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Split-Quaternion Analyticity and (2+1)-Electrodynamics

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It is shown that the analyticity condition, applying to the invariant construction of split quaternions, is equivalent to some system of differential equations for quaternionic spinors and vectors. Assuming that the derivatives by extra time-like coordinate of (2+2)-space of split quaternions generate triality (supersymmetric) rotations, the analyticity equations is reduced to the exact Dirac-Maxwell system in 3-dimensional Minkowski space-time.

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1. Introduction

On physical application of quaternions one can find thousands of publications [1]. The most commonly quaternions are used in the areas of computer graphics, navigation systems, quantum physics and kinematics (see [2–5] and references therein). It is known that ordinary (Hamilton's) and split quaternions can be used to describe 3-dimensional Euclidean and Minkowski spaces, respectively. Here we consider vector and spinor representations in (2+2)-space of split quaternions, which generates kinematics of 3-dimensional Minkowski space-times [6].

Algebra of split quaternions over reals is an associative, non-commutative ring, the general element of which can be written as the linear combination of four basis elements, $e_0 = 1$ and e_k ($k = 1, 2, 3$), with the real coefficients q_0 and q_k,

$$q = q_0 + q^k e_k = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \, . \qquad (k = 1, 2, 3) \tag{1}$$

The first basis element $e_0 = 1$ is real, while the other three, e_k, are anti-commuting hyper-complex units. Using algebra of e_k it can be shown that one of the hyper-complex basis elements, for example the last one e_3, is possible to define as the product of other two,

$$e_3 = e_1 e_2 = -e_2 e_1 \, . \tag{2}$$

The quaternion algebra is associative and therefore can be represented by matrices. For example, we get the simplest non-trivial representation of the basis element of split quaternions by the unit matrix and the three traceless (2×2)-matrices of the $SL(2, R)$-algebra,

$$e_0 = (1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad e_3 = e_1 e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3}$$

In this representations split quaternion (1) uniquely corresponds to the 2×2 real matrix,

$$q_{SL(2,R)} = \begin{pmatrix} (q_0 + q_1) & (q_2 + q_3) \\ (q_2 - q_3) & (q_0 - q_1) \end{pmatrix} . \tag{4}$$

To represent the basis units of split quaternions in (3), instead of the three real $SL(2, R)$-matrices one can use the complex $SU(1, 1)$-matrices,

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} , \tag{5}$$

where i is the ordinary complex unit. So, any split quaternion (1) has the following $SU(1, 1)$-matrix representation:

$$q_{SU(1,1)} = \begin{pmatrix} (q_0 - iq_3) & (q_2 - iq_1) \\ (q_2 + iq_1) & (q_0 + iq_3) \end{pmatrix} , \tag{6}$$

which can be obtained from the $SL(2, R)$-matrix representation (4) by the replacements $q_1 \to -iq_3$ and $q_3 \to -iq_1$.

In analogy with complex numbers, the quaternionic conjugation, which changes the sign of the hyper-complex basis elements, is defined as:

$$e_0^* = e_0 \, , \quad e_1^* = -e_1 \, , \quad e_2^* = -e_2 \, , \quad e_3^* = (e_1 e_2)^* = e_2^* e_1^* = -e_1 e_2 = -e_3 \, . \tag{7}$$

Then the inverted (conjugated) quaternion can be constructed:

$$q^* = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 \ . \tag{8}$$

Note that the two main basis units of split quaternions have the feature of unit polar vectors,

$$e_1^2 = e_2^2 = 1 \ , \tag{9}$$

while the third hyper-complex basis element, $e_3 = e_1 e_2$, behaves like a complex unit,

$$e_3^2 = (e_1 e_2)(e_1 e_2) = -e_1^2 e_2^2 = -1 \ , \qquad e_3^* = -e_3 \ . \tag{10}$$

Because of these distinct properties of basis elements, the norm of a split quaternion,

$$N = \sqrt{|qq^*|} = \sqrt{|q_0^2 - q_1^2 - q_2^2 + q_3^2|} \ , \tag{11}$$

has (2+2)-signature and in general qq^* is not positively defined.

For geometrical applications, we can define the line element in the space of split quaternions in the form [6]:

$$s = \lambda + e_1 x + e_2 y + e_3 t \ , \tag{12}$$

where the four real parameters that multiply basis units, 1 and e_k, denotes: some time-like quantity λ, the spatial coordinates x and y and the time coordinate t (for speed of light we set $c = 1$). Using the conjugation rules one can find that the norm of (12) (the interval),

$$ss^* = s^* s = \lambda^2 - x^2 - y^2 + t^2 \ , \tag{13}$$

has (2+2)-signature and in general is not positively defined. As in the standard relativity we require

$$ss^* \geq 0 \ , \tag{14}$$

i.e. we restrict ourself to time-like split quaternions.

2. Zero divisors

In split algebras the special singular objects, zero divisors, can be constructed [7]. These critical elements of the algebra, which are similar to light-cone variables in Minkowski space-time, could serve as the unit signals characterizing physical events. The norms of split quaternions (11) have (2+2)-signature and thus we have two different types of 'light-cones'. Correspondingly, two types of zero divisors, idempotent elements (projection operators) and nilpotent elements (Grassmann numbers) [7].

At first let us consider the idempotent quaternions, which have the property that they coincide with their squares. A non-zero split quaternion D fulfills this condition if and only if $N_D = 0$ and $q_0 = 1/2$. In the algebra there exist two classes (totally four) primitive idempotents,

$$D_1^{\pm} = \frac{1}{2}(1 \pm e_1) \ , \qquad D_2^{\pm} = \frac{1}{2}(1 \pm e_2) \ . \tag{1}$$

These two classes do not commute with each other. The commuting ones with the standard properties of projection operators:

$$D^{\pm}D^{\mp} = 0 \,, \qquad D^{\pm}D^{\pm} = D^{\pm} \,, \tag{2}$$

are only the pairs (D_1^+, D_1^-), or (D_2^+, D_2^-).

The operators $D_{1,2}^+$ and $D_{1,2}^-$ differ from each other by the reflection of hyper-complex basis element and thus correspond to the direct and reverse critical signals along one of the two real directions, e_1 or e_2, and turn into each other by quaternionic conjugations,

$$(D_1^{\pm})^* = \frac{1}{2}(1 \mp e_1) = D_1^{\mp} \,, \qquad (D_2^{\pm})^* = \frac{1}{2}(1 \mp e_2) = D_2^{\mp} \,. \tag{3}$$

They satisfy also to the conditions:

$$D_1^+ + D_1^- = D_2^+ + D_2^- = 1 \,. \tag{4}$$

We can characterize D_1 and D_2 as primitive idempotent quaternions, since, in contrast to 1, may no longer be decomposed into the sum of idempotents and non-zero quaternions. One can show that D_1^- and D_2^- are the only primitive quaternion that is independent of D_1^+ and D_2^+, respectively.

In the algebra of split quaternions we have also two classes (totally four) of primitive nilpotents,

$$G_1^{\pm} = \frac{1}{2}(e_2 \pm e_3) = \frac{1}{2}(1 \pm e_1)e_2 \,, \qquad G_2^{\pm} = \frac{1}{2}(e_1 \mp e_3) = \frac{1}{2}(1 \pm e_2)e_1 \,, \tag{5}$$

with the properties

$$G^{\pm}G^{\mp} = D^{\pm} \,, \qquad D^{\pm}G^{\pm} = G^{\pm}D^{\mp} = G^{\pm} \,, \qquad G^{\pm}G^{\pm} = D^{\pm}G^{\mp} = G^{\pm}D^{\pm} = 0 \,, \tag{6}$$

for each index, 1 or 2. Quaternionic conjugations of the nilpotents give the same element with the opposite signs,

$$(G_1^{\pm})^* = -\frac{1}{2}(e_2 \pm e_3) = -G_1^{\pm} \,, \qquad (G_2^{\pm})^* = -\frac{1}{2}(e_1 \mp e_2) = -G_2^{\pm} \,. \tag{7}$$

3. Quaternions and triality

The triality algebra of split quaternions can be defined as:

$$\alpha_1(pq) = \alpha_2(p)q + p\alpha_3(q) \,, \tag{1}$$

where p and q denote some split quaternions and α_1, α_2 and α_3 are unit quaternions that generate $SO(2,1)$ rotations from the full $SO(2,2)$ group of symmetry of the norms of split quaternions .

In the formulations of $SO(2,2)$ representations of vector, A, covariant spinor, ξ, and contravariant spinor, χ, in terms of quaternions, the relationships between the three may be expressed without gamma matrices. For example, a covariant and a contravariant spinor can be used to form a vector

$$A = \xi\chi^* \,, \tag{2}$$

or a contravariant spinor can be made from a vector and a covariant spinor

$$\chi = A^* \xi \,, \tag{3}$$

or a covariant spinor can be made from a vector and a contravariant spinor

$$\xi = A\chi \,. \tag{4}$$

Supposing that all three objects A, ξ and χ are transformed by a priori unrelated unit quaternions α_1, α_2 and α_3 and still insisting that A is made from ξ and χ, these transformations would have to be related. This relation is exactly the condition that defines triality (1) and we are led to the conclusion that the triality transformations is the largest group that preserves (2). However, these transformations are exactly of the form used in supersymmetry (see, for example [8]), so it is only natural that the overall symmetry of these theories is given by the triality algebras.

If A, ξ and χ are quaternionic vector and covariant and contravariant spinors, respectively, then the triality construction

$$\mathbf{Tri} \equiv \chi A \xi \tag{5}$$

is $SO(2,1)$ invariant. This can be checked explicitly by applying to (5) spinor and vector transformation rules,

$$\chi' = \chi\alpha^* \,, \qquad \xi' = \alpha\xi \,, \qquad A' = \alpha A \alpha^* \,. \tag{6}$$

were α is a unit quaternion, i.e. $\alpha\alpha^* = 1$.

The construction of division algebras from trialities has tantalizing links to physics. In the Standard Model of particle physics, all particles (other than the Higgs boson) transform either as vectors or spinors, and the interaction between matter and the forces is described by a trilinear map (5), involving two spinors and one vector. Moreover, split normed algebras naturally introduce pseudo-Euclidean spaces which are needed to describe physical spinors and vectors that are associated to Lorentz-type groups.

4. Quaternionic gradient operator

In physical applications it is important to define the quaternionic gradient operator. Although there have been some derivations of this operator in literature with different level of details (see, [9–18] and references therein), it is still not fully clear how this operator can be written in the most general case and how it can be applied to various quaternion-valued functions.

Employ analogy with complex analysis, we can define the split quaternionic derivative operator in the form [12–14],

$$\frac{d}{ds} = \frac{1}{2}\left(\partial_\lambda + e_1\partial_x + e_2\partial_y + e_3\partial_t\right) \,, \tag{1}$$

such that its action upon interval quaternion (12) is equal to one,

$$\frac{ds}{ds} = \frac{1}{2}\left(1 + e_1^2 + e_2^2 + e_3^2\right) = 1 \,, \tag{2}$$

while if applied to conjugated interval element,

$$s^* = \lambda - xe_1 - ye_2 - te_3 \,, \tag{3}$$

it gives zero,

$$\frac{ds^*}{ds} = \frac{1}{2}\left(1 - e_1^2 - e_2^2 - e_3^2\right) = 0 \,. \tag{4}$$

Similarly, the conjugated gradient can be defined by the operator

$$\frac{d}{ds^*} = \frac{1}{2}\left(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t\right) , \tag{5}$$

which annihilates s. Thus from the definitions of quaternionic gradients, (1) and (5), we find

$$\frac{ds^*}{ds} = \frac{ds}{ds^*} = 0 \,. \tag{6}$$

5. Analyticity condition

The main obstacle in physical applications of quaternions is that the real-valued functions of quaternion variables $\Phi(q)$ are not analytic according to quaternion analysis [9–18]. To bypass the issue of non-existent derivatives of real functions of quaternion variables, current applications typically rewrite $\Phi(q)$ in terms of the four real components $\phi^a(q^a)$ $(a = 0, 1, 2, 3)$ of four quaternion variables:

$$\Phi(s, s^*) = \phi_\lambda + e_1\phi_x + e_2\phi_y + e_3\phi_t \;, \tag{1}$$

and take the real derivatives with respect to q^a [14]. Thus, analogously to the Cauchy-Riemann equations from complex analysis,

$$\partial_{z^*} f(z, z^*) = 0 \,, \qquad (z = x + iy) \tag{2}$$

the Cauchy-Riemann-Fueter condition of analyticity for quaternionic functions of quaternionic variables can be written as [12–14],

$$\frac{d\Phi(s, s^*)}{ds^*} = \frac{1}{2}\left(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t\right)\Phi = 0 \,. \tag{3}$$

This statement, that quaternionic functions should be independent of the variable s^*, represents the condition that quaternionic derivative be independent of direction along which it is evaluated.

Note that the simple case of the quaternionic analyticity condition for interval vectors (6), due to the existence of the relations of the type (4), holds only for split quaternions. This justifies in split quaternionic calculus the use of the simple gradient operators (1) introduced in [12, 13]. For ordinary quaternions, even the polynomial functions do not satisfy these Cauchy-Riemann-Fueter conditions and several generalizations are necessary [14–18].

The coordinate transformations, e.g. $q' = \alpha q$, in general do not preserve the property of analyticity, since unit split quaternions representing rotations and boosts are not analytic functions. Thus, to construct analytic and invariant structures (Lagrangians, Superpotentials, *etc*), we need combinations of spinor-like and vector-like split quaternions. For instance, if the transformation laws of χ, ξ and A have the forms,

$$\chi' = \chi\alpha^* \,, \qquad \xi' = \alpha\xi \,, \qquad A' = \alpha A\alpha^* \,, \tag{4}$$

the invariant constructions are products of the type $\chi\xi$ and $\chi A\xi$. Indeed,

$$
\begin{aligned}
(\chi\xi)' = \chi'\xi' = \chi\ \alpha^*\alpha\ \xi = \chi\xi\ , \\
(\chi A\xi)' = \chi\ \alpha^*\alpha\ A\ \alpha^*\alpha\ \xi = \chi A\xi\ .
\end{aligned} \tag{5}
$$

Then, using validity of the distribution law for quaternionic gradient operators (1) and (5) [12–14], we can write Cauchy-Riemann-Fueter analyticity condition (3) for the triality invariant construction (5) in the form:

$$
\frac{d(\chi A\xi)}{ds^*} = \frac{d\chi}{ds^*}A\xi + \chi\frac{dA}{ds^*}\xi + \chi A\frac{d\xi}{ds^*} = 0\ . \tag{6}
$$

This condition is equivalent to the system of equations for quaternionic vector and spinors of the first and second kind (covariant and contravariant),

$$
\frac{dA}{ds^*} = 0\ , \qquad \frac{d\chi}{ds^*} = 0\ , \qquad \frac{d\xi}{ds^*} = 0\ . \tag{7}
$$

Below we shall show that this system can be reduce to the equations of (2+1)-electrodynamics, i.e. to the system of standard Dirac and Maxwell equations in 3-dimensional Minkowski space-time.

6. Quaternionic Dirac equation

Let us demonstrate that the algebraic Cauchy-Riemann-Fueter condition (7) for the covariant spinor ξ can be understand as the Dirac equation. Analogous logic can be applied for the case of the second kind (contravariant) spinor χ to obtain Dirac equation for conjugated spinors.

Consider the first kind spinor ξ^+,

$$
\xi^+ = (q_0 - iq_3)D^+ + (q_2 + iq_1)G^-\ , \tag{1}
$$

expressed in terms of the idempotent and nilpotent elements in $SU(1,1)$-representation, where q^n are some real functions of λ, x, y and t. We can make the replacement,

$$
\xi^+(\lambda, x, y, t) \quad \to \quad e^{m\lambda}\xi^+(x, y, t)\ , \tag{2}
$$

were m is some real parameter. Then the condition (7) for the covariant spinor ξ takes the form:

$$
\left(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t + m\right)\xi^+(x, y, t) = 0\ . \tag{3}
$$

Next we assume that the derivative of the covariant split quaternion $\xi^+(x, y, t)$ by the extra time-like coordinate λ generates the supersymmetric (triality) transformation:

$$
\partial_\lambda\xi^+ = B\chi^{+*}\ , \tag{4}
$$

where B is some vector-type split quaternion, while χ^{+*} is the conjugated contravariant quaternionic spinor in $SU(1,1)$-representation:

$$
\chi^{+*} = (q_0 - iq_3)D^+ - (q_2 + iq_1)G^-\ . \tag{5}
$$

Then the quaternionic Cauchy-Riemann condition (3) obtains the form:

$$
\left(e_1\partial_x + e_2\partial_y + e_3\partial_t - m\right)\xi^+ - B\chi^{+*} = 0\ , \tag{6}
$$

or in matrix notation,

$$\left[\begin{pmatrix} -i\partial_t + m & \partial_y - i\partial_x \\ \partial_y + i\partial_x & i\partial_t + m \end{pmatrix} + \begin{pmatrix} B_0 - iB_3 & B_2 - iB_1 \\ B_2 + iB_1 & B_0 + iB_3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} q_0 - iq_3 \\ q_2 + iq_1 \end{pmatrix} = 0 . \tag{7}$$

From the other hand, the (2+1)-dimensional Dirac equation for a massive particle has the form,

$$\left[i\gamma^k(\partial_k + iA_k) - m\right]\Psi(t,x,y) = 0 , \qquad (k = 1,2,3) \tag{8}$$

where i denotes the standard complex unit, A_k is the vector-potential and γ^3 plays the role of the γ^0 matrix. As the (2+1)-gamma-matrices, we can use the split quaternionic basis elements in $SU(1,1)$-matrix representation,

$$i\gamma^1 = e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad i\gamma^2 = e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad i\gamma^3 = e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} , \tag{9}$$

and identify the complex Dirac spinors $\Psi(t,x,y)$ with the complex $SU(1,1)$-representation of the quaternionic spinor $\xi^+(t,x,y)$. Then the matrix form of (2+1)-Dirac equation (8) obtains the form:

$$\left[\begin{pmatrix} -i\partial_t + m & \partial_y - i\partial_x \\ \partial_y + i\partial_x & i\partial_t + m \end{pmatrix} + \begin{pmatrix} A_t & A_x + iA_y \\ -A_x + iA_y & -A_t \end{pmatrix}\right]\begin{pmatrix} q_0 - iq_3 \\ q_2 + iq_1 \end{pmatrix} = 0 , \tag{10}$$

where $q^n(t,x,y)$ denote the four real components of the complex Dirac spinor $\xi^+(t,x,y)$.

Than assuming

$$B_0 = A_t , \qquad B_1 = A_y , \qquad B_2 = -A_x , \qquad B_3 = 0 , \tag{11}$$

the quaternionic analyticity conditions (7) becomes equivalent to the complex (2+1)-Dirac equation (10).

7. Quaternionic Maxwell equations

To write Maxwell's equations in (2+1)-spaces with the signature

$$\eta_{km} = \mathbf{diag}(+1,+1,-1) , \qquad (k,m = x,y,t) \tag{1}$$

let us define the 3-potential $A^k = (A^x, A^y, A^t)$ and the Faraday tensor,

$$F^{km} = \partial^k A^m - \partial^m A^k , \tag{2}$$

which has only three independent components. The relation between the vector potential and the magnetic and electric fields can be done through

$$H = \partial_x A_y - \partial_y A_x , \qquad E_x = \partial_x A_t - \partial_t A_x , \qquad E_y = \partial_y A_t - \partial_t A_y . \tag{3}$$

In covariant form, Maxwell's equations in (2+1)-spaces are then

$$\partial_k F^{mk} = j^m , \qquad \partial_k \tilde{F}^k = 0 , \tag{4}$$

where j_t is the surface charge and j_x and j_y are surface currents. One can work out also the differential Maxwell's equations in (2+1)-spaces in terms of the fields,

$$\begin{aligned} \partial_x E_x + \partial_y E_y = j_t \,, \qquad & \partial_t E_x - \partial_y H = -j_x \,, \\ \partial_x E_y - \partial_y E_x + \partial_t H = 0 \,, \qquad & \partial_t E_y + \partial_x H = -j_y \,. \end{aligned} \tag{5}$$

The split quaternion that contains the electromagnetic potentials we write as

$$A = A_0 + e_1 A_x + e_2 A_y + e_3 A_t \,, \tag{6}$$

where A_x, A_y and A_t are components of the (2+1)-vector and A_0 corresponds to extra degree of freedom in the quaternionic algebra. Using quaternionic gradient operator (5), the Cauchy-Riemann-Fueter analyticity condition (7) for the vector potential (6) can be written in the form:

$$\left(\partial_\lambda - e_1\partial_x - e_2\partial_y - e_3\partial_t\right)A = \left[\partial_\lambda A - \left(e_1\partial_x + e_2\partial_y + e_3\partial_t\right)A_0 - F\right] = 0 \,. \tag{7}$$

Here F denotes the quaternionic electro-magnetic field,

$$F = \left(e_1\partial_x + e_2\partial_y + e_3\partial_t\right)\left(e_1 A_x + e_2 A_y + e_3 A_t\right) = -\partial_k A^k - e_1 E_y + e_2 E_x + e_3 H \,, \tag{8}$$

where H, E_x and E_y are components of the magnetic and electric fields (3).

As in (2), we separate the variable λ in the scalar part of the vector-potential (6) (with the unit 'charge', $m = 1$),

$$\partial_\lambda A_0 = A_0 \,. \tag{9}$$

Similar to the case of spinors (4), we also assume that the derivative of the vector part of (6) by the extra time-like coordinate λ generates the supersymmetric (triality) transformations ($A \to \xi\chi$),

$$\partial_\lambda\left(e_1 A_x + e_2 A_y + e_3 A_t\right) = e_1 j_x + e_2 j_y - e_3 j_t \,, \tag{10}$$

where j_k are the components of the current vector, which is constructed by the covariant and contravarint spinors $j \sim \xi\chi$.

Using (9) and (10) the Cauchy-Riemann-Fueter analyticity condition (7) can be written as the first order Maxwell system [19–21]:

$$\begin{aligned} F &= A_0 \,, \\ \left(e_1\partial_x + e_2\partial_y + e_3\partial_t\right)A_0 &= -e_1 j_x - e_2 j_y + e_3 j_t \,. \end{aligned} \tag{11}$$

Utilize the auxiliary function A_0 this system can be transferred to the single second order Maxwell's equation in (2+1)-space,

$$\left(e_1\partial_x + e_2\partial_y + e_3\partial_t\right) F = -e_1 j_x - e_2 j_y + e_3 j_t \,. \tag{12}$$

Indeed, the left side of this equation under the Lorenz gauge,

$$\partial_k A^k = \partial_x A_x + \partial_y A_y - \partial_t A_t = 0 \,, \tag{13}$$

in $SU(1,1)$-matrix representation has the form:

$$\begin{aligned} &\begin{pmatrix} -i\partial_t & (\partial_y - i\partial_x) \\ (\partial_y + i\partial_x) & i\partial_t \end{pmatrix}\begin{pmatrix} -iB & (E_x + iE_y) \\ (E_x - iE_y) & iH \end{pmatrix} = \\ &= \left(\partial_y E_x - \partial_x E_y - \partial_t H\right)e_0 + \left(\partial_t E_x - \partial_y H\right)e_1 + \left(\partial_t E_y + \partial_x H\right)e_2 + \left(\partial_x E_x + \partial_y E_y\right)e_3 \,. \end{aligned} \tag{14}$$

Equating to zero the coefficients in front of the four quaternionic basis units, we obtain the system of four real equations which are identical to the system of the Maxwell equations (5).

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8. Conclusions

In thi paper it was shown that the quaternion analyticity condition, analog of the Cauchy-Riemann equations from complex analysis, applied to the triality invariant construction of split quaternions, is equivalent to some system of differential equations for quaternionic spinors and vectors. Assuming that derivatives by the extra time-like coordinate of quaternionic (2+2)-space generate triality (supersymmetric) rotations of vectors and spinors, the analyticity equations is reduced to the exact Dirac-Maxwell system in 3-dimensional Minkowski space-time.

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