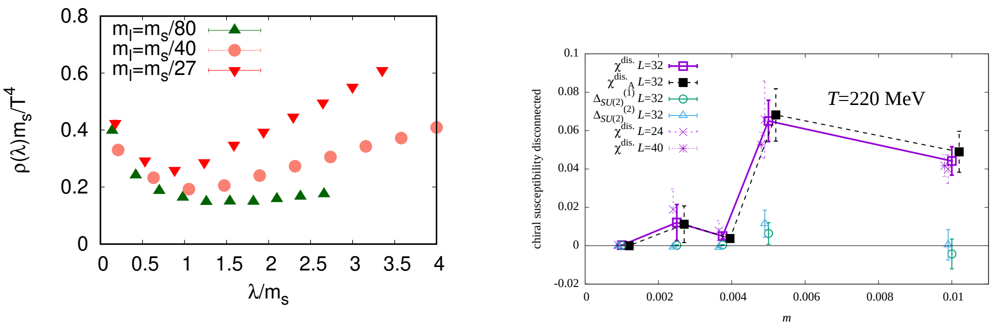

Figure 11: Left: Renormalized eigenvalue density of the appropriately mass-tuned valence overlap operator on (2+1)-flavor HISQ ensembles for three different light quark masses [29]. Right: Disconnected part of the chiral susceptibility from a calculation from reweighted overlap Dirac spectrum on Möbius domain-wall sea quarks, within a fixed scale approach [30].

in the chiral limit $U_A(1)$ remains broken just above the transition temperature.

In the right panel of fig. 11 the disconnected part of the chiral susceptibility, obtained from a calculation of the reweighted overlap Dirac spectrum on Möbius domain-wall sea quarks within a fixed scale approach [30], is shown. χ_{disc} is consistent with zero for the lowest quark mass and no significant volume dependence is seen, leading to the conclusion that $U_A(1)$ is restored in the chiral limit.

One important thing to note from all the studies mentioned above is that *most* of the those probe the temperature range above the pseudo-critical temperature T_{pc} and consequently well above the chiral critical temperature T_c. One has to keep in mind that at high temperatures the $U_A(1)$ breaking anyway becomes small being consistent with the dilute instanton gas approximation. The effective restoration or breaking of $U_A(1)$ thus needs to be checked close to the chiral critical temperature while going towards the chiral limit. This is difficult!

In the left panel of fig. 12 the temperature variation of $\chi_\pi - \chi_{a_0}$ is shown calculated using the meson correlation function with HISQ action with $N_\tau = 8$ [31]. The apparent increase with smaller quark masses at low temperature is expected from the Ward identity. At high temperatures this difference becomes small according to the expectation, and upon looking carefully one can realize that the mass dependence is inverted w.r.t. the dependence at low temperature. The exact same behavior can also be noticed in the temperature variation of the disconnected part of the chiral susceptibility, χ_{disc} [20]. It is not hard to realize that a given fixed (intermediate) temperature simultaneously belongs to the chirally restored phase and chirally broken phase for comparatively large and small masses, respectively. This makes the mass dependence of any $U_A(1)$ breaking measure highly non-monotonic for temperatures above but close to the chiral critical temperature, T_c. This can be seen in a much clearer manner through the right panel of fig. 12 where the mass variation of the renormalized χ_{disc} is plotted for various temperature above but close to T_c which is ~ 144 MeV for $N_\tau = 8$ [15]. It is clearly seen for comparatively higher temperatures, $T \gtrsim 165$ MeV, in the right panel of fig. 12. χ_{disc} monotonically decreases towards the chiral limit which is easier to handle w.r.t. chiral extrapolation. On the other hand, for lower temperatures one can clearly see the onset of the non-monotonic behavior w.r.t. the quark mass - with decreasing quark masses χ_{disc} first increases to a certain point in mass then turns and start decreasing for even lower

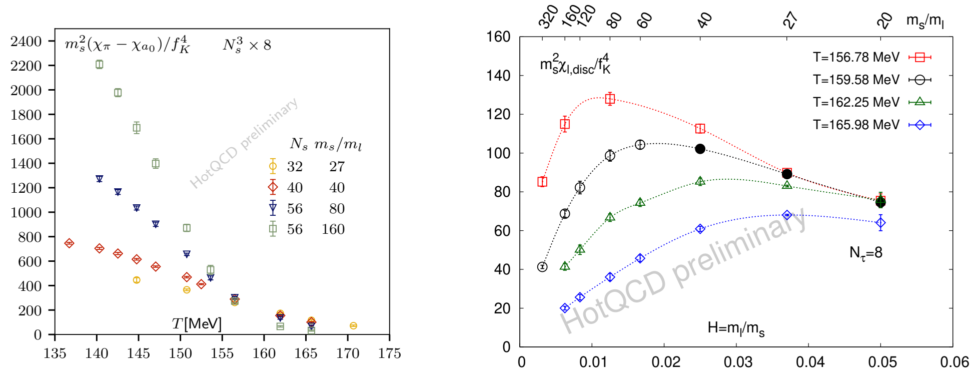

Figure 12: Left: Temperature variation of $\chi_\pi - \chi_{a_0}$, calculated using the meson correlation function using the HISQ action with $N_\tau = 8$. Plot is taken from [31]. Right: Variation of χ_{disc} w.r.t. light quark mass for various (fixed) temperatures above but close to the chiral critical temperature. Calculations have been performed with the (2+1)-flavor HISQ action with $N_\tau = 8$.

masses. Moreover it can be seen that this turning around towards the low values shifts to smaller masses and the slope in quark mass becomes larger when the temperature approaches towards T_c from above which makes it really hard to comment about the chiral limit value of χ_{disc} without having calculations at really small masses. Exactly same behavior has also been observed in the study of $U_A(1)$ breaking/restoration through mesonic susceptibilities [31]. This is one of the main difficulties in the study of effective restoration of $U_A(1)$ for temperatures above but close to T_c.

Interestingly the above mentioned behaviors of the $U_A(1)$ breaking measures have a striking similarity with scaling expectation of the order parameter susceptibility and the difference between so-called the transverse and the longitudinal susceptibilities [32]. The latter one has a leading quadratic dependence on the symmetry breaking field and the former has also a quadratic dependence but in the next-to-leading order. It can also be shown, that this quadratic behavior can only be realized for arbitrarily small values of the symmetry breaking field when one approaches the critical temperature from above. At this point it is tempting to connect the discussed quadratic behavior with the same found in ref. [26] and could be seen in fig. 12 very close to the chiral limit, although one has to admit, it will be a bit ambiguous to associate the scaling behavior with the part(s) of the order parameter susceptibilities which appears in the $U_A(1)$ breaking measures. Another difficulty comes through the fact that the pseudo-critical temperatures T_{pc} depend on N_τ, meaning for a fixed mass a given temperature can simultaneously be below and above the pseudo-critical temperature for a smaller and larger N_τ, respectively, which makes the continuum extrapolations very difficult. I hope I could convince the reader why it is way more difficult to study the breaking/restoration of $U_A(1)$ closer to T_c compared to away from it.

2.3 $N_f = 3$

Pisarski and Wilczek originally argued that for three massless flavors the chiral transition will be of first order, which seems natural given that a cubic term in an effective chiral framework can be realized for $N_f = 3$ or through the critical fluctuations, even when the strength of the anomaly is quite small near the chiral transition temperature [7]. This finding was seconded by a study based on renormalization-group flow where the chiral transition for $N_f = 3$ is found to be of first order irrespective of the effective restoration of $U_A(1)$ [11]. In this regard, it appeared that the assumption of the irrelevance of the gauge degrees of freedom in the Landau-Ginzburg-Wilson type approaches also should be given a serious second thought [33, 34]. Given these apparent concerns it is important to ask the questions what is the order of the chiral phase transition in QCD for three massless flavors? Moreover, if it is first order, then what is the critical pion mass at which the chiral transition is of second order belonging to Z_2 universality class?

The community is trying to answer this question for quite a while and since it is impossible to mention all of them in the following I try to mention some of the recent efforts in this direction. Let me start with a calculation with unimproved staggered actions which found a finite critical pion mass [35], although it was realized that the critical pion mass is very sensitive to cutoff effects. In a later calculation with the HISQ action [36] there was no evidence of a first order transition down to pion mass of 80 MeV, and scaling arguments pushed the bound on critical pion mass down to 50 MeV. Still, a possibility of a small first order region could not be ruled out. Almost at the same time, a calculation based on stout-smeared rooted staggered quarks [37] observed a significant decrease in the critical pion mass with increasing amount of smearing, raising the possibility of having a second order phase transition in the three flavor corner of the Columbia plot.

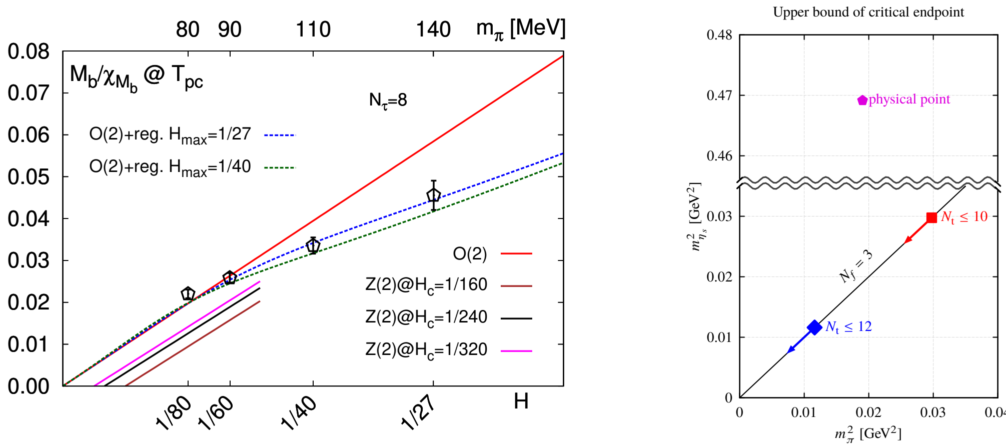

Figure 13: Left: Ratio of the non-subtracted chiral order parameter and its susceptibility calculated at the peak of the susceptibility shown as a function of mass for three degenerate flavors and compared with the scaling expectations. The comparison with the singular part is parameter free as expected from eq. 3. Plot is taken from ref. [38]. Right: Columbia-like plot in terms of hadron masses to show the critical point belonging to the Z_2 universality class from calculations with $O(a)$ improved Wilson fermions. The strong cutoff dependence of the critical pion mass is vivid. Plot is taken from ref. [39].

Recently in a calculation using the standard staggered formulation treating N_f as a real continuous parameter [18], it was found that the critical pion mass for having a second order transition decreases with increasing N_τ, and the scaling analysis suggests that this critical mass eventually vanishes for finite values of N_τ, implying that the chiral transition for three massless flavors will be of second order in the continuum.

In a very recent study with the HISQ action no direct evidence of a first order phase transition was found [38]. Moreover it was found that the behavior of various chiral observables made out of the chiral condensate and its susceptibility can be very satisfactorily described assuming a second order phase transition at the three flavor chiral point. Further support arises from an analysis described through Eq. 3 also for $N_f = 3$. It can be clearly seen from the left panel of fig. 13 that even a calculation with a fixed value of N_τ seems to prefer a second order chiral phase transition with the correct universality class, which is $O(2)$ in this case. The critical temperature was found to be $T_c = 98^{+3}_{-6}$ MeV, within the scale setting of (2+1)-flavor QCD.

Coming to the Wilson fermion side, the chronology has a similar character. In all of the calculations with $O(a)$-improved Wilson fermions, a finite value of critical mass is found either assuming or establishing the Z_2 universality class through the calculation of finite size scaling of the kurtosis of the chiral condensate [39–41]. It is found that although the critical temperature does not suffer much from cutoff effects, the critical pion mass suffers from huge cutoff effects. In the right panel of fig. 13, I borrowed a Columbia-like plot from ref. [39], which clearly shows the strong cutoff dependence of the critical pion mass leaving only a tiny room for a first order phase transition in the chiral limit. An interesting observation is that although of different nature, the transition temperature found in staggered calculations [38] is quite comparable with what is calculated from the Wilson studies [39, 41]. Recently it has been shown [18] that the existing result for critical pion masses from Wilson fermion calculations also show tri-critical scaling and leads to vanishing pion masses at a finite value of N_τ before reaching the continuum, implying a second order chiral phase transition.

I would like to end this part with the remark that at this point discussions regarding the chiral transition for $N_f = 3$ are standing at a very interesting and crucial crossing. In the following I try to list a few comments or questions regarding the chiral transition for three massless flavors which I collected during the discussion:

- An increasing amount of evidence is coming up in favor of a second order chiral phase transition.
- What is the reason that the first order transition, expected for a long time, doesn't show up in QCD calculations?
- Related to the earlier question, what is the possible reason that the tri-linear coupling in the chiral effective framework originated from the anomaly contribution becomes unimportant?
- The origin of the observed tri-critical scaling may lie in a hexa-linear coupling in Landau-Ginzburg-Wilson type theories, as suggested in ref. [18].
- What is the role of gauge symmetries in the Landau-Ginzburg-Wilson type approaches?

Hopefully these will motivate many interesting studies in future.

2.4 $N_f > 3$

Theories with large number of quark flavors have also been a point of interest of the community for various reasons. According to Pisarski and Wilczek [7] the chiral transition is also expected to be of first order for $N_f > 3$. In the following I try to mention some recent studies and my excuses if I miss some.

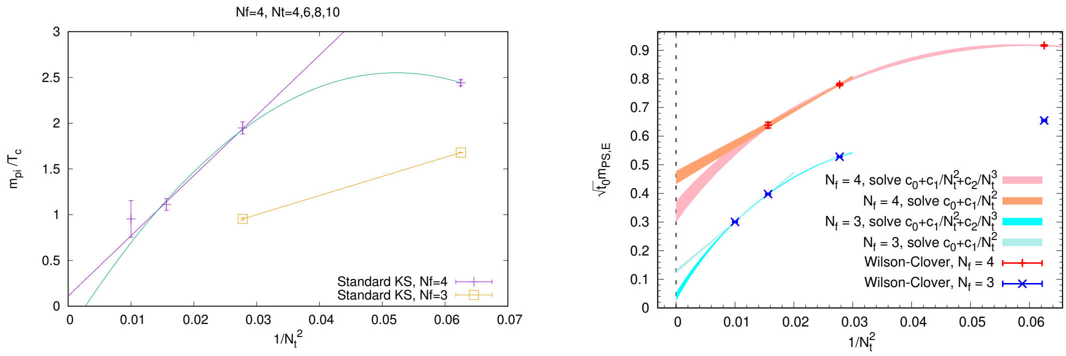

Figure 14: Left: Critical pion mass (in units of the critical temperature T_c) for various N_τ from a $N_f = 4$ calculation with unrooted standard staggered fermions. For comparison the critical pion mass for $N_f = 3$ calculations are also shown. Plot is taken from ref. [42]. Right: Critical pion mass for various N_τ from a $N_f = 4$ calculation with the on-perturbatively improved Wilson-Clover fermion action. Plot is taken from ref. [43].

It is usually expected that with increasing number of flavors the first order region gets larger, which indicates to a larger value of the critical pion mass. This expectation seems to be fulfilled in calculations with $N_f = 4$, performed with unrooted standard staggered fermions [42]. The strong reduction of the critical pion mass has been realized also for $N_f = 4$, and the cutoff dependence seems to be even larger compared to $N_f = 3$, which can be seen from the left panel of fig. 14. This study also clarifies one important criticism about calculations with staggered fermions that the strong reduction of the extent of the first order region in the Columbia plot is not due to rooting.

Coming to the calculations with Wilson fermions the situation is quite similar to staggered one, meaning the critical pion mass for $N_f = 4$ is found to be larger compared to the same for $N_f = 3$ [43], which can be seen from the right panel of fig. 14, where apparently a non-zero critical pion mass can be seen in the continuum limit. Since the cutoff effects are not fully under control in ref. [43] an update has been planned [44] to this analysis.

Going over to even higher numbers of flavors it can be seen from the right panel of fig. 4 that the cutoff effects become larger with increasing number of flavors. It is found that the critical pion masses calculated for $N_f = 5$ also prefers tri-critical scaling, and a clear tension can be realized while trying to describe the critical masses with an ansatz compatible to a first order scenario. This can only be avoided given a change in the curvature arbitrarily close to the continuum, which would be rather surprising. Given all these observations and realizations it is argued that for $N_f \lesssim 6$ the first order transition can not persist to the continuum implying that in the continuum the chiral transition for $N_f \lesssim 6$ is going to be of second order [18]. This finding may have a bigger perspective w.r.t. to the discussions about a conformal window of QCD when N_f exceeds some critical value [45, 46].

2.5 Energy-like observables for $N_f = 2 + 1$.

For the universality classes of our interest, namely $O(N)$ or Z_2, in the infinite volume limit, there exits two relevant scaling fields in the RG sense. One, corresponding to the 'magnetic'-direction, breaks the symmetry explicitly and is proportional to the symmetry breaking fields/parameters in the leading order of Taylor expansion around the critical point. Derivatives of the partition function w.r.t. this symmetry breaking field give rise to the "magnetization-like" observables *e.g.* the order parameter and its susceptibility. The other relevant scaling field, called "temperature-like", corresponds to the 'thermal'-direction in RG space and does not break the symmetry. This "temperature-like" scaling field is proportional to the so-called reduced temperature to the leading order. Important consequences arise from the realization that any parameter which does not break the corresponding symmetry of the theory under consideration must appear in the temperature-like scaling field. A derivative of the the partition function w.r.t. any such parameter, eventually being proportional to a derivative w.r.t. the temperature-like scaling field, defines an "energy-like" observable. The behavior of these observables proportional to the first and second derivatives of the partition function w.r.t. the temperature-like scaling field, can be related to that of the energy and specific heat in the spin model [32], respectively.

Let's start with the Polyakov loop (PL), which was first proposed as an order parameter of the confinement-deconfinement phase transition in the quenched or infinite quark mass limit of QCD [47–49]. Later on there were attempts to use the inflection point of PL or the peak of the PL susceptibility to comment about the deconfinement crossover. Some of these studies found that the crossover temperature defined through observables derived from the chiral condensate are quite close to the same defined through the various PL observables [50–53], whereas others found that the latter estimators give higher temperatures compared to the former ones [54–57].

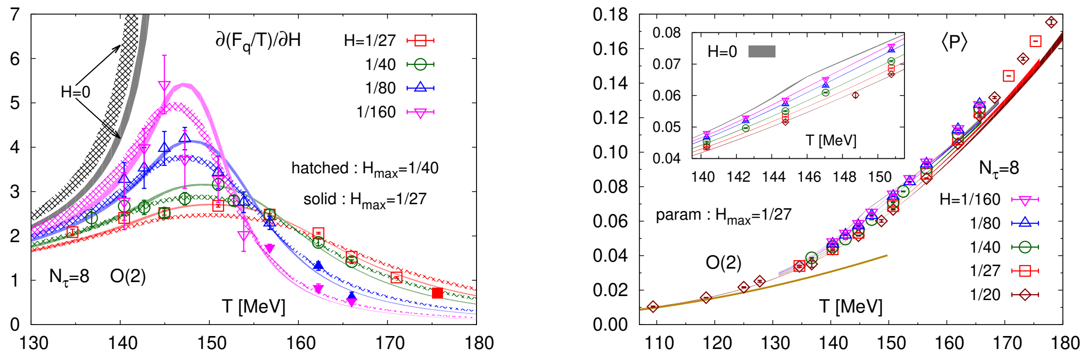

Figure 15: Left: Scaling fit to the quark mass derivative of the HQFE. Right: Description of the PL using the fit parameters from the scaling fits of the mixed-susceptibility and the HQFE. Calculations are performed with HISQ action with $N_\tau = 8$. Both plots taken from ref. [58].

Interestingly at some point it was thought that the PL is not directly sensitive to the singular structure and none of its features are related to any critical exponent of chiral criticality [52]. However, evolving thoughts with a gap of more than a decade gave a completely fresh perspective of the PL w.r.t. the chiral phase transition [58]. Being a purely gluonic operator the PL remains invariant under global chiral rotations. Since the PL does not appear in the QCD action directly, it will be helpful to think of an effective theory of QCD near the critical point. In such an effective theory the PL will appear as an energy-like operator because it does not explicitly break the chiral symmetry. Given this expectation the behavior of the PL and the related heavy quark free energy (HQFE) are expected to behave as energy-like observables w.r.t. the chiral criticality of QCD. Being energy-like, the HQFE itself is not expected to diverge at the chiral limit. Rather a 'mixed' susceptibility, defined through the quark mass derivative of the HQFE which is proportional to the correlation between the PL and the chiral condensate, diverges moderately (slower compared to the magnetic susceptibility) in the chiral limit [58, 59]. In the left panel of Fig. 15 the behavior of this mixed-susceptibility is shown towards the chiral limit and it can be clearly seen that the mass dependence can very well be described by the scaling expectations. Non-universal parameters extracted from the scaling fits of the mixed-susceptibility and the regular part parameters extracted from the HQFE scaling fit describe the mass and temperature dependence of the PL very satisfactorily, as is evident from the right panel of Fig. 15 ensuring the fact that the PL is indeed an energy-like observable w.r.t. the chiral phase transition of QCD which calls into question treating the PL as an 'indicator' of the deconfinement at physical and lower than physical pion masses. I shall come back later to this point while discussing "specific heat-like" observables.

Following the discussion in the beginning of this section, one can realize that the baryon chemical potential (μ_B) does not break the chiral symmetry and it should appear in the temperature like scaling field [60, 61]:

$$t \propto \left(\frac{T}{T_c} - 1\right) + \kappa_B^0 \left(\frac{\mu_B}{T_c}\right)^2 \Rightarrow \kappa_B^0 \frac{\partial}{\partial T} = \frac{T_c}{2} \frac{\partial^2}{\partial \mu_B^2}, \text{ within the scaling window }, \tag{5}$$

where the first term in the RHS of the proportionality relation is the usual reduced temperature and the quadratic dependence on μ_B is to regard the $\mathcal{CP}$ conjugation. The non-universal parameter κ_B^0 is the curvature of the critical line in the $T - \mu_B$ plane. This proportionality relation implies that within the scaling window, one temperature derivative is equivalent (with correct dimensions, of course) to a second order derivative w.r.t. μ_B, which immediately tells us that the second order baryon number susceptibility will be an energy-like observable w.r.t. the chiral phase transition. Singular contributions to the energy-like observables, being scaled as $H^{(1-\alpha)/\beta\delta}$ with H being proportional to the light quark masses, vanish in the chiral limit [60, 61].

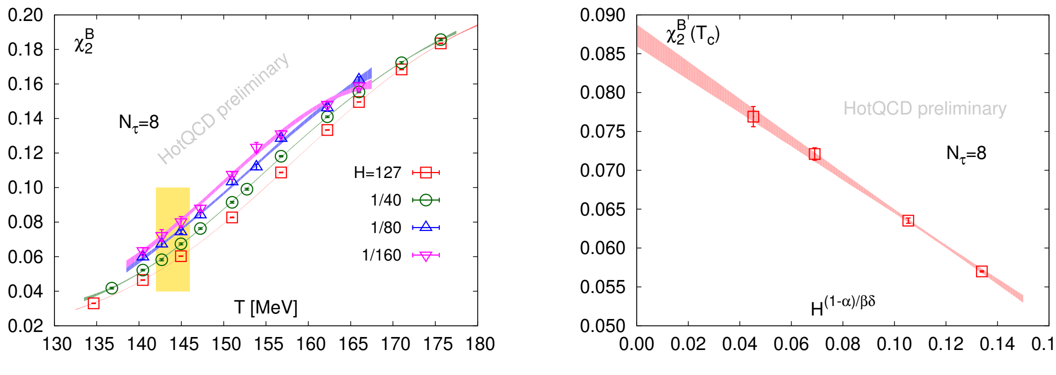

Figure 16: Left: Temperature variation of the baryon number susceptibility for various light quark masses, calculated with the HISQ action for $N_\tau = 8$. The yellow band represents the chiral critical temperature, $T_c = 144(2)$ MeV for $N_\tau = 8$ [15, 16, 20]. Right: Scaling fit to the quark mass variation of the baryon number susceptibility from the left panel, evaluated at T_c, for various quark masses.

In the left panel of Fig. 16, the baryon number susceptibility, χ_2^B, for various light quark masses, calculated with HISQ action for $N_\tau = 8$, is shown. In the right panel of Fig. 16, it can be seen that the scaling expectation of χ_2^B being proportional to $H^{(1-\alpha)/\beta\delta}$ is satisfied very well. The same

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expectation also holds for strangeness susceptibility since the strangeness chemical potential does not break the chiral symmetry either. Other aspects of this analysis can be found in ref. [60, 61].

Similar to chemical potentials, in (2+1)-flavor QCD, the strange quark mass m_s also does not break the 2-flavor chiral limit. Hence m_s should also appear in the temperature like scaling variable and one derivative w.r.t. m_s should also be equivalent (again, with proper dimensions) to one temperature derivative in the scaling regime. As a result of this conjecture, the strange quark condensate should behave as energy-like observable w.r.t. the chiral transition for two massless flavors which can be ensured by calculating another mixed-susceptibility that is essentially the correlation between condensates of light and strange quarks [61]. Note that the fact that strange quark condensate is energy-like, may have an impact on the scaling analysis of the subtracted chiral condensate.

Unill now I discussed some aspects of various energy-like observables. Going to the next level, observables related to fourth order derivatives w.r.t. chemical potentials or one temperature and two chemical potential derivatives or second order derivative w.r.t. strange quark mass of logarithm of the partition function, should behave like a specific heat, which is obtained by two temperature derivatives of the logarithm of the partition function. All these observables are expected to show the characteristic cusp of the $O(N)$ systems at T_c whereas for small but finite quark masses they may show a rounded peak near the corresponding T_{pc}. There is of course a subtlety in the form that the appearance of peaks at small masses are highly subjected to the regular background which differs among various observables significantly. Such a background from energy density rising with temperature was pointed out [62] and confirmed [63] in a calculation of the specific heat itself. An example can also be found from the conserved charge fluctuations where the fourth order baryon number susceptibility shows a peak around T_{pc} but the fourth order strangeness susceptibility monotonically rises across the T_{pc}. It is also difficult to locate a peak in the PL susceptibility in calculations done with improved actions, close to the continuum limit whereas the μ_B response of the HQFE [64] clearly resolves a peak already at the physical mass [65, 66]. These apparently contradicting behaviors of various specific heat-like observables regarding the peak structure needs more detailed analyses, probably focused to understanding namely whether it is some kind of background contribution which is present in one class of observables and not in the other.

I end the discussion on the specific heat-like observables by discussing a special one - the temperature derivative of HQFE. When calculated through temperature interpolation performed mass-by-mass, this shows a peak close to the corresponding T_{pc}, and this peak height decreases with quark mass. This behavior was proposed as a possible indicator of deconfinement and interpreted as a hint for the possible coincidence of the chiral and deconfinement crossover [67, 68]. On the other hand the consistent description of mass and temperature variation of the HQFE and the mixed-susceptibility by the corresponding scaling functions, renders a scenario where peaks at finite quark masses eventually grow (expected at quark masses which are at least two orders of magnitude smaller than what is being used in the current day simulations) towards the chiral limit and show the characteristic $O(N)$ cusp at T_c [58]. This asks for a serious reconsideration of the temptation to carry the interpretation of deconfinement through the free energy or entropy of a test charge from the quenched corner to the regime of physical and lower than physical pion masses.

I drop the scene by discussing recent efforts of calculating the curvature of the (pseudo-)critical lines. There are mainly three avenues to calculate the curvature - using Taylor expansion, analytic

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continuation of calculations done at imaginary chemical potentials or using scaling arguments. In the former method one can choose a physical condition at $\mu_B = 0$ and follow that condition over the $T - \mu_B$ plane through the Taylor expansion using the parametrization of the of $T_{\rm pc}$ on μ_B;

$$T_{\rm pc}\left(\mu_B\right) = T_{\rm pc}\left(0\right)\left[1 - \kappa_B^H\left(\frac{\mu_B}{T_{\rm pc}\left(0\right)}\right)^2\right]. \tag{6}$$

Then collecting terms in various orders in μ_B will define the curvature within the same order of the Taylor expansion in terms of the Taylor coefficients; *e.g.* the quadratic term will give the expression of κ_B^H for this specific value of quark mass (proportional to H). The choice of the physical condition may have a wide variety, either by fixing the value of pressure, energy density or entropy to its value at $\mu_B = 0$ and $T_{\rm pc}\left(0\right)$ [69] or choosing a pseudo-critical condition *e.g.* the inflection point of the chiral condensate or peak of the chiral susceptibility [9, 70]. This calculation, when performed mass-by-mass, can directly show the quark mass dependence of the curvature of the pseudo-critical lines (represented by the black dashed line in the $T - \mu_B$ plane in fig. 1). A direct calculation of the curvature for imaginary chemical potentials using a parametrization of eq. 6 gives compatible results [10, 71] which rely on the validity of the analytic continuation.

The third method, which can give directly the curvature of the critical line (represented by the red solid line in fig. 1) by exploiting the scaling relation of the mixed-susceptibility, involves the mixture of magnetic and thermal derivatives [72]. There also exists a simpler way to calculate estimators of κ_B^0 using the implied equality in eq. 5; taking the ratio between two μ_B derivative of any chiral observables and one T derivative of of the same gives an estimator of κ_B^0 for a given H. In fig. 17 such a calculation with non-subtracted chiral condensate is shown for various quark masses. The calculations have been done with HISQ action for $N_\tau = 8$. The value of the ratio around T_c, shown by a yellow band, is what should be looked for. In the same plot the continuum extrapolated estimate of the curvature of the pseudo-critical line for physical pion mass, obtained through the Taylor expansion method [9], is indicated by the enclosed vertical interval between two black lines. This preliminary comparison tends to suggests that the curvature may not change dramatically towards the chiral limit [73].

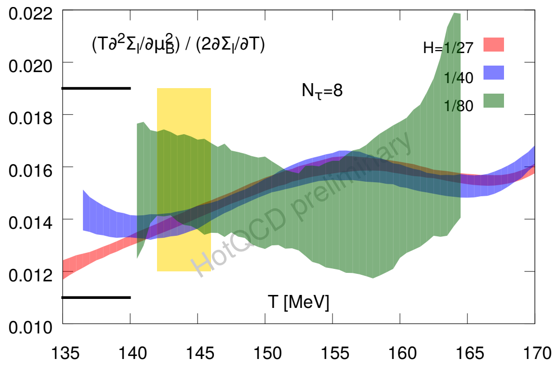

Figure 17: Estimators of the curvature of the critical line in the $T - \mu_B$ plane. Calculations are done with HISQ action for $N_\tau = 8$. The vertical range enclosed by the pair of black lines shows the continuum extrapolated estimate of the curvature of the pseudo-critical line for physical pion mass [9].

An interesting point to note is that the ratio shown in fig. 17 to calculate the estimators of κ_B^0 for a given H can also be used used to estimate κ_B^H for the same H from the Taylor expansion method, corresponding to the condition that the chiral condensate is fixed to its $\mu_B = 0$ value at $T_{\rm pc}\left(0\right)$, over the $T - \mu_B$ plane [69]. Although then one has to focus around the pseudo-critical temperature, $T_{\rm pc}$, for that particular value of H, not around T_c as is done fig. 17. This suggests that the change of κ_B^H in the scaling regime will be $\sim 10\%$ while going to its chiral limit value κ_B^0.

3. Take home messages

I hope that through this write-up of my talk I could convince the reader that study of QCD towards the chiral limit is one of the most interesting and important fields of research. Various groups in the lattice community have been actively working over the years on a variety of problems and hopefully this will continue in future. In this write-up I tried to review some of the aspects of this very dynamic field. With this said I would like to conclude by pointing out some take home messages in the following:

- There have been some progress ....
  - The chiral transition temperature for $N_f = 2$ has been found to be around 130 MeV.
  - The CEP for physical world has to be searched for $T < 130$ MeV and correspondingly for $\mu_B > 400$ MeV.
  - The curvature of (pseudo-)critical lines seems to have very weak quark mass dependence towards the chiral plane.
  - The Polyakov loop behaves as an energy-like observable w.r.t. the chiral phase transition, calling into question its relation with deconfinement, even at physical mass.
  - Various conserved charge fluctuations and the strange condensate also behave as energy-like observables.
- Feel the heat ....
  - Effective restoration of $U_A(1)$ is yet to be settled among various fermion discretizations.
  - Significantly more attention needed for temperatures higher than but close to T_c.
  - The possibility of having a 1^{st}-order region in the $N_f = 3$ corner getting is feeble.
  - Studies of many flavor QCD are going to be interesting in future, especially w.r.t. the existence of a conformal window.

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