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Split Octonions and Triality in (4+4) Space

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Property called triality that manifests itself as equivalence of vectors and chiral spinors is demonstrated in (4+4) space. It is shown that split octonionic representation of this phenomenon respects the triality symmetry.

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1. Introduction

Octonion and split octonion algebras are not commonly seen in theoretical and mathematical physics. However their use has been suggested in various domains, include GUT models [1–3], formalization of quark structure and color symmetry [4, 5], description of elementary particle generations [6, 7], formulation of quantum mechanics [8], associator quantization [9], M-theory [10], electrodynamics [11], gravity [12] and geometry [13–16]. These algebras have been also used for representation of some of the Lie groups with subgroups significant to physics: $SU(3)$, $SU(2) \otimes SU(2)$ and $SO(3,1)$ [4, 17].

Octonions are far from having an established place in physics unlike the other three normed division algebras – quaternions, complex numbers and reals. This is primarily because they lack the associative property, which makes them hard to work with. Applications of split octonions are considered even less often, which might be due to the fact that, apart from associative property, they also lack divisibility. However, all non-invertible split octonions constitute a manifold containing copies of relativistic light cones, which is precisely why they are interesting to study in the context of geometry [15].

Here we highlight one of the unique properties of (4+4)-space concerning the equivalence of vectors and chiral spinors in this space. This peculiarity, referred to as triality, was noted in [18] for the Euclidean space of the same dimension. Larger aim of the research is investigating a possible unification of internal and external symmetries under split octonion algebra.

2. Overview

Before presenting novel results let us briefly discuss some established mathematical results.

2.1 Cayley-Dickson constructions and Hurwitz algebras

Octonions $\mathbb{O}$ and split octonions $\mathbb{O}'$ are Cayley-Dickson constructions obtained by doubling the real number algebra three times. Doubling procedure is as follows: given involution algebra $(\mathbb{A}_n, +, \times)$, the algebra $(\mathbb{A}_{n+1}, +, \times)$ is constructed for $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$, where

$$\begin{aligned}
(a,b) + (c,d) &= (a+c, b+d)\ , && (\text{with } a,b,c,d \in \mathbb{A}_n)\\
(a,b) \times (c,d) &= (ac - \gamma d^* b, da + bc^*)\ , && (\text{with } \gamma = \pm 1)\\
(a,b)^* &= (a^*, -b)\ .
\end{aligned} \tag{1}$$

For example starting from $\mathbb{A}_0 = \mathbb{R}$ with $\gamma = 1$ the following chain of algebras is obtained: complex numbers $\mathbb{A}_1 = \mathbb{C}$, Hamilton's quaternions $\mathbb{A}_2 = \mathbb{H}$, octonions $\mathbb{A}_3 = \mathbb{O}$, sedenions $\mathbb{A}_4 = \mathbb{S}$ and so on. Each doubling removes some properties of real number field. For instance, $\mathbb{A}_3$ is the last algebra in which division is always possible, making it the largest of four Hurwitz algebras – $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, out of which only first three are associative and hence have matrix representations. Setting $\gamma = -1$ produces split algebras, including split octonions.

2.2 Clifford algebras $C\ell_{p,q}(\mathbb{R})$

Clifford algebras are a standard tool for dealing with spinors and also many other geometric problems. Simplest motivating example is derived from requiring square of a vector $\mathbf{x} = x_1 e_1 +$

$x_2 e_2 \in \mathbb{R}^2$ to be equal to its length $\mathbf{x}^2 = x_1^2 + x_2^2$. This leads to an algebraic relation

$$e_m e_n + e_n e_m = 2g_{mn} , \tag{2}$$

that generalizes to any dimension and diagonal metric g_{mn} with (p, q) signature. Familiar examples are Pauli σ_n matrices that generate the algebra of physical space $C\ell_{3,0}(\mathbb{R})$ and Dirac γ^μ matrices that generate spacetime algebra $C\ell_{1,3}(\mathbb{R})$.

Using the algebra $C\ell_{p,q}(\mathbb{R})$, group transformations of $SO(p, q)$ are represented as

$$\mathbf{x}' = \exp\left(-\frac{1}{2}\tilde{\mathbf{a}}\right) \mathbf{x} \exp\left(\frac{1}{2}\tilde{\mathbf{a}}\right) , \tag{3}$$

where $\tilde{\mathbf{a}}$ is a bivector. Corresponding spin group $Spin(p, q)$ is represented as

$$\psi' = \exp\left(-\frac{1}{2}\tilde{\mathbf{a}}\right) \psi , \tag{4}$$

where spinor ψ is identified as minimal left ideal of the algebra, in other words ψ is an element from subalgebra which is closed under left multiplication by general $C\ell_{p,q}(\mathbb{R})$ element. Minimal left ideals are easy to find after determining matrix representation of the Clifford algebra under consideration since they are always a single column vector, or some portion of it when matrix representations are block diagonal.

Number of free parameters of spinor scales exponentially with the dimension of space, i.e. dimension of physical vectors in that space. Coincidences in the number of parameters of spinors and vectors only happen in four cases – when dimensions of space matches dimension of one of the four normed division algebras. Largest such case is in 8 dimensions where two chiral spinors are described by 8 real parameters each.

For more details about this subject and method of finding matrix representations of Clifford algebras see [19].

2.3 Split octonions

General split octonion $\mathrm{A} \in \mathbb{O}'$ is written as

$$\mathrm{A} = A_0 + A_1 j_1 + A_2 j_2 + A_3 j_3 + A_4 I + A_5 J_1 + A_6 J_2 + A_7 J_3 , \tag{5}$$

where $A_0, A_1, \ldots \in \mathbb{R}$. Multiplication rules are determined through the following algebraic relations

$$\begin{aligned} & j_m j_n = -\delta_{mn} + \sum_k \epsilon_{mnk} j_k , \\ & I^2 = 1 , \\ & J_m J_n = \delta_{mn} + \sum_k \epsilon_{mnk} j_k , \qquad (m, n, k = 1, 2, 3) \\ & j_n I = J_n , \\ & J_m j_n = \delta_{mn} I - \sum_k \epsilon_{mnk} J_k , \end{aligned} \tag{6}$$

which is succinctly thumbed up in a Fano plane mnemonic (Figure 1).

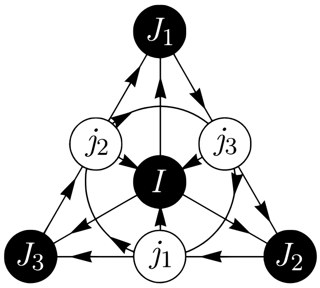

Figure 1: Fano plane structure of split octonion multiplication table.

Modulus of a split octonion is obtained through a quadratic form

$$\left\|\mathrm{A}\right\| = A_0^2 + A_1^2 + A_2^2 + A_3^2 - A_4^2 - A_5^2 - A_6^2 - A_7^2 \tag{7}$$

and can be calculated using split octonionic conjugate

$$\overline{\mathrm{A}} = A_0 - A_1 j_1 - A_2 j_2 - A_3 j_3 - A_4 I - A_5 J_1 - A_6 J_2 - A_7 J_3 \tag{8}$$

as $\left\|\mathrm{A}\right\| = \overline{\mathrm{A}}\mathrm{A}$. The inner product $\cdot : \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ between two split octonions $\mathrm{A}, \mathrm{B} \in \mathbb{O}'$ generalizes the modulus as

$$\mathrm{A} \cdot \mathrm{B} = \frac{1}{2}\left(\overline{\mathrm{A}}\mathrm{B} + \overline{\mathrm{B}}\mathrm{A}\right) . \tag{9}$$

3. Triality in (4+4) space

Triality symmetry of (4+4)-space can be demonstrated using matrix representation of $C\ell_{4,4}(\mathbb{R})$. Generating Γ_μ-matrices of $C\ell_{4,4}(\mathbb{R})$ are 16 dimensional and can be obtained from generating A_μ matrices of $C\ell_{8,0}(\mathbb{R})$ as follows

$$\begin{aligned} \Gamma_\mu &= A_\mu \quad \text{for} \quad \mu = 0, 1, 2, 3 , \\ \Gamma_\mu &= i A_\mu \quad \text{for} \quad \mu = 4, 5, 6, 7 . \end{aligned} \tag{1}$$

This changes Euclidean metric into the split metric. Matrices A_μ are Hermitian and have the form

$$A_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \alpha_\mu^\dagger & 0 \end{pmatrix} , \tag{2}$$

where

$$\alpha_0 = \left(\begin{array}{cc|cc|cc|cc} -1 & & & & & & & \\ & 1 & & & & & & \\ \hline & & 1 & & & & & \\ & & & 1 & & & & \\ \hline & & & & -1 & & & \\ & & & & & -1 & & \\ \hline & & & & & & -1 & \\ & & & & & & & 1 \end{array}\right), \quad \alpha_1 = \left(\begin{array}{cc|cc|cc|cc} i & & & & & & & \\ & i & & & & & & \\ \hline & & i & & & & & \\ & & & i & & & & \\ \hline & & & & i & & & \\ & & & & & i & & \\ \hline & & & & & & i & \\ & & & & & & & i \end{array}\right),$$

$$\alpha_2 = \left(\begin{array}{cc|cc|cc|cc} & 1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & -1 & & & \\ & & & & & -1 & & \\ \hline & & -1 & & & & & \\ & & & -1 & & & & \\ \hline & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right), \quad \alpha_3 = \left(\begin{array}{cc|cc|cc|cc} & -i & & & & & & \\ i & & & & & & & \\ \hline & & & & i & & & \\ & & & & & i & & \\ \hline & & -i & & & & & \\ & & & -i & & & & \\ \hline & & & & & & & -i \\ & & & & & & i & \end{array}\right),$$

$$\alpha_4 = \left(\begin{array}{cc|cc|cc|cc} & & 1 & & & & & \\ & & & & 1 & & & \\ \hline 1 & & & & & & & \\ & & & & & & -1 & \\ \hline & 1 & & & & & & \\ & & & & & & & -1 \\ \hline & & & -1 & & & & \\ & & & & & -1 & & \end{array}\right), \quad \alpha_5 = \left(\begin{array}{cc|cc|cc|cc} & & i & & & & & \\ & & & & i & & & \\ \hline -i & & & & & & & \\ & & & & & & -i & \\ \hline & -i & & & & & & \\ & & & & & & & -i \\ \hline & & & i & & & & \\ & & & & & i & & \end{array}\right),$$

$$\alpha_6 = \left(\begin{array}{cc|cc|cc|cc} & & & 1 & & & & \\ & & & & & 1 & & \\ \hline & & & & & & 1 & \\ 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & 1 & & & & & & \\ \hline & & 1 & & & & & \\ & & & & 1 & & & \end{array}\right), \quad \alpha_7 = \left(\begin{array}{cc|cc|cc|cc} & & & -i & & & & \\ & & & & & -i & & \\ \hline & & & & & & -i & \\ i & & & & & & & \\ \hline & & & & & & & -i \\ & i & & & & & & \\ \hline & & i & & & & & \\ & & & & i & & & \end{array}\right).$$

Using Γ_μ matrices the (4+4)-vector can be written as

$$\mathcal{X} = \sum_{\nu=0}^{7} x_\nu \Gamma_\nu \,, \tag{3}$$

which squares to

$$\mathcal{X}^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 \,. \tag{4}$$

Matrix $\mathcal{X}$ that represents a vector is transposed by $B = -\gamma_1\gamma_3\gamma_5\gamma_7$ as

$$\mathcal{X}^T = B\mathcal{X}B \,. \tag{5}$$

Choosing particular basis of spinor $\xi = \phi + \psi$,

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi_2 + i\phi_3 \\ \phi_0 - i\phi_1 \\ -\phi_7 - i\phi_6 \\ -\phi_5 + i\phi_4 \\ -\phi_5 - i\phi_4 \\ \phi_7 - i\phi_6 \\ -\phi_0 - i\phi_1 \\ -\phi_2 - i\phi_3 \\ \psi_2 - i\psi_3 \\ -\psi_0 - i\psi_1 \\ -\psi_7 - i\psi_6 \\ -\psi_5 + i\psi_4 \\ \psi_5 + i\psi_4 \\ -\psi_7 + i\psi_6 \\ -\psi_0 + i\psi_1 \\ -\psi_2 - i\psi_3 \end{pmatrix},$$

it can be observed that chiral spinors square to the same quadratic form as the vector

$$\begin{aligned} \phi^T B\phi &= \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2 - \phi_5^2 - \phi_6^2 - \phi_7^2 \,, \\ \psi^T B\psi &= \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2 - \psi_5^2 - \psi_6^2 - \psi_7^2 \,. \end{aligned} \tag{6}$$

The following trilinear form $\mathcal{F} : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}$ on vector and spinors

$$\mathcal{F}(\phi, \mathcal{X}, \psi) = \phi^T B\mathcal{X}\psi \,, \tag{7}$$

is invariant under $SO(4,4)$ and $Spin(4,4)$ group transformations, i.e $\mathcal{F}(\phi', \mathcal{X}', \psi') = \mathcal{F}(\phi, \mathcal{X}, \psi)$. Furthermore, roles of vector $\mathcal{X}$ and chiral spinors ϕ and ψ are completely interchangeable. This phenomenon is called triality. For demonstrating this let us consider the transformation

$$L_{01}(\vartheta) = \exp\left(-\frac{1}{2}\vartheta\Gamma_0\Gamma_1\right) \,, \tag{8}$$

in the tangential space

$$L_{01}(\vartheta) \simeq 1 - \frac{1}{2}\vartheta\Gamma_0\Gamma_1 \,, \tag{9}$$

which is written in components as

$$\begin{cases} x_0' = x_0 - \vartheta x_1 \\ x_1' = x_1 + \vartheta x_0 \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = x_4 \\ x_5' = x_5 \\ x_6' = x_6 \\ x_7' = x_7 \end{cases} \quad \begin{cases} \phi_0' = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\ \phi_1' = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\ \phi_2' = \phi_2 - \frac{1}{2}\vartheta\phi_3 \\ \phi_3' = \phi_3 + \frac{1}{2}\vartheta\phi_2 \\ \phi_4' = \phi_4 - \frac{1}{2}\vartheta\phi_5 \\ \phi_5' = \phi_5 + \frac{1}{2}\vartheta\phi_4 \\ \phi_6' = \phi_6 + \frac{1}{2}\vartheta\phi_7 \\ \phi_7' = \phi_7 - \frac{1}{2}\vartheta\phi_6 \end{cases} \quad \begin{cases} \psi_0' = \psi_0 + \frac{1}{2}\vartheta\psi_1 \\ \psi_1' = \psi_1 - \frac{1}{2}\vartheta\psi_0 \\ \psi_2' = \psi_2 + \frac{1}{2}\vartheta\psi_3 \\ \psi_3' = \psi_3 - \frac{1}{2}\vartheta\psi_2 \\ \psi_4' = \psi_4 + \frac{1}{2}\vartheta\psi_5 \\ \psi_5' = \psi_5 - \frac{1}{2}\vartheta\psi_4 \\ \psi_6' = \psi_6 - \frac{1}{2}\vartheta\psi_7 \\ \psi_7' = \psi_7 + \frac{1}{2}\vartheta\psi_6 \end{cases} . \tag{10}$$

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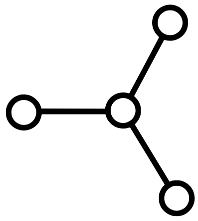

Figure 2: Dynkin diagram D_4

At this point it is easy to notice that transformations on vector $\mathcal{X}$ can be constructed in a way which will exactly repeat spinorial transformations. This is achieved by

$$L_{10}\left(\frac{\vartheta}{2}\right)L_{23}\left(\frac{\vartheta}{2}\right)L_{54}\left(\frac{\vartheta}{2}\right)L_{67}\left(\frac{\vartheta}{2}\right) \simeq 1 - \frac{1}{4}\vartheta\left(\gamma_1\gamma_0 + \gamma_2\gamma_3 + \gamma_5\gamma_4 + \gamma_6\gamma_7\right) . \tag{11}$$

After applying these transformations the behaviors of $\mathcal{X}$,ϕ and ψ get cyclically swaps

$$\begin{cases} x_0' = x_0 + \frac{1}{2}\vartheta x_1 \\ x_1' = x_1 - \frac{1}{2}\vartheta x_0 \\ x_2' = x_2 - \frac{1}{2}\vartheta x_3 \\ x_3' = x_3 + \frac{1}{2}\vartheta x_2 \\ x_4' = x_4 - \frac{1}{2}\vartheta x_5 \\ x_5' = x_5 + \frac{1}{2}\vartheta x_4 \\ x_6' = x_6 + \frac{1}{2}\vartheta x_7 \\ x_7' = x_7 - \frac{1}{2}\vartheta x_6 \end{cases} , \quad \begin{cases} \phi_0' = \phi_0 + \frac{1}{2}\vartheta \phi_1 \\ \phi_1' = \phi_1 - \frac{1}{2}\vartheta \phi_0 \\ \phi_2' = \phi_2 + \frac{1}{2}\vartheta \phi_3 \\ \phi_3' = \phi_3 - \frac{1}{2}\vartheta \phi_2 \\ \phi_4' = \phi_4 + \frac{1}{2}\vartheta \phi_5 \\ \phi_5' = \phi_5 - \frac{1}{2}\vartheta \phi_4 \\ \phi_6' = \phi_6 - \frac{1}{2}\vartheta \phi_7 \\ \phi_7' = \phi_7 + \frac{1}{2}\vartheta \phi_6 \end{cases} , \quad \begin{cases} \psi_0' = \psi_0 - \vartheta \psi_1 \\ \psi_1' = \psi_1 + \vartheta \psi_0 \\ \psi_2' = \psi_2 \\ \psi_3' = \psi_3 \\ \psi_4' = \psi_4 \\ \psi_5' = \psi_5 \\ \psi_6' = \psi_6 \\ \psi_7' = \psi_7 \end{cases} . \tag{12}$$

This symmetry of triality is directly connected to the symmetry of $SO(8)$ and $SO(4,4)$ Dynkin diagram (Figure 2).

4. Triality and split octonionions

As seen above, (4+4)-vector $\mathcal{X}$ and chiral spinors ϕ and ψ are represented by different kinds of objects when described by $C\ell_{4,4}(\mathbb{R})$, namely $\mathcal{X}$ is a matrix and spinors are column vectors. This is despite the fact that distinction between the spinors and the vector is arbitrary due to triality symmetry. However, representing these objects with split octonions respects the triality symmetry and all three objects become of the same type

$$\begin{aligned} \mathrm{X} &= x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3 + x_4 I + x_5 J_1 + x_6 J_2 + x_7 J_3 , \\ \Phi &= \phi_0 + \phi_1 j_1 + \phi_2 j_2 + \phi_3 j_3 + \phi_4 I + \phi_5 J_1 + \phi_6 J_2 + \phi_7 J_3 , \\ \Psi &= \psi_0 + \psi_1 j_1 + \psi_2 j_2 + \psi_3 j_3 + \psi_4 I + \psi_5 J_1 + \psi_6 J_2 + \psi_7 J_3 . \end{aligned} \tag{1}$$

Furthermore, quadratic forms are also calculated identically to each other which can be seen from the following connection between the two representations

$$\begin{aligned} \overline{\mathrm{X}}\mathrm{X} &= \mathcal{X}^2 , \\ \overline{\Phi}\Phi &= \phi^T B \phi , \\ \overline{\Psi}\Psi &= \psi^T B \psi . \end{aligned} \tag{2}$$

In this representation trilinear form $\mathcal{F} : \mathbb{O}' \times \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ is evaluated as follows

$$\mathcal{F}(\Phi, X, \Psi) = -\overline{\Phi} \cdot (X\Psi) \ .$$

5. Summary and outlook

It was shown that in (4+4)-space vectors and chiral spinors are equivalent and invariant trilinear form evaluated on them was explicitly written. We had also demonstrated that in this space the vector and the two spinors can be parameterized by a single split octonionic number, which respects the triality symmetry unlike the Clifford algebraic description.

In further works split octonionic representation of $SO(4,4)$ and $Spin(4,4)$ groups should be established. Also, application of analyticity condition to the trilinear form must be studied as it is expected to produce generalization of field equations.

Acknowledgments

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