PROCEEDINGS OF SCIENCE

$O(d,d)$ transformations preserve classical integrability

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In this article we summarize the relation between the classically integrable structure of two-dimensional sigma models and global $O(d,d)$ transformations. After giving a brief review of the classical integrability and the doubled formalism, we present a recipe for constructing Lax pairs in the $O(d,d)$-deformed models. The key point in our analysis is to apply the so-called $O(d,d)$(-duality) map involving winding coordinates. As an example we discuss the $O(2,2)$ transformation of the $SU(2)$ WZNW model, corresponding to the marginal $J\bar{J}$ deformation.

Corfu Summer Institute 2019 "School and Workshops on Elementary Particle Physics and Gravity" (CORFU2019)
31 August - 25 September 2019
Corfù, Greece

*Speaker.

PoS(CORFU2019)107

1. Introduction

The deformation technique of string backgrounds has commanded considerable attention in various contexts, ranging from the stringy realization of a supersymmetric (deformed) gauge theory to the integrable deformation of type IIB string theory in the $\mathrm{AdS}_5 \times S^5$ background, where the classical integrability is preserved [1]. The integrable deformation of string theory has been one of the significant research directions to push forward the confirmation of the gauge/gravity duality using the integrability techniques (see a review [2]). In particular, the so-called Yang-Baxter deformation is one of the systematic ways of performing such integrable deformations [3, 4, 5, 6]. Given a classical r-matrix, which solves the (modified) classical Yang-Baxter equations, the Yang-Baxter sigma model straightforwardly gives a corresponding deformed string background, where the classical integrability is always guaranteed by the existence of Lax pairs.

Using the Yang-Baxter deformation, there has been produced a variety of examples of the gauge/gravity duality so far. Interestingly, some of them can be interpreted in terms of geometrical transformations together with the T-duality transformations. For instance, Abelian r-matrices [7, 8, 9, 10] correspond to the so-called TsT transformation, which consists of two T-dualities and a shift transformation, acting on a two-torus fiber in the background [11, 12, 13, 14]. Besides, others can be interpreted as generated by the non-Abelian T-duality [15, 16]. Furthermore, the non-Abelian r-matrices lead to the solutions for the generalized supergravity equations of motion [17], and they are involved with the generalized T-duality [18]. The generalized supergravity is an extended supergravity framework by the addition of the non-geometric Q-fluxes, and its recent developments are summarized in [19].

More recently, some of Yang-Baxter deformations turn out to be interpreted as a current-current $J\bar{J}$ deformation [20]. Current-current deformations are traditional marginal deformations of two-dimensional conformal field theories. See [21, 22], for instance. They are known to be generated by transformations $g \in O(d) \times O(d) \subset O(d,d)$. In the case of a WZNW model on a compact Lie group, all maximal Abelian subgroups are pairwise conjugated by inner automorphisms so the complete deformation space is $D = O(r,r)/(O(r) \times O(r))$, where r is the rank of the group [23]. The feature of this deformation is that the flow of the deformed action with respect to the deformation parameter consists of the product of conserved (chiral/anti-chiral) currents J and $\bar{J}$ in the deformed model. In a separate development, in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the duality between the $J\bar{J}$ deformation on the AdS bulk and the $\mathrm{T\bar{T}}$ deformation on the boundary was shown [24]. The latter deformation has recently drawn much attention due to its nature of irrelevant but integrable deformations (see *e.g.* [25, 26] and also lecture notes [27]).

Admittedly, the Yang-Baxter deformation is a powerful method of generating integrable deformed string backgrounds. However, this machinery is a top-down approach in the sense that it is not so trivial to immediately interpret the resulting target space geometry in terms of geometrical transformations and stringy dualities. In our work [28], we take an opposite direction. Inspired by the recent interplay between Yang-Baxter deformations and $J\bar{J}$ deformations, we directly start from the $J\bar{J}$ deformation. Then we extract its classically integrable structure by constructing the Lax pair. In this bottom-up approach, we do not use the notion of Yang-Baxter deformations.

To construct the Lax pair in the $O(d,d)$-deformed sigma models, we studied the doubled sigma model [29, 30, 31, 32]. It is based on the doubled torus, which consists of the torus fiber in the background and its T-dual. In the doubled geometry, we thereby deal with the diffeomorphism, the B-field gauge transformations, and T-duality transformations on the same footing under the $O(d,d)$ group. The doubled sigma model action is invariant under global $O(d,d)$ duality transformations, and thus the action of $O(d,d)$ transformations can be well-controlled in this unifying description. Still, the doubled formalism is not a physical description by construction. To go back to a physical sigma model, the so-called self-duality constraint is imposed. This enables us to acquire the equivalence for the field equations between doubled and physical sigma models and plays a crucial role in our analysis. Historically, the T-duality invariant approach to string theory was first discussed in [33, 34], but here we adopt the framework by Hull [29, 30].

The rest of the article is organized as follows. In Section 2, we review the classical integrability of the WZNW model. In Section 3, we start from a brief review of the doubled formalism and construct the $O(d,d)$ map. Then we construct a recipe for obtaining the $O(d,d)$-deformed Lax pair in Section 4. In Section 5, we present an example of $J\bar{J}$-deformations generated by the $O(2,2)$ transformation. In Section 6, we make concluding remarks on further investigations and on the interplay between the duality invariant formulation and integrable deformations.

2. Classical integrability of WZNW models

Let us begin with the WZNW model action. The whole action is labelled by a group element $g \in G$ of our interest as follows:

$$\begin{aligned} S[g] &= -\frac{1}{4}\int_{\Sigma_2} \mathrm{Tr}\big[g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\big] + \frac{i\kappa}{3!}\int_{\mathscr{V}_3} \mathrm{Tr}\big[g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big] \\ &= -\frac{1}{4}\int_{\Sigma_2} \mathrm{Tr}[\omega_{\mathrm{L}} \wedge \star\omega_{\mathrm{L}}] + \frac{i\kappa}{3!}\int_{\mathscr{V}_3} \mathrm{Tr}[\omega_{\mathrm{L}} \wedge \omega_{\mathrm{L}} \wedge \omega_{\mathrm{L}}] \\ &= -\frac{1}{4}\int_{\Sigma_2} \mathrm{Tr}[\omega_{\mathrm{R}} \wedge \star\omega_{\mathrm{R}}] - \frac{i\kappa}{3!}\int_{\mathscr{V}_3} \mathrm{Tr}[\omega_{\mathrm{R}} \wedge \omega_{\mathrm{R}} \wedge \omega_{\mathrm{R}}], \end{aligned} \tag{2.1}$$

where for $\kappa = 1$ the model lies at the conformal point and the left(right)-invariant Maurer–Cartan one-forms $\omega_{\mathrm{L/R}}$ are respectively given by

$$\omega_{\mathrm{L}} = g^{-1}\mathrm{d}g, \qquad \omega_{\mathrm{R}} = -\mathrm{d}g\,g^{-1}. \tag{2.2}$$

satisfying the Maurer–Cartan equations $\mathrm{d}\omega + \omega \wedge \omega = 0$.

Varyng the action (2.1) with respect to g, we find the equations of motion to be

$$\mathrm{d}\star(1 - i\kappa\star)\,\omega_{\mathrm{L}} = 0, \qquad \mathrm{d}\star(1 + i\kappa\star)\,\omega_{\mathrm{R}} = 0, \tag{2.3}$$

where the Hodge duality on the world-sheet satisfies $\star^2 = 1$. The above equations of motion can be seen as the conservation laws for the following currents:

$$J_{\mathrm{L}} = (1 - i\kappa\star)\,\omega_{\mathrm{L}}, \qquad J_{\mathrm{R}} = (1 + i\kappa\star)\,\omega_{\mathrm{R}}. \tag{2.4}$$

It is important to note that these conserved currents also satisfy the flatness (or zero-curvature) condition irrespective of any value of κ:

$$\mathrm{d}J_{\mathrm{L}} + J_{\mathrm{L}} \wedge J_{\mathrm{L}} = (1 + \kappa^2)(\mathrm{d}\omega_{\mathrm{L}} + \omega_{\mathrm{L}} \wedge \omega_{\mathrm{L}}) = 0. \tag{2.5}$$

The same applies for J_{R}. Thus, in the WZNW model, the flat currents can be easily constructed.

The existence of flat currents is essential for proving the classical integrability of a model. Suppose that our model possesses a flat conserved current J. Then we define the Lax pair as

$$\mathscr{L}_\lambda = a_\lambda J + b_\lambda \star J, \tag{2.6}$$

where $\lambda \in \mathbb{R}$ is the spectral parameter. We require $\mathscr{L}_\lambda$ to satisfy the flatness condition

$$\mathrm{d}\mathscr{L}_\lambda + \mathscr{L}_\lambda \wedge \mathscr{L}_\lambda = 0 \tag{2.7}$$

on shell, so that the coefficients a_λ, b_λ must be determined to be

$$a_\lambda = \frac{1}{2}(1 \pm \cosh(\lambda)), \qquad b_\lambda = \frac{1}{2}\sinh(\lambda). \tag{2.8}$$

Having an infinite number of conserved charges is referred to as classical integrability. Let us look at how to generate infinitely many conserved charges using the Lax pair [35]. Once we have defined the Lax pair (2.6), we define the so-called monodromy matrix as the path-ordered exponential of the spatial component of the Lax connection:

$$\mathscr{T}(t;\lambda) = \mathrm{P}\left[\exp\left(-\int_{-\infty}^{+\infty} \mathscr{L}_{\lambda,x}(x')\mathrm{d}x'\right)\right]. \tag{2.9}$$

The derivative of the monodromy matrix for the world-sheet time t is given by

$$\frac{\mathrm{d}\mathscr{T}}{\mathrm{d}t}(t;\lambda) = -\,\partial_t \mathscr{L}_\lambda|_{x\to+\infty}\mathscr{T}(t;\lambda) + \mathscr{T}(t;\lambda)\partial_t\,\mathscr{L}_\lambda|_{x\to-\infty}, \tag{2.10}$$

where we used the flatness condition (2.7). Therefore, if the Lax pair satisfies the boundary conditions $\partial_t\mathscr{L}_\lambda \to 0$ as $x \to \pm\infty$, the whole monodromy matrix (2.9) is conserved.

Moreover, if we expand $\mathscr{T}(t;\lambda)$ around $\lambda = 0$,

$$\mathscr{T}(t;\lambda) = 1 + \sum_{n=0}^{\infty} \lambda^{n+1} Q^{(n)}(t), \tag{2.11}$$

we see that the conservation law for $\mathscr{T}(t;\lambda)$ is understood as the conservation of each coefficient $Q^{(n)}(t)$, corresponding to an infinite tower of conserved charges:

$$\frac{\mathrm{d}Q^{(n)}}{\mathrm{d}t}(t) = 0, \qquad \forall n = 0, 1, \ldots. \tag{2.12}$$

For the WZNW model (2.1), taking the flat currents $J_{\mathrm{L/R}}$ (2.4), we can construct the corresponding Lax pairs both in left and right sectors. Thus, the WZNW model is shown as classically integrable.

3. Doubled sigma model and $O(d,d)$ transformations

Here we give the basics of the doubled sigma model and $O(d,d)$ transformations. For simplicity, let us restrict ourselves to an NSNS string background in D dimensions with a metric G and a two-form B. For our construction of $O(d,d)$-deformed Lax pairs, the dilaton does not get involved. Then we write a string sigma model as

$$S[G,B] = \frac{1}{2}\int G_{\hat{\imath}\hat{\jmath}}(X)\,\mathrm{d}X^{\hat{\imath}} \wedge \star\mathrm{d}X^{\hat{\jmath}} + B_{\hat{\imath}\hat{\jmath}}(X)\,\mathrm{d}X^{\hat{\imath}} \wedge \mathrm{d}X^{\hat{\jmath}}, \tag{3.1}$$

where $X^{\hat{\imath}}$, $\hat{\imath} = 1,\dots,D$ denotes a set of local coordinates on the patch of the target space.

Suppose that our target space geometry $\mathscr{M}$ admits a group of isometries G. Then we focus on a submanifold $M \subset \mathscr{M}$, whose maximal torus $T^d \subset G$, $d \leq D$ of the full isometry group acts freely. Thus, local coordinates X^i on the torus, called adapted coordinates, generate Abelian Killing vectors $\partial/\partial X^i$. Then we take our target space, labelled by $X^\imath$, to be T^d-fibered over some base manifold parametrized by Y.

Focusing on the torus fiber, we extract the components of the metric G_{ij} and B-field B_{ij}, where $i,j = 1,\dots,d$, along T^d from the action (3.1). Then they are merged into the so-called generalized metric

$$\mathscr{H}(G,B)_{\hat{I}\hat{J}} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}_{\hat{I}\hat{J}}, \qquad \hat{I},\hat{J} = 1,\dots,2d\,, \tag{3.2}$$

which itself belongs to the $O(d,d)$ group:

$$\mathscr{H}^t L \mathscr{H} = L\,, \tag{3.3}$$

with the indefinite matrix L

$$L = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}. \tag{3.4}$$

The generalized metric is encoded in the doubled sigma model as the curved background metric.

Below we embed the above construction of the target space into the doubled formalism [29]. The doubled manifold is locally described by a patch of M and a patch of the T-dual $\widetilde{M}$. Note that both M and $\widetilde{M}$ are originally not independent of each other, but T-dual to each other. Let us introduce a set of local coordinates on a patch of the doubled manifold by $\mathbb{X}^I, I = 1,\dots,2d$. This is called a generalized vector $\mathbb{X}^I$, the doublet of local coordinates on M and $\widetilde{M}$:

$$\mathbb{X}^I = \begin{pmatrix} X^i & \widetilde{X}_i \end{pmatrix}^t, \qquad i = 1,\dots,d\,. \tag{3.5}$$

Note that when we go to a standard sigma model from the doubled formalism, we need to introduce a polarization to select physical coordinates from $\mathbb{X}$. For simplicity, we will use the so-called standard polarization, which takes the first half components of $\mathbb{X}$ to be the physical ones.

Now we are in a position to reveal the action of the doubled sigma model [29, 30, 31, 32]:

$$S_d = \int \frac{1}{2}\mathscr{H}_{IJ}\mathrm{d}\mathbb{X}^I \wedge \star\mathrm{d}\mathbb{X}^J + \mathrm{d}\mathbb{X}^I \wedge \star\mathscr{J}_I(Y) + \mathscr{L}(Y)\,, \tag{3.6}$$

where the first term corresponds to the kinetic term with a canonical prefactor $\frac{1}{2}$ while the second term is called a source term $\mathscr{J}_I$ dependent only on Y. The Lagrangian density on the base

$\mathscr{L}(Y)$ depends only on Y. As remarked earlier, the above action is manifestly invariant under the following $O(d,d;\mathbb{Z})$ transformations for $g \in O(d,d;\mathbb{Z})$:

$$\mathscr{H} \to g^t \mathscr{H} g, \qquad \mathrm{d}\mathbb{X} \to g^{-1}\mathrm{d}\mathbb{X}, \qquad \mathscr{J} \to g^t \mathscr{J}. \tag{3.7}$$

The first rule implies that we can redefine the transformed generalized metric as a new one in terms of new background fields

$$g^t \mathscr{H} g \equiv \mathscr{H}'(G',B') \tag{3.8}$$

in a new generalized vector $\mathbb{X}'$, whose derivatives satisfy

$$\mathrm{d}\mathbb{X}' = g^{-1}\mathrm{d}\mathbb{X}. \tag{3.9}$$

The above actions of $O(d,d;\mathbb{Z})$ (3.7) can be naturally extended to those of $O(d,d;\mathbb{R})$. Then an $O(d,d;\mathbb{R})$ element g includes a real continuous deformation parameter, say α. The field redefinition (3.8) plays a role in a solution generating technique to give rise to a new metric G' and a new B-field B' deformed by α.

Finally, let us comment on the constraint on the doubled sigma model. Here we set the source term in the action (3.6) to be zero for simplicity. By construction, the doubled formalism has redundant degrees of freedom from the perspective of the standard sigma model. Still, both sigma models must describe the same field equations for consistency. To achieve the right number of physical degrees of freedom and the equivalence for dynamics, we need to impose the so-called self-duality constraint:

$$\mathrm{d}\mathbb{X}^I = L^{IJ}\mathscr{H}_{JK} \star \mathrm{d}\mathbb{X}^K. \tag{3.10}$$

Note that as long as this constraint holds, the equations of motion of the doubled sigma model (3.6) are always satisfied.

Using the self-duality constraint, let us observe how to remove the winding coordinates $\widetilde{X}_i$, which are unphysical in the standard polarization. Using the expressions (3.2), (3.4), and (3.5), the first component of the constraint reads

$$\mathrm{d}\widetilde{X}_i = \star\left(G_{ij}\mathrm{d}X^j + B_{ij}\star\mathrm{d}X^j\right) = \star J_i, \tag{3.11}$$

where J_i is the conserved currents associated to the freely acting $U(1)^d$ isometries along the commuting Killing vectors $\partial/\partial X^i$. The result states that the differentials of the winding coordinates $\widetilde{X}_i$ turn into the Hodge duals of Noether currents J_i. The fact that $\mathrm{d}\widetilde{X}_i$ is exact is consistent with the conservation of the currents J_i:

$$\mathrm{d}^2\widetilde{X}_i = 0 = \mathrm{d}\star J_i, \tag{3.12}$$

which corresponds to an on-shell condition along the physical torus fiber.

Recall that $O(d,d)$ transformations act on both $\mathrm{d}X^i$ and $\mathrm{d}\widetilde{X}_i$ according to the second rule in (3.7). Therefore, after $O(d,d;\mathbb{R})$ deformations, the (differentials of) winding coordinates $\mathrm{d}\widetilde{X}_i$ should contribute to the resulting Lax pairs. Still, to rewrite the deformed Lax connections using the physical coordinates $\mathrm{d}X'^i$ only, we should construct a direct relation between $\mathrm{d}X^i$ and $\mathrm{d}X'^i$. For this purpose, let us first take g to be a generic $O(d,d)$ element. It is given by

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{3.13}$$

where α, β, γ and δ are $d\times d$ matrices with the following the index structures:

$$\alpha^i{}_j, \qquad \beta^{ij}, \qquad \gamma_{ij}, \qquad \delta_i{}^j, \qquad i,j=1,\dots,d. \tag{3.14}$$

By definition g preserves the indefinite metric (3.4):

$$g^t L g = L. \tag{3.15}$$

Then the inverse of g is given by

$$g^{-1} = \begin{pmatrix} \delta^t & \beta^t \\ \gamma^t & \alpha^t \end{pmatrix}. \tag{3.16}$$

Therefore, under general $O(d,d)$ transformations as well as the constraint (3.11), the field redefinition for $\mathrm{d}\mathbb{X}^I$ (3.9) can be explicitly rewritten as

$$\begin{aligned} \mathrm{d}X'^i &= (\delta^t)^i{}_j \mathrm{d}X^j + \left(\beta^t\right)^{ik} \star J_k, \\ \mathrm{d}\widetilde{X}'_i &= \star J'_i = (\gamma^t)_{ij}\mathrm{d}X^j + (\alpha^t)_i{}^k \star J_k, \end{aligned} \tag{3.17}$$

whereas their inverse is

$$\begin{aligned} \mathrm{d}X^i &= \alpha^i{}_j \mathrm{d}X'^j + \beta^{ij}\mathrm{d}\widetilde{X}'_j = \alpha^i{}_j \mathrm{d}X'^j + \beta^{ik}\star J'_k, \\ \mathrm{d}\widetilde{X}_i &= \star J_i = \gamma_{ij}\mathrm{d}X'^j + \delta_i^k \star J'_k. \end{aligned} \tag{3.18}$$

Using the constraint (3.11) once more, we can deduce a relation between the physical coordinates of the original and of the $O(d,d)$-deformed model:

$$\mathrm{d}X^i = \alpha^i{}_j\mathrm{d}X'^j + \beta^{ik}\star J'_k = \beta^{ij}G'_{jk}\star\mathrm{d}X'^k + (\alpha+\beta B')^i{}_j\mathrm{d}X'^j, \tag{3.19}$$

where the deformed metric and B-field are computed using (3.8) to be [36]

$$\begin{aligned} G' &= \rho_1^{-1}G(\rho_1^{-1})^t \qquad \text{with} \quad \rho_1 = \delta + (G-B)\beta, \\ B' &= \rho_1^{-1}\left[\rho_1\rho_2^t - G\right](\rho_1^{-1})^t \qquad \text{with} \quad \rho_2 = \gamma + (G-B)\alpha. \end{aligned} \tag{3.20}$$

For $g\in O(d,d;\mathbb{Z})$, the relation (3.19) is the so-called $O(d,d)$-"duality" map derived in [37]. On the other hand, for $g\in O(d,d;\mathbb{R})$, it is merely the coordinate transformation. In the spirit of [38], it can be called a field-dependent diffeomorphism. It is intriguing to note that the $O(d,d)$ map (3.19) is non-local in the sense that the Hodge duals of the conserved currents J_i are involved. We will exploit this map to directly construct the $O(d,d)$-deformed Lax pair in the next sections.

4. Recipe for $O(d,d)$-deformed Lax pairs

We have seen so far the construction of the Lax pair in the initial model, the action of the $O(d,d)$ group on the background fields, and the $O(d,d)$ map. Now we are in a position to present a recipe for Lax connections in the $O(d,d)$-deformed models.

Suppose that the initial sigma model is integrable and the initial Lax pair $\mathscr{L}_\lambda$ is explicitly given. They may explicitly depend on local coordinates $X^i, i=1,\dots d$ on the T^d fiber even if the background admits $U(1)$ isometries along T^d. If this is the case, we have to remove the explicit dependence on X^i from the Lax pair, as the $O(d,d)$ map (3.19) acts on the adapted coordinates only through their derivatives. To make the Lax pair manifestly invariant under the the $U(1)$ isometries, we need to perform a proper gauge transformation as follows:

$$\mathscr{L}_\lambda \to \hat{\mathscr{L}}_\lambda = h^{-1}\mathscr{L}_\lambda h + h^{-1}\mathrm{d}h, \tag{4.1}$$

where $h \in G$ with G a symmetry group of the initial model. Then the gauged Lax pair $\hat{\mathscr{L}}_\lambda$ satisfies the flatness condition on shell,

$$\mathrm{d}\hat{\mathscr{L}}_\lambda + \hat{\mathscr{L}}_\lambda \wedge \hat{\mathscr{L}}_\lambda = h^{-1}(\mathrm{d}\mathscr{L}_\lambda + \mathscr{L}_\lambda \wedge \mathscr{L}_\lambda)h = 0, \tag{4.2}$$

where the last equality trivially holds due to the flatness of the initial Lax pair.

Once we have prepared the gauged Lax pair which does not explicitly depend on the adapted coordinates, we apply the $O(d,d)$ map (3.19) to find the deformed Lax pair $\mathscr{L}'_\lambda$

$$\hat{\mathscr{L}}_\lambda(\mathrm{d}X^i) \to \mathscr{L}'_\lambda(\mathrm{d}X'^i) = \hat{\mathscr{L}}_\lambda(\mathrm{d}X^i \to \alpha^i{}_j \mathrm{d}X'^j + \beta^{ik} \star J'_k). \tag{4.3}$$

Let us check that $\mathscr{L}'_\lambda$ fulfills the zero-curvature condition on shell in the $O(d,d)$-deformed model. First, we note that the zero-curvature condition of the Lax pair $\hat{\mathscr{L}}_\lambda$ can be understood as a linear combination of the equations of motion of the initial model,

$$\mathrm{d}\hat{\mathscr{L}}_\lambda + \hat{\mathscr{L}}_\lambda \wedge \hat{\mathscr{L}}_\lambda = \sum_m \mathrm{EoM}_m(\mathrm{d}X^i, Y) = 0, \tag{4.4}$$

Then the new Lax pair $\mathscr{L}'_\lambda$ should satisfy by construction

$$\mathrm{d}\mathscr{L}' + \mathscr{L}' \wedge \mathscr{L}' = \sum_m \mathrm{EoM}_m(\mathscr{D}(\mathrm{d}X), Y), \tag{4.5}$$

where $\mathscr{D}$ denotes the $O(d,d)$ map (3.19). If this vanishes on shell in the $O(d,d)$-deformed model, the right hand side should be rewritten as a set of the equations of motion in the deformed system:

$$\{\mathrm{EoM}(\mathscr{D}(\mathrm{d}X), Y)\} = \{\mathrm{EoM}'(\mathrm{d}X', Y)\}, \tag{4.6}$$

where EoM' denotes the equations of motion for X' and Y. In other words, the $O(d,d)$ map has to keep the equivalence between the sets of field equations both in the initial and deformed systems.

To see this explicitly, we observe that the equations of motion of the standard sigma model are equivalent to those of the doubled sigma model under the self-duality constraint (3.10). For example, the field equations for X'^i are related to those for X^i via

$$0 = \mathrm{d}^2\widetilde{X}'_i = \mathrm{d}\star J'_i = \left(\gamma^t\right)_{ij} \mathrm{d}^2 X^j + \left(\alpha^t\right)_i{}^k \mathrm{d}\star J_k = \alpha^k{}_i \mathrm{d}\star J_k, \tag{4.7}$$

where the last equality leads to a linear combination of the conservation laws for the initial $U(1)$ currents J_k. On the other hand, the equivalence of the equations of motion for the base coordinate Y under $O(d,d)$ transformations is clear from the $O(d,d)$ invariance of the doubled sigma model.

Consequently, the flatness condition of the transformed Lax pair can be written as a linear combination of the equations of motion of the deformed model and is fulfilled on shell

$$\mathrm{d}\mathscr{L}'_\lambda + \mathscr{L}'_\lambda \wedge \mathscr{L}'_\lambda = \sum_m \mathrm{EoM}_m(\mathscr{D}(\mathrm{d}X^i), Y) = \sum_m \Lambda^n{}_m \mathrm{EoM}'_n(\mathrm{d}X'^i, Y) = 0. \tag{4.8}$$

In summary, by rewriting $\mathrm{d}X^i$ in the gauged Lax pair $\hat{\mathscr{L}}_\lambda$ via the $O(d,d)$ map (3.19), we have straightforwardly constructed the $O(d,d)$-deformed Lax pair $\mathscr{L}'_\lambda{}'$. Like this, the classical integrability of the string sigma model is preserved on general grounds under general $O(d,d)$ transformations including the $O(d,d;\mathbb{Z})$ dualities and the solution generating techniques by $O(d,d;\mathbb{R})$.

5. An example: the $J\bar{J}$ deformation of the $SU(2)$ WZNW model

As an application of our recipe for the $O(d,d)$-deformed Lax pair, we discuss an $O(2,2)$ deformation of the $SU(2)$ WZNW model. It corresponds to the $J\bar{J}$ deformation [21, 22, 23].

5.1 $SU(2)$ WZNW model

Let us start to parametrize $g \in SU(2)$ as

$$g = e^{-(Z_1+Z_2)T_2} e^{Y T_1} e^{+(Z_1-Z_2)T_2}, \tag{5.1}$$

where the generators T_α, $\alpha = 1,2,3$ are defined in terms of the usual Pauli matrices,

$$T_\alpha = -\frac{i}{2}\sigma_\alpha, \qquad \alpha = 1,2,3, \tag{5.2}$$

satisfying $[T_\alpha, T_\beta] = \varepsilon_{\alpha\beta\gamma}T_\gamma$ and $\mathrm{Tr}(T_\alpha T_\beta) = -\frac{1}{2}\delta_{\alpha\beta}$, where $\varepsilon_{123} = 1$.

Using the above parametrization, we can write down the action (2.1) for $\kappa = 1$ as follows:

$$\begin{aligned}
-\frac{1}{4}\mathrm{Tr}\left[g^{-1}\mathrm{d}g \wedge \star g^{-1}\mathrm{d}g\right] &= \frac{1}{8}\mathrm{d}Y \wedge \star \mathrm{d}Y + \frac{1}{2}\sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1 \wedge \star \mathrm{d}Z_1 + \frac{1}{2}\cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_2 \wedge \star \mathrm{d}Z_2, \\
\frac{i}{3!}\mathrm{Tr}\left[g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right] &= -i\sin\left(\frac{Y}{2}\right)\cos\left(\frac{Y}{2}\right)\mathrm{d}Z_1 \wedge \mathrm{d}Z_2 \wedge \mathrm{d}Y,
\end{aligned} \tag{5.3}$$

from which we can read off the components of the metric and B-field along (Z_1, Z_2) as

$$G_{ij} = \begin{pmatrix} \sin^2\left(\frac{Y}{2}\right) & 0 \\ 0 & \cos^2\left(\frac{Y}{2}\right) \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} 0 & \cos^2\left(\frac{Y}{2}\right) \\ -\cos^2\left(\frac{Y}{2}\right) & 0 \end{pmatrix}. \tag{5.4}$$

Moreover, the equation of motion for Y reads

$$\mathrm{d}\star\mathrm{d}Y - \sin(Y)(\mathrm{d}Z_1 \wedge \star\mathrm{d}Z_1 - \mathrm{d}Z_2 \wedge \star\mathrm{d}Z_2 - 2i\mathrm{d}Z_1 \wedge \mathrm{d}Z_2) = 0\,. \tag{5.5}$$

Using the above choice of the B-field (5.4), the corresponding sigma model admits two $U(1)$ isometries along $\partial/\partial Z_1$ and $\partial/\partial Z_2$. We will choose Z_1 and Z_2 to be adapted coordinates in the $O(2,2)$ doubled formalism.

5.2 Gauged Lax pairs

The Lax pairs in the initial system can be explicitly obtained via the flat currents (2.4). It is not hard to see that the results explicitly depend on the adapted coordinates Z_1, Z_2. Thus, we need to perform suitable gauge transformations both in the left and right sectors. The right gauge choice for the left sector is

$$h = e^{-Z_- T_2}, \tag{5.6}$$

then the gauged Lax pair (4.1) decomposes on the basis of T_α as[1]

$$\begin{aligned}
\hat{\mathscr{L}}^1_{\mathrm{L}} &= +((ib-a)+(ia-b)\star)\,\mathrm{d}Y, \\
\hat{\mathscr{L}}^2_{\mathrm{L}} &= -((ib-a)+(ia-b)\star)\,(\mathrm{d}Z_- - \cos(Y)\mathrm{d}Z_+) - \mathrm{d}Z_-, \\
\hat{\mathscr{L}}^3_{\mathrm{L}} &= +((ib-a)+(ia-b)\star)\sin(Y)\mathrm{d}Z_+,
\end{aligned} \tag{5.7}$$

[1] We leave out the subscript λ denoting the spectral parameter in (2.6) and (2.8).

where a,b are given in (2.8) and we introduce the coordinates $Z_\pm = Z_1 \pm Z_2$.

On the other hand, if we choose for the right sector

$$h = e^{-Z_+ T_2}, \tag{5.8}$$

then we obtain

$$\begin{aligned}
\hat{\mathscr{L}}^1_{\rm R} &= +((ib+a)+(ia+b)\star)\,\mathrm{d}Y,\\
\hat{\mathscr{L}}^2_{\rm R} &= +((ib+a)+(ia+b)\star)\,(\mathrm{d}Z_+ - \cos(Y)\mathrm{d}Z_-) - \mathrm{d}Z_+,\\
\hat{\mathscr{L}}^3_{\rm R} &= +((ib+a)+(ia+b)\star)\sin(Y)\mathrm{d}Z_-.
\end{aligned} \tag{5.9}$$

Note that for both sectors, the first components $\hat{\mathscr{L}}^1_{\rm L/R}$ do not depend on $\mathrm{d}Z_\pm$ at all. Thus they will not be subject to the $O(2,2)$ map obtained below.

For clarity, we explicitly calculate the curvatures of the gauged Lax pairs (5.7) and (5.9) as

$$\begin{aligned}
\mathrm{d}\hat{\mathscr{L}}^1_{\rm L} + \hat{\mathscr{L}}^2_{\rm L}\wedge\hat{\mathscr{L}}^3_{\rm L} &= (ia-b)\left[\mathrm{d}\star\mathrm{d}Y - \sin(Y)\,(\mathrm{d}Z_1\wedge\star\mathrm{d}Z_1 - \mathrm{d}Z_2\wedge\star\mathrm{d}Z_2 - 2i\mathrm{d}Z_1\wedge\mathrm{d}Z_2)\right],\\
\mathrm{d}\hat{\mathscr{L}}^2_{\rm L} + \hat{\mathscr{L}}^3_{\rm L}\wedge\hat{\mathscr{L}}^1_{\rm L} &= -2(ia-b)\,(\mathrm{d}\star J_1 - \mathrm{d}\star J_2),\\
\mathrm{d}\hat{\mathscr{L}}^3_{\rm L} + \hat{\mathscr{L}}^1_{\rm L}\wedge\hat{\mathscr{L}}^2_{\rm L} &= 2(ia-b)\left[\cot\left(\frac{Y}{2}\right)\mathrm{d}\star J_1 + \tan\left(\frac{Y}{2}\right)\mathrm{d}\star J_2\right],
\end{aligned} \tag{5.10}$$

whereas

$$\begin{aligned}
\mathrm{d}\hat{\mathscr{L}}^1_{\rm R} + \hat{\mathscr{L}}^2_{\rm R}\wedge\hat{\mathscr{L}}^3_{\rm R} &= (ia+b)\left[\mathrm{d}\star\mathrm{d}Y - \sin(Y)\,(\mathrm{d}Z_1\wedge\star\mathrm{d}Z_1 - \mathrm{d}Z_2\wedge\star\mathrm{d}Z_2 - 2i\mathrm{d}Z_1\wedge\mathrm{d}Z_2)\right],\\
\mathrm{d}\hat{\mathscr{L}}^2_{\rm R} + \hat{\mathscr{L}}^3_{\rm R}\wedge\hat{\mathscr{L}}^1_{\rm R} &= 2(ia+b)\,(\mathrm{d}\star J_1 + \mathrm{d}\star J_2),\\
\mathrm{d}\hat{\mathscr{L}}^3_{\rm R} + \hat{\mathscr{L}}^1_{\rm R}\wedge\hat{\mathscr{L}}^2_{\rm R} &= +2(ia+b)\left[\cot\left(\frac{Y}{2}\right)\mathrm{d}\star J_1 - \tan\left(\frac{Y}{2}\right)\mathrm{d}\star J_2\right],
\end{aligned} \tag{5.11}$$

where J_1 and J_2 denote the conserved currents for the $U(1)^2$ isometries along $\partial/\partial Z_1$ and $\partial/\partial Z_2$, respectively:

$$J_1 = \sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z_1 - i\sin^2\left(\frac{Y}{2}\right)\star\mathrm{d}Z_2, \qquad J_2 = \cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z_2 + i\sin^2\left(\frac{Y}{2}\right)\star\mathrm{d}Z_1. \tag{5.12}$$

Therefore, the gauged Lax pairs vanish under the equations of motion (5.5) and the conservation laws for J_1 and J_2.

5.3 The $O(2,2)$ doubled formalism and deformed action

As mentioned before, by taking Z_1 and Z_2 to be adapted coordinates on the T^2 fiber, we introduce the generalized vector

$$\mathbb{X}^I = \begin{pmatrix} Z_1 & Z_2 & \widetilde{Z}_1 & \widetilde{Z}_2 \end{pmatrix}^t, \tag{5.13}$$

and the corresponding generalized metric (3.2) is computed using (5.4) as

$$\mathscr{H} = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & +\cot^2\left(\frac{Y}{2}\right) & -\cot^2\left(\frac{Y}{2}\right) & 0\\ 0 & -\cot^2\left(\frac{Y}{2}\right) & +\csc^2\left(\frac{Y}{2}\right) & 0\\ 1 & 0 & 0 & \sec^2\left(\frac{Y}{2}\right) \end{pmatrix}. \tag{5.14}$$

Let us compute the $J\bar{J}$-deformed sigma model action. To realize the deformation of our interest, the following $O(2,2;\mathbb{R})$ element is used (see [21] for detail):

$$g_{J\bar{J}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \tan(\alpha) \\ 0 & \frac{\cos(\alpha)}{\cos(\alpha)+\sin(\alpha)} & -\frac{\sin(\alpha)}{\cos(\alpha)+\sin(\alpha)} & 0 \\ 0 & \frac{\cos(\alpha)}{\cos(\alpha)+\sin(\alpha)} & \frac{\cos(\alpha)}{\cos(\alpha)+\sin(\alpha)} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{5.15}$$

for a real parameter $\alpha \in \mathbb{R}$. Then we find the new generalized metric (3.8) to be

$$\mathscr{H}' = \begin{pmatrix} \tan^2\left(\frac{Y}{2}\right) & 0 & 0 & -\tan^2\left(\frac{Y}{2}\right) \\ 0 & \frac{\cos^2(\alpha)}{(\cos(\alpha)+\sin(\alpha))^2} & \frac{\cos^2(\alpha)}{(\cos(\alpha)+\sin(\alpha))^2} & 0 \\ 0 & \frac{\cos^2(\alpha)}{(\cos(\alpha)+\sin(\alpha))^2} & \frac{\Delta_1}{(\cos(\alpha)+\sin(\alpha))^2\sin^2\left(\frac{Y}{2}\right)} & 0 \\ -\tan^2\left(\frac{Y}{2}\right) & 0 & 0 & \frac{\Delta_1}{\cos^2(\alpha)\cos^2\left(\frac{Y}{2}\right)} \end{pmatrix}, \tag{5.16}$$

where

$$\Delta = \cos^2(\alpha) + \cos^2\left(\frac{Y}{2}\right)\sin(\alpha)(\sin(\alpha)+2\cos(\alpha)). \tag{5.17}$$

Thus, we read off the deformed background data, and write the deformed action as

$$\begin{aligned} S'_\alpha[Y,Z'_1,Z'_2] = \frac{1}{2}\int_{\Sigma_2} \frac{1}{4}\mathrm{d}Y\wedge\star\mathrm{d}Y + \frac{1}{\Delta}(\cos(\alpha)+\sin(\alpha))^2\sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z'_1\wedge\star\mathrm{d}Z'_1 \\ + \frac{1}{\Delta}\cos^2(\alpha)\cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z'_2\wedge\star\mathrm{d}Z'_2 - \frac{2i}{\Delta}\cos^2(\alpha)\sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z'_1\wedge\mathrm{d}Z'_2 . \end{aligned} \tag{5.18}$$

The equations of motion for Z'_1, Z'_2 can be expressed as

$$\mathrm{d}\star J_1(\alpha) = 0, \qquad \mathrm{d}\star J_2(\alpha) = 0, \tag{5.19}$$

using the $U(1)^2$ conserved currents of the deformed model

$$J_1(\alpha) = \frac{1}{\Delta}\left[(\cos(\alpha)+\sin(\alpha))^2\sin^2\left(\frac{Y}{2}\right)\mathrm{d}Z'_1 - i\cos^2(\alpha)\sin^2\left(\frac{Y}{2}\right)\star\mathrm{d}Z'_2\right], \tag{5.20}$$

$$J_2(\alpha) = \frac{1}{\Delta}\cos^2(\alpha)\left[\cos^2\left(\frac{Y}{2}\right)\mathrm{d}Z'_2 + i\sin^2\left(\frac{Y}{2}\right)\star\mathrm{d}Z'_1\right]. \tag{5.21}$$

Obviously, they are associated with the commuting Killing vectors $\partial/\partial Z'_1$ and $\partial/\partial Z'_1$, respectively.

5.4 The $O(2,2)$ map and deformed Lax pairs

Finally, we construct the Lax pairs in the $J\bar{J}$-deformed model. Using (5.15), we explicitly compute the $O(2,2)$ map (3.19) as

$$\mathrm{d}Z_1 = \mathrm{d}Z'_1 + \tan(\alpha)\star J_2(\alpha), \qquad \mathrm{d}Z_2 = \frac{1}{1+\tan(\alpha)}\left[\mathrm{d}Z'_2 - \tan(\alpha)\star J_1(\alpha)\right]. \tag{5.22}$$

Therefore, the $J\bar{J}$-deformed Lax pair $\mathscr{L}' = \hat{\mathscr{L}}(\mathrm{d}Z_{1/2} \to \mathrm{d}Z'_{1/2})$ is obtained as follows:

$$\begin{aligned}
\mathscr{L}'^{1}_{\mathrm{L}} &= \Big((ib-a)+(ia-b)\star\Big)\mathrm{d}Y, \\
\mathscr{L}'^{2}_{\mathrm{L}} &= -2\Big((ib-a)+(ia-b)\star\Big)\left[\begin{pmatrix}\sin^2\left(\frac{Y}{2}\right) & \frac{-\cos^2\left(\frac{Y}{2}\right)}{1+\tan(\alpha)}\end{pmatrix}\begin{pmatrix}\mathrm{d}Z'_1+\tan(\alpha)\star J_2(\alpha) \\ \mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)\end{pmatrix}\right] \\
&\qquad -\mathrm{d}Z'_1-\tan(\alpha)\star J_2(\alpha)+\frac{1}{1+\tan(\alpha)}\left(\mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)\right), \\
\mathscr{L}'^{3}_{\mathrm{L}} &= \Big((ib-a)+(ia-b)\star\Big)\sin(Y)\left[\mathrm{d}Z'_1+\tan(\alpha)\star J_2(\alpha)+\frac{\mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)}{1+\tan(\alpha)}\right],
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
\mathscr{L}'^{1}_{\mathrm{R}} &= \Big((ib+a)+(ia+b)\star\Big)\mathrm{d}Y, \\
\mathscr{L}'^{2}_{\mathrm{R}} &= 2\Big((ib+a)+(ia+b)\star\Big)\left[\begin{pmatrix}\sin^2\left(\frac{Y}{2}\right) & \frac{+\cos^2\left(\frac{Y}{2}\right)}{1+\tan(\alpha)}\end{pmatrix}\begin{pmatrix}\mathrm{d}Z'_1+\tan(\alpha)\star J_2(\alpha) \\ \mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)\end{pmatrix}\right] \\
&\qquad -\mathrm{d}Z'_1-\tan(\alpha)\star J_2(\alpha)-\frac{1}{1+\tan(\alpha)}\left(\mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)\right), \\
\mathscr{L}'^{3}_{\mathrm{R}} &= +\Big((ib+a)+(ia+b)\star\Big)\sin(Y)\left[\mathrm{d}Z'_1+\tan(\alpha)\star J_2(\alpha)-\frac{\mathrm{d}Z'_2-\tan(\alpha)\star J_1(\alpha)}{1+\tan(\alpha)}\right],
\end{aligned} \tag{5.24}$$

where the inner product of the two-component vectors is used in $\mathscr{L}'^{2}_{\mathrm{L/R}}$. Thus we were able to explicitly construct the Lax pairs of the $J\bar{J}$-deformed model based on our general recipe in Section 4. Note that the $O(2,2;\mathbb{R})$ deformation breaks the original isometry $SU(2)\times SU(2)$ to $U(1)^2$ while the deformed Lax pairs (5.23) and (5.24) still form the flatness condition based on the $SU(2)\times SU(2)$ algebra. In this sense the deformed Lax pairs form the hidden $SU(2)\times SU(2)$ algebra.

6. Conclusions and outlook

In this article, we reviewed our work on the proof that any $O(d,d)$ transformation preserves the classical integrability, provided that the initial model is classically integrable. One of our motivations is that some Yang-Baxter deformations were recently identified with the current-current deformation [20]. The latter is described by $O(d,d)$ transformations. However, we took an opposite direction and approached the classical integrability of $O(d,d)$-deformed models by constructing the corresponding Lax pairs without using the mathematical notion of Yang-Baxter deformations.

Here we presented only the example of the $O(2,2)$ deformation of the $SU(2)$ WZNW-model. However, our recipe for constructing Lax connections in the deformed models in Section 4 can be applied to any global $O(d,d)$ transformation. This recipe corresponds to the generalization of the one in [39], which discussed the T-dual Lax pairs of string sigma models. Their T-dual Lax pair in the $\mathrm{AdS}_5\times S^5$ background is related to the algebra of a dual conformal symmetry hidden in the T-dual of $\mathrm{AdS}_5\times S^5$ geometry [40]. On the other hand, the Lax pairs in the $O(d,d;\mathbb{R})$ deformed AdS_5 background constructed following our recipe would lead to the generalization of such a hidden algebra. Thus, the algebra of $O(d,d)$-deformed Lax pairs and non-local charges in the $O(d,d)$-deformed models would be worth investigating.

Our recipe of deformed Lax pairs via the $O(d,d)$ map (3.19) is based on the doubled formalism, which is the Abelian T-duality invariant formulation of string theory. It would be interesting to naturally uplift our construction to the cases of other T-dualities such as non-Abelian T-duality [41], and the Poisson-Lie T-duality [42, 43]. This issue should be related to the generalization of global $O(d,d;\mathbb{R})$ transformations to local $O(d,d;\mathbb{R})$ transformations, whose elements are field-dependent. For further investigations in this direction, see the related work [44, 45, 46].

Finally, as commented in [47], the author of this article believes that various integrable deformed models arise depending on how to take slices in the duality invariant formalism. All the integrable $J\bar{J}$, $J\bar{T}$, and $\mathrm{T}\bar{\mathrm{T}}$ deformations of string sigma models and even more extensions thereof should be unified based on the (generalization) of the doubled formalism. In particular, as remarked below (3.20), there is a similarity of the $O(d,d)$ map to the field-dependent diffeomorphism in the $\mathrm{T}\bar{\mathrm{T}}$ deformation [38]:

$$\mathrm{d}X^{\hat{\imath}} = \mathrm{d}X'^{\hat{\imath}} - \alpha\,\varepsilon^{\hat{\imath}}{}_{\hat{m}}\,\varepsilon^{\hat{n}}{}_{\hat{\jmath}}\,T(X')^{\hat{m}}{}_{\hat{n}}\,\mathrm{d}X'^{\hat{\jmath}}. \tag{6.1}$$

This relation raises a question how to extend the doubled sigma model such that the physical constraint should correspond to the coordinate transformation involving the stress-energy tensor. Then it would be interesting to see the resulting underlying geometry of such an extension.[2]

Acknowledgments: It is my great pleasure to thank Jean-Pierre Derendinger for instructive and inspiring discussions to study the topic summarized in this article. I wish to thank Domenico Orlando and Susanne Reffert for helpful collaborations and their patience during my PhD period. I thank Kentaroh Yoshida for a collaboration. I am very grateful to Carlo Angelantonj, Paolo Aschieri, Marco Billò, Falk Hassler, Chris M. Hull, Sylvain Lacroix, Usman Naseer, Holger Bech Nielsen, Carlos Núñez, Franco Pezzella, Vladimir Procházka, Konstantinos Siampos, Roberto Tateo, Daniel C. Thompson, and Satoshi Watamura for enlightening and fruitful discussions and comments. I acknowledge the hospitality of DESY, Galileo Galilei Institute, University of Turin, Uppsala University, Swansea University, and Andreas Burckhardt Foundation. Y.S. is supported by the Swiss National Science Foundation (SNF) under both PP00P2_183718/1 and P1BEP2_188137.

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