PROCEEDINGS OF SCIENCE

Planck scale from broken local conformal invariance in Weyl geometry

Ichiro Oda*
Department of Physics, Faculty of Science, University of the Ryukyus,
Nishihara, Okinawa 903-0213, Japan
E-mail: ioda@sci.u-ryukyu.ac.jp

We show that in a quadratic gravity based on Weyl's conformal geometry, the Planck mass scale can be generated from quantum effects of the gravitational field and the Weyl gauge field via the Coleman-Weinberg mechanism where a local scale symmetry is broken. At the same time, the Weyl gauge field acquires a mass less than the Planck mass by absorbing the scalar graviton. The shape of the effective potential is almost flat owing to a gravitational character and high symmetries, so our model would provide for an attractive model for the inflationary universe. We also present a toy model showing spontaneous symmetry breakdown of a global scale symmetry by moving from the Jordan frame to the Einstein one, and point out its problems.

Corfu Summer Institute 2019 "School and Workshops on Elementary Particle Physics and Gravity" (CORFU2019)
31 August - 25 September 2019
Corfu, Greece

*Speaker.

1. Introduction

One of the most important problems in modern particle physics is to understand the origin of not only the mass of elementary particles but also different mass scales existing in nature. This understanding is also important for attacking unsolved problems such as the origin of the Higgs potential, the gauge hierarchy problem and the cosmological constant problem etc.

In order to understand the origin of the mass and various mass scales, it is natural to start with a theory without intrinsic mass scales and consider how the mass is generated from a massless world via dynamical symmetry breaking mechanism. At this point, let us recall that the mass, or equivalently, the energy, couples to a gravitational field through the energy-momentum tensor in a universal manner, so we are forced to take a gravity into consideration for undertanding the origin of the mass. Moreover, it is worthwhile to point out that there naturally appears a local or global scale symmetry in a theory having no intrinsic mass scales. However, since as stressed in [1], global symmetries are in general against the spirit of general relativity (GR) owing to no-hair theorem of black holes [2], one should work with a gravitational theory which is invariant under not global but local scale transformation as well as the general coordinate transformation at very short distances.[1]

In this article, we therefore would like to consider a problem of how we could generate the Planck mass scale, beyond which the concept of the space-time does not make sense, by beginning with conformally invariant gravitational theories. From the success of the standard model (SM) of elementary particles, we are confident of the existence of at least two mass scales, those are, the electroweak scale around $10^2 GeV$ by the Higgs condensation and the QCD scale around $10^2 MeV$ by chiral symmetry breaking. These mass scales should be generated via dynamical symmetry breakings as well after the Planck mass scale is generated.

By the way, which conformally invariant gravitational theory is most interesting from the geometrical viewpoint? We think that it is a Weyl conformal gravity. About one hundred years ago, shortly after the advent of GR by Einstein, a conformally invariant extension of GR was proposed by Weyl on the basis of his conformal geometry, what we call, the Weyl geometry [4, 5].[2] The Weyl geometry is defined as a geometry equipped with a real symmetric metric tensor $g_{\mu\nu}$ as in GR and a symmetric connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$, which is related to the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ by the relation Eq. (3.5) as seen shortly. It turns out that the Weyl geometry reduces to the Riemann geometry when the Weyl gauge field S_μ is vanishing, or more precisely speaking, S_μ is a gradient, i.e., pure gauge.

In geometrical terms, the Weyl geometry critically differs from the Riemann one in that only angles, but not lengths, are preserved under parallel transport. To put differently, parallel displacement of a vector field changes its length in such a way that the very notion of lengths becomes path-dependent. For instance, one can envisage a space traveller, who travels to a distant star and then returns to the earth, being surprised to know not only that people in the earth have aged much rather than him as predicted by GR in the Riemann geometry but also that the clock on the rocket runs at a different rate from those in the earth as understood by Weyl conformal gravity in the Weyl geometry, what is called, "the second clock problem" [32]. Based on this very striking geometry, Weyl has attempted to geometrize the electromagnetic theory in the space-time geometry, but his

[1] In this article, we call a global scale symmetry simply *scale symmetry* while we refer to a local scale transformation as *conformal symmetry* by following the terminology of the textbook [3].

[2] See Ref. [6] for historical review on the Weyl geometry and various related works [7]-[31].

attempt has failed since it turned out later that the electromagnetic theory is based on a compact $U(1)$ gauge group whereas the Weyl geometry deals with conformal symmetry which is essentially a non-compact Abelian group [5]. Nevertheless, it seems that a Weyl quadratic gravity has recently revived as a theory predicting an elementary particle constituting dark matter, which is the Weyl gauge field interacting with only the graviton and the Higgs particle [30].

In Section 2, we present a toy model which shows spontaneous symmetry breakdown (SSB) of a global scale symmetry [3, 28, 30, 31]. The key idea is that we begin with a scale invariant scalar-tensor gravity in the Jordan frame and then move to the Einstein frame. In the process of moving from the Jordan frame to the Einstein frame, we need to introduce a constant with mass dimension to compensate for the mass dimension of a scalar field, thereby triggering the SSB of the scale symmetry. But we also point out problems of this SSB [31]. In Section 3, we briefly review a Weyl's conformal geometry. In Section 4, we present an action of a quadratic gravity in the Weyl geometry, for which we calculate the one-loop effective potential in the Coleman-Weinberg formalism [33] in Section 5. Section 6 is devoted to the conclusion.

2. Spontaneous symmetry breakdown of scale symmetry

There is a well-known mechanism of spontaneous symmetry breakdown of a global scale symmetry [3, 28, 30, 31]. In this section, we shall briefly review a scale invariant scalar-tensor gravity with two scalar fields, explain how the scale symmetry is broken spontaneously, and then point out unsatisfactory points of this SSB mechanism.

As a model of a scale invariant scalar-tensor gravity with two scalar fields, let us work with the following Lagrangian density in the Jordan frame[3]:

$$\mathscr{L} = \sqrt{-g}\left(\frac{1}{2}\xi\phi^2 R - \frac{1}{2}\varepsilon g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{\lambda_1}{4}\phi^4 - \frac{\lambda_2}{2}\phi^2\Phi^2 - \frac{\lambda_3}{4}\Phi^4\right), \quad (2.1)$$

where ξ is a constant, and ε takes the value $+1$ for ϕ being a normal field while it does -1 for ϕ being a ghost field. Moreover, ϕ and Φ are two distinct scalar fields, and $\lambda_i (i = 1,2,3)$ are dimensionless coupling constants. As often taken in the application for the BSM [34], we assume that $\lambda_1 > 0, \lambda_3 > 0$ and $\lambda_2 < 0$, and furthermore $|\lambda_2| \ll \lambda_1, \lambda_3 \approx \mathscr{O}(0.1)$. The conformally invariant scalar-tensor gravity corresponds to either the case of $\xi = \frac{1}{6}$ and $\varepsilon = -1$ or the case of $\xi = -\frac{1}{6}$ and $\varepsilon = 1$. In this section, since we consider only a globally scale invariant theory, we assume $\xi > 0$ and $6 + \frac{\varepsilon}{\xi} > 0$.

From this Lagrangian density, it is straightforward to derive the field equations for the metric tensor $g_{\mu\nu}$ and the two scalar fields ϕ, Φ whose result is written as

$$\begin{aligned} &2\varphi G_{\mu\nu} + 2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\varphi = T_{\mu\nu}, \\ &\xi\phi R + \varepsilon\Box\phi - \lambda_1\phi^3 - \lambda_2\phi\Phi^2 = 0, \\ &\Box\Phi - \lambda_2\phi^2\Phi - \lambda_3\Phi^3 = 0, \end{aligned} \quad (2.2)$$

where we have defined

$$\varphi = \frac{1}{2}\xi\phi^2, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \qquad \Box\varphi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi),$$

[3]We follow the conventions and notation adopted in the MTW textbook [2].

$$T_{\mu\nu} = \varepsilon\partial_\mu\phi\partial_\nu\phi + \partial_\mu\Phi\partial_\nu\Phi + g_{\mu\nu}\Big(-\frac{1}{2}\varepsilon g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi - \frac{\lambda_1}{4}\phi^4 - \frac{\lambda_2}{2}\phi^2\Phi^2 - \frac{\lambda_3}{4}\Phi^4\Big). \tag{2.3}$$

Using these field equations, one can derive the following equation:

$$\Box(\varphi + \frac{\zeta^2}{2}\Phi^2) = 0, \tag{2.4}$$

where we have defined $\zeta^{-2} \equiv 6 + \frac{\varepsilon}{\xi} > 0$.

The key step for the SSB of scale invariance is to move from the Jordan frame (J-frame) to the Einstein frame (E-frame) by applying a conformal transformation, i.e., a local scale transformation:

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \qquad \phi \to \phi' = \Omega^{-1}(x)\phi, \qquad \Phi \to \Phi' = \Omega^{-1}(x)\Phi. \tag{2.5}$$

After some calculations, we can derive the transformation rule for scalar curvature [3]:

$$R = \Omega^2(x)(R' + 6\Box' f - 6g'^{\mu\nu} f_\mu f_\nu), \tag{2.6}$$

where we have defined

$$f = \log\Omega, \qquad f_\mu = \partial_\mu f, \qquad \Box' f = \frac{1}{\sqrt{-g'}}\partial_\mu(\sqrt{-g'}g'^{\mu\nu}\partial_\nu f). \tag{2.7}$$

Using these relations, we find that the Lagrangian density (2.1) can be cast to the form in a new conformal frame:

$$\mathscr{L} = \sqrt{-g'}\Big[\frac{1}{2}\xi\phi'^2(R' + 6\Box' f - 6g'^{\mu\nu}f_\mu f_\nu) - \frac{1}{2}\varepsilon\Omega^{-2}g'^{\mu\nu}\partial_\mu(\Omega\phi')\partial_\nu(\Omega\phi') - \frac{1}{2}\Omega^{-2}g'^{\mu\nu}\partial_\mu(\Omega\Phi')\partial_\nu(\Omega\Phi') - \frac{\lambda_1}{4}\phi'^4 - \frac{\lambda_2}{2}\phi'^2\Phi'^2 - \frac{\lambda_3}{4}\Phi'^4\Big]. \tag{2.8}$$

Moving to the E-frame requires us to choose the scalar field ϕ' to[4]

$$\phi' = \frac{M_{Pl}}{\sqrt{\xi}}, \tag{2.9}$$

where M_{Pl} is the (reduced) Planck mass defined as $M_{Pl} = \frac{1}{\sqrt{8\pi G}} = 2.44 \times 10^{18} GeV$ with G being the Newton constant. Then, in the E-frame, up to a total derivative, the Lagrangian density (2.8) reduces to the form:

$$\mathscr{L} = \sqrt{-g'}\Big(\frac{M_{Pl}^2}{2}R' - \frac{1}{2}g'^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g'^{\mu\nu}\mathscr{D}_\mu\Phi'\mathscr{D}_\nu\Phi' - \frac{\lambda_1}{4}\frac{M_{Pl}^4}{\xi^2} - \frac{\lambda_2}{2}\frac{M_{Pl}^2}{\xi}\Phi'^2 - \frac{\lambda_3}{4}\Phi'^4\Big). \tag{2.10}$$

[4]In case of conformal symmetry, this condition is called the "Einstein gauge" or "unitary gauge".

Here we have defined

$$\Omega(x) = e^{\frac{\zeta}{M_{Pl}}\sigma(x)}, \qquad \mathscr{D}_\mu \Phi' = \left(\partial_\mu + \frac{\zeta}{M_{Pl}} \partial_\mu \sigma \right) \Phi', \tag{2.11}$$

where a scalar field σ is called "dilaton".

Now, owing to our assumption $\lambda_1 > 0, \lambda_3 > 0$ and $\lambda_2 < 0$, we have a Higgs potential given by

$$\begin{aligned} V(\Phi') &= \frac{\lambda_3}{4}\Phi'^4 + \frac{\lambda_2}{2}\frac{M_{Pl}^2}{\xi}\Phi'^2 + \frac{\lambda_1}{4}\frac{M_{Pl}^4}{\xi^2} \\ &= \frac{\lambda_3}{4}\left(\Phi'^2 - \frac{|\lambda_2|}{\lambda_3}\frac{M_{Pl}^2}{\xi}\right)^2 + \frac{1}{4}\left(\lambda_1 - \frac{\lambda_2^2}{\lambda_3}\right)\frac{M_{Pl}^4}{\xi^2}, \end{aligned} \tag{2.12}$$

which determines a vacuum expectation value (VEV):

$$\langle \Phi' \rangle = \sqrt{\frac{|\lambda_2|}{\lambda_3}\frac{M_{Pl}^2}{\xi}}. \tag{2.13}$$

Expanding as $\Phi' = \langle \Phi' \rangle + \tilde{\Phi}'$ with $\tilde{\Phi}'$ being a quantum fluctuation, we have

$$\begin{aligned} \mathscr{L} &= \sqrt{-g'}\Bigg[\frac{M_{Pl}^2}{2}R' - \frac{1}{2}g'^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g'^{\mu\nu}\partial_\mu\tilde{\Phi}'\partial_\nu\tilde{\Phi}' - \frac{1}{2}m_\Phi^2\tilde{\Phi}'^2 \\ &- \frac{\zeta}{M_{Pl}}g'^{\mu\nu}\tilde{\Phi}'\partial_\mu\tilde{\Phi}'\partial_\nu\sigma - \frac{\zeta^2}{2M_{Pl}^2}g'^{\mu\nu}\tilde{\Phi}'^2\partial_\mu\sigma\partial_\nu\sigma - \sqrt{\frac{\lambda_3}{2}}m_\Phi\tilde{\Phi}'^3 \\ &- \frac{\lambda_3}{4}\tilde{\Phi}'^4 - \frac{\lambda_1}{4}\frac{M_{Pl}^4}{\xi^2}\Bigg], \end{aligned} \tag{2.14}$$

where we have simplified the equations by using the relation $|\lambda_2| \ll \lambda_1, \lambda_3 \approx \mathscr{O}(0.1)$ and we have defined $m_\Phi = \sqrt{\frac{2|\lambda_2|}{\xi}}M_{Pl}$.

As is obvious from (2.14), the SSB of scale symmetry has occurred and as a result the scalar field $\tilde{\Phi}'$ becomes massive while the "dilaton" σ remains massless, which is nothing but a Nambu-Goldstone field. Also notice that the dilaton couples to the scalar field $\tilde{\Phi}'$ with derivatives which is one of characteristic features of the dilaton. To establish that σ really plays a role of the Nambu-Goldstone field, it is useful to derive the *dilatation current* associated with scale invariance, for which the scale factor Ω becomes a constant independent of the coordinates x^μ. It is then convenient to consider an infinitesimal transformation given by

$$\Omega = e^\Lambda, \tag{2.15}$$

where $|\Lambda| \ll 1$. Using the Lagrangian density (2.1) and the infinitesimal scale transformation (2.5) with (2.15), we find that via the Noether theorem the dilatation current J^μ reads

$$J^\mu = \frac{1}{\zeta^2}\sqrt{-g}g^{\mu\nu}\partial_\nu\left(\varphi + \frac{\zeta^2}{2}\Phi^2\right). \tag{2.16}$$

The dilatation current is certainly conserved

$$\partial_\mu J^\mu = \frac{1}{\zeta^2}\sqrt{-g}\Box\left(\varphi + \frac{\zeta^2}{2}\Phi^2\right) = 0, \tag{2.17}$$

where we have used the equation (2.4). In the E-frame, this current can be written as

$$J^{\mu} = \frac{1}{2}\sqrt{-g'}g'^{\mu\nu}\left[\frac{2M_{Pl}}{\zeta}\partial_\nu\sigma + \left(\partial_\nu + \frac{2\zeta}{M_{Pl}}\partial_\nu\sigma\right)\Phi'^2\right]. \tag{2.18}$$

Provided that one defines the dilatation charge as $Q = \int d^3xJ^0$, owing to the linear term in σ its charge fails to annihilate the vacuum $|0\rangle$

$$Q|0\rangle \neq 0, \tag{2.19}$$

which shows that the dilaton σ is the Nambu-Goldstone boson arising from the SSB of scale invariance.

To close this section, let us summarize the scenario of the SSB explained above and comment on its problems. We have started with a scale invariant gravitational theory involving two kinds of scalar fields and only dimensionless coupling constants. In the process of moving from the J-frame to the E-frame, we had to introduce a dimensional constant, which is the Planck mass in the present context, to compensate for the mass dimension of the scalar field. This introduction of the Planck mass has triggered the SSB of scale symmetry. Let us note that in the conventional scenario of the SSB, there is a potential inducing the SSB whereas we have no such a potential in the SSB under consideration. Nevertheless, the very presence of a solution with dimensional constants justifies the claim that the present scenario of the SSB is also nothing but a spontaneous symmetry breakdown. Actually, this fact was explicitly verified by the dilatation charge, which does not annihilate the vacuum due to the presence of a linear dilaton.

There are, however, at least two problems in this scenario of the SSB. First, it is impossible to apply this scenario for the conformally invariant scalar-tensor gravity, for which we must take either $\xi = \frac{1}{6}$ and $\varepsilon = -1$ or $\xi = -\frac{1}{6}$ and $\varepsilon = 1$, due to $\zeta^{-2} \equiv 6 + \frac{\varepsilon}{\xi} = 0$. The second problem arises from the lack of the suitable potential in the sense that we cannot single out a solution realizing the SSB on the stability argument [3]. Incidentally, though it might be possible that the cosmological argument would pick up an appropriate VEV of a scalar field, it is not plausible that the macroscopic physics like cosmology could determine a microscopic configuration such as the VEV. These two problems have been recently studied in Ref. [31].

3. Review of Weyl conformal geometry

We briefly review the basic concepts and definitions of the Weyl conformal geometry. In the Weyl geometry, the Weyl gauge transformation, which is the sum of a local scale transformation for a generic field $\Phi(x)$ and a gauge transformation for the Weyl gauge field $S_\mu(x)$, is defined as

$$\Phi(x) \rightarrow \Phi'(x) = e^{w\Lambda(x)}\Phi(x), \qquad S_\mu(x) \rightarrow S'_\mu(x) = S_\mu(x) - \frac{1}{f}\partial_\mu\Lambda(x), \tag{3.1}$$

where w is called the "Weyl weight", or simply "weight" henceforth, f is the coupling constant for the non-compact Abelian gauge group, and $\Lambda(x)$ is a local parameter for the Weyl transformation. The Weyl gauge transformation for various fields is explicitly given by

$$\begin{aligned} g_{\mu\nu}(x) &\rightarrow g'_{\mu\nu}(x) = e^{2\Lambda(x)}g_{\mu\nu}(x), \qquad \phi(x) \rightarrow \phi'(x) = e^{-\Lambda(x)}\phi(x), \\ \psi(x) &\rightarrow \psi'(x) = e^{-\frac{3}{2}\Lambda(x)}\psi(x), \qquad A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x), \end{aligned} \tag{3.2}$$

where $g_{\mu\nu}(x)$, $\phi(x)$, $\psi(x)$ and $A_\mu(x)$ are the metric tensor, scalar, spinor, and electromagnetic gauge fields, respectively. The covariant derivative D_μ for the Weyl gauge transformation for a generic field $\Phi(x)$ of weight w is defined as

$$D_\mu \Phi \equiv \partial_\mu \Phi + w f S_\mu \Phi, \tag{3.3}$$

which transforms covariantly under the Weyl transformation:

$$D_\mu \Phi \to (D_\mu \Phi)' = e^{w\Lambda(x)} D_\mu \Phi. \tag{3.4}$$

The Weyl geometry is defined as a geometry with a real symmetric metric tensor $g_{\mu\nu}(= g_{\nu\mu})$ and a symmetric connection $\tilde{\Gamma}^\lambda_{\mu\nu}(= \tilde{\Gamma}^\lambda_{\nu\mu})$ which is defined as[5]

$$\begin{aligned} \tilde{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho} \left(D_\mu g_{\nu\rho} + D_\nu g_{\mu\rho} - D_\rho g_{\mu\nu} \right) \\ &= \Gamma^\lambda_{\mu\nu} + f \left(S_\mu \delta^\lambda_\nu + S_\nu \delta^\lambda_\mu - S^\lambda g_{\mu\nu} \right), \end{aligned} \tag{3.5}$$

where

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right), \tag{3.6}$$

is the Christoffel symbol in the Riemann geometry. The most important difference between the Riemann geometry and the Weyl one lies in the fact that in the Riemann geometry the metric condition is satisfied

$$\nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} = 0, \tag{3.7}$$

while in the Weyl geometry we have

$$\tilde{\nabla}_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho_{\lambda\mu} g_{\rho\nu} - \tilde{\Gamma}^\rho_{\lambda\nu} g_{\mu\rho} = -2 f S_\lambda g_{\mu\nu}, \tag{3.8}$$

where ∇_μ and $\tilde{\nabla}_\mu$ are covariant derivatives for diffeomorphisms in the Riemann and Weyl geometries, respectively. Since the metric condition (3.7) implies that both length and angle are preserved under parallel transport, Eq. (3.8) shows that only angle, but not length, is preserved by the Weyl connection.

The general covariant derivative for both diffeomorphisms and the Weyl gauge transformation, for instance, for a covariant vector of weight w, is defined as

$$\begin{aligned} \mathscr{D}_\mu V_\nu &\equiv D_\mu V_\nu - \tilde{\Gamma}^\rho_{\mu\nu} V_\rho \\ &= \tilde{\nabla}_\mu V_\nu + w f S_\mu V_\nu \\ &= \nabla_\mu V_\nu + w f S_\mu V_\nu - f (S_\mu \delta^\rho_\nu + S_\nu \delta^\rho_\mu - S^\rho g_{\mu\nu}) V_\rho \\ &= \partial_\mu V_\nu + w f S_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho - f (S_\mu \delta^\rho_\nu + S_\nu \delta^\rho_\mu - S^\rho g_{\mu\nu}) V_\rho. \end{aligned} \tag{3.9}$$

One can verify that using the general covariant derivative, the following metric condition is satisfied:

$$\mathscr{D}_\lambda g_{\mu\nu} = 0. \tag{3.10}$$

[5]We often use the tilde characters to express quantities belonging to the Weyl geometry.

Moreover, under the Weyl gauge transformation the general covariant derivative for a generic field Φ of weight w transforms in a covariant manner as desired:

$$\mathscr{D}_\mu \Phi \to (\mathscr{D}_\mu \Phi)' = e^{w\Lambda(x)} \mathscr{D}_\mu \Phi, \tag{3.11}$$

because the Weyl connection is invariant under the Weyl gauge transformation, i.e., $\tilde{\Gamma}'^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu}$.

As in the Riemann geometry, in the Weyl geometry one can also construct a Weyl invariant curvature tensor $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ via a commutator of the covariant derivative $\tilde{\nabla}_\mu$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] V_\rho = \tilde{R}_{\mu\nu\rho}{}^{\sigma} V_\sigma. \tag{3.12}$$

Calculating this commutator, one finds that

$$\begin{aligned} \tilde{R}_{\mu\nu\rho}{}^{\sigma} &= \partial_\nu \tilde{\Gamma}^{\sigma}_{\mu\rho} - \partial_\mu \tilde{\Gamma}^{\sigma}_{\nu\rho} + \tilde{\Gamma}^{\alpha}_{\mu\rho} \tilde{\Gamma}^{\sigma}_{\alpha\nu} - \tilde{\Gamma}^{\alpha}_{\nu\rho} \tilde{\Gamma}^{\sigma}_{\alpha\mu} \\ &= R_{\mu\nu\rho}{}^{\sigma} + 2f \left(\delta^{\sigma}_{[\mu} \nabla_{\nu]} S_\rho - \delta^{\sigma}_{\rho} \nabla_{[\mu} S_{\nu]} - g_{\rho[\mu} \nabla_{\nu]} S^\sigma \right) \\ &+ 2f^2 \left(S_{[\mu} \delta^{\sigma}_{\nu]} S_\rho - S_{[\mu} g_{\nu]\rho} S^\sigma + \delta^{\sigma}_{[\mu} g_{\nu]\rho} S_\alpha S^\alpha \right), \end{aligned} \tag{3.13}$$

where $R_{\mu\nu\rho}{}^{\sigma}$ is the curvature tensor in the Riemann geometry, and we have defined the antisymmetrization by the square bracket, e.g., $A_{[\mu} B_{\nu]} \equiv \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)$. Then, it is straightforward to prove the following identities:

$$\tilde{R}_{\mu\nu\rho}{}^{\sigma} = -\tilde{R}_{\nu\mu\rho}{}^{\sigma}, \qquad \tilde{R}_{[\mu\nu\rho]}{}^{\sigma} = 0, \qquad \tilde{\nabla}_{[\lambda} \tilde{R}_{\mu\nu]\rho}{}^{\sigma} = 0. \tag{3.14}$$

The curvature tensor $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ has 26 independent components, twenty of which are possessed by $R_{\mu\nu\rho}{}^{\sigma}$ and six by the Weyl invariant field strength $H_{\mu\nu} \equiv \partial_\mu S_\nu - \partial_\nu S_\mu$.

From $\tilde{R}_{\mu\nu\rho}{}^{\sigma}$ one can define a Weyl invariant Ricci tensor:

$$\begin{aligned} \tilde{R}_{\mu\nu} &\equiv \tilde{R}_{\mu\rho\nu}{}^{\rho} \\ &= R_{\mu\nu} + f \left(-2\nabla_\mu S_\nu - H_{\mu\nu} - g_{\mu\nu} \nabla_\alpha S^\alpha \right) \\ &+ 2f^2 \left(S_\mu S_\nu - g_{\mu\nu} S_\alpha S^\alpha \right). \end{aligned} \tag{3.15}$$

Let us note that

$$\tilde{R}_{[\mu\nu]} \equiv \frac{1}{2}(\tilde{R}_{\mu\nu} - \tilde{R}_{\nu\mu}) = -2f H_{\mu\nu}. \tag{3.16}$$

Similarly, one can define a not Weyl invariant but Weyl covariant scalar curvature:

$$\tilde{R} \equiv g^{\mu\nu} \tilde{R}_{\mu\nu} = R - 6f \nabla_\mu S^\mu - 6f^2 S_\mu S^\mu. \tag{3.17}$$

One finds that under the Weyl gauge transformation, $\tilde{R} \to \tilde{R}' = e^{-2\Lambda(x)} \tilde{R}$ while $\tilde{\Gamma}^{\lambda}_{\mu\nu}, \tilde{R}_{\mu\nu\rho}{}^{\sigma}$ and $\tilde{R}_{\mu\nu}$ are all invariant.

Even in the Weyl geometry, it is possible to write out a generalization of the Gauss-Bonnet topological invariant which can be described as

$$\begin{aligned} I_{GB} &\equiv \int d^4x \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} \tilde{R}_{\mu\nu}{}^{\alpha\beta} \tilde{R}_{\rho\sigma}{}^{\gamma\delta} \\ &= -2 \int d^4x \sqrt{-g} \left(\tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\rho\sigma\mu\nu} - 4\tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + \tilde{R}^2 - 12 f^2 H_{\mu\nu} H^{\mu\nu} \right) \\ &= -2 \int d^4x \sqrt{-g} \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right). \end{aligned} \tag{3.18}$$

We close this section by discussing a spinor field as an example of matter fields in the Weyl geometry [13, 14]. As is well known, to describe a spinor field it is necessary to introduce the vierbein e^a_μ, which is defined as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{3.19}$$

where $a, b, \cdots$ are local Lorentz indices taking $0, 1, 2, 3$ and $\eta_{ab} = diag(-1, 1, 1, 1)$.

Now the metric condition (3.10) takes the form

$$\mathscr{D}_\mu e^a_\nu \equiv D_\mu e^a_\nu + \tilde{\omega}^a{}_{b\mu} e^b_\nu - \tilde{\Gamma}^\rho_{\mu\nu} e^a_\rho = 0, \tag{3.20}$$

where the general covariant derivative is extended to include the local Lorentz transformation whose gauge connection is the spin connection $\tilde{\omega}^a{}_{b\mu}$ of weight 0 in the Weyl geometry, and $D_\mu e^a_\nu = \partial_\mu e^a_\nu + f S_\mu e^a_\nu$ since the vierbein e^a_μ has weight 1. Solving the metric condition (3.20) leads to the expression of the spin connection in the Weyl geometry

$$\tilde{\omega}_{ab\mu} = \omega_{ab\mu} + f e^c_\mu (\eta_{ac} S_b - \eta_{bc} S_a), \tag{3.21}$$

where $\omega_{ab\mu}$ is the spin connection in the Riemann geometry and we have defined $S_a \equiv e^\mu_a S_\mu$. Then, the general covariant derivative for a spinor field Ψ of weight $-\frac{3}{2}$ reads

$$\mathscr{D}_\mu \Psi = D_\mu \Psi + \frac{i}{2} \tilde{\omega}_{ab\mu} S^{ab} \Psi, \tag{3.22}$$

where $D_\mu \Psi = \partial_\mu \Psi - \frac{3}{2} f S_\mu \Psi$ and the Lorentz generator S^{ab} for a spinor field is defined as $S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$. Here we define the gamma matrices to satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$. Since the spin connection $\tilde{\omega}^a{}_{b\mu}$ has weight 0, the covariant derivative $\mathscr{D}_\mu \Psi$ transforms covariantly under the Weyl gauge transformation

$$\mathscr{D}_\mu \Psi \to (\mathscr{D}_\mu \Psi)' = e^{-\frac{3}{2}\Lambda(x)} \mathscr{D}_\mu \Psi. \tag{3.23}$$

Then, the Lagrangian density for a massless Dirac spinor field is of form

$$\mathscr{L} = \frac{i}{2} e \, e^\mu_a (\bar{\Psi} \gamma^a \mathscr{D}_\mu \Psi - \mathscr{D}_\mu \bar{\Psi} \gamma^a \Psi), \tag{3.24}$$

where $e \equiv \sqrt{-g}, \bar{\Psi} \equiv \Psi^\dagger \gamma^0$, and $\mathscr{D}_\mu \bar{\Psi}$ is given by

$$\mathscr{D}_\mu \bar{\Psi} = D_\mu \bar{\Psi} - \bar{\Psi} \frac{i}{2} \tilde{\omega}_{ab\mu} S^{ab}. \tag{3.25}$$

Inserting Eqs. (3.22) and (3.25) to the Lagrangian density (3.24), we find that

$$\begin{aligned} \mathscr{L} &= \frac{i}{2} e \Big[e^\mu_a \left(\bar{\Psi} \gamma^a \partial_\mu \Psi - \partial_\mu \bar{\Psi} \gamma^a \Psi + \frac{i}{2} \omega_{bc\mu} \bar{\Psi} \{\gamma^a, S^{bc}\} \Psi \right) \\ &+ \frac{i}{2} f (\eta_{ab} S_c - \eta_{ac} S_b) \bar{\Psi} \{\gamma^a, S^{bc}\} \Psi \Big]. \end{aligned} \tag{3.26}$$

The last term identically vanishes owing to the relation

$$\{\gamma^a, S^{bc}\} = -\varepsilon^{abcd} \gamma_5 \gamma_d, \tag{3.27}$$

where we have defined as $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon^{0123} = +1$. Thus, as is well known, the Weyl gauge field S_μ does not couple minimally to a spinor field Ψ. Technically speaking, it is the absence of imaginary unit i in the covariant derivative $D_\mu\Psi = \partial_\mu\Psi - \frac{3}{2}fS_\mu\Psi$ that induced this decoupling of the Weyl gauge field from the spinor field. Without the imaginary unit, the terms including the Weyl gauge field cancel out each other in Eq. (3.24). In a similar manner, we can prove that the Weyl gauge field does not couple to a gauge field either such as the electromagnetic potential A_μ. On the other hand, the Weyl gauge field can couple to a scalar field such as the Higgs field as well as a graviton. In such a situation, we cannot help identifying the Weyl gauge field with an elementary particle that constitutes dark matter. It seems that the Weyl gauge theory was rejected as a unified theory of gravitation and electromagnetism but it has revived as a geometrical theory which predicts the existence of dark matter.

4. Quadratic gravity in Weyl geometry

In this section, we will present a gravitational theory on the basis of the Weyl geometry outlined in the previous section. It is of interest to notice that if only the metric tensor is allowed to use for the construction of a gravitational action, the action invariant under the Weyl transformation must be of form of quadratic gravity, but not be of the Einstein-Hilbert type. Using the topological invariant (3.18), one can write out a general action of quadratic gravity, which is invariant under the Weyl transformation, as follows:

$$S_{QG} = \int d^4x\sqrt{-g}\left[-\frac{1}{2\xi^2}\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} + \frac{\lambda}{4!}\tilde{R}^2\right] \equiv \int d^4x\sqrt{-g}\,\mathscr{L}_{QG}, \tag{4.1}$$

where ξ and λ are dimensionless coupling constants. And a generalization of the conformal tensor, $\tilde{C}_{\mu\nu\rho\sigma}$, in the Weyl geometry is defined as in $C_{\mu\nu\rho\sigma}$ in the Riemann geometry:

$$\begin{aligned}\tilde{C}_{\mu\nu\rho\sigma} &\equiv \tilde{R}_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}\tilde{R}_{\nu\sigma} + g_{\nu\sigma}\tilde{R}_{\mu\rho} - g_{\mu\sigma}\tilde{R}_{\nu\rho} - g_{\nu\rho}\tilde{R}_{\mu\sigma}\right) + \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\tilde{R} \\ &= C_{\mu\nu\rho\sigma} + f\left[-g_{\rho\sigma}H_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}H_{\nu\sigma} + g_{\nu\sigma}H_{\mu\rho} - g_{\mu\sigma}H_{\nu\rho} - g_{\nu\rho}H_{\mu\sigma}\right)\right].\end{aligned} \tag{4.2}$$

This conformal tensor in the Weyl geometry has the following properties:

$$\tilde{C}_{\mu\nu\rho\sigma} = -\tilde{C}_{\nu\mu\rho\sigma}, \qquad \tilde{C}_{\mu\nu\rho}{}^{\nu} = 0, \qquad \tilde{C}_{\mu\nu\rho}{}^{\rho} = -4fH_{\mu\nu}. \tag{4.3}$$

Next, by introducing a scalar field ϕ and using the classical equivalence, let us rewrite $\tilde{R}^2$ in the action (4.1) in the form of the scalar-tensor gravity plus $\lambda\phi^4$ interaction [25, 27] whose Lagrangian density takes the form

$$\begin{aligned}\frac{1}{\sqrt{-g}}\mathscr{L}_{QG} &= -\frac{1}{2\xi^2}\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} + \frac{\lambda}{12}\phi^2\tilde{R} - \frac{\lambda}{4!}\phi^4 \\ &= -\frac{1}{2\xi^2}\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} + \frac{1}{12}\phi^2\tilde{R} - \frac{\lambda_\phi}{4!}\phi^4 \\ &= -\frac{1}{2\xi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{12}\phi^2 R - \frac{\lambda_\phi}{4!}\phi^4 - \frac{3f^2}{\xi^2}H_{\mu\nu}^2 \\ &\quad - \frac{1}{2}\phi^2(f\nabla_\mu S^\mu + f^2 S_\mu S^\mu),\end{aligned} \tag{4.4}$$

where in the second equality we have redefined $\sqrt{\lambda}\phi \to \phi$ and set $\lambda = \frac{1}{\lambda_\phi}$. It is straightforward to write down a standard model (SM) or physics beyond the standard model (BSM) action which is invariant under the Weyl transformation, but we will omit to do it in this article and present the detail in a separate publication.

5. Emergence of Planck scale

At low energies, general relativity (GR) describes various gravitational and astrophysical phenomena neatly, so the Weyl invariant Lagrangian density (4.4) of quadratic gravity should be reduced to that of GR at low energies. To do that, we need to break the Weyl symmetry at any rate by some method. One method is to appeal to the procedure of spontaneous symmetry breakdown (SSB) explained in terms of a toy model in Section 2. However, as emphasized there, since there is no potential to induce this SSB in the theory, we have no idea which solution we should pick up among many of configurations from the stability argument.

The other simple procedure is to take a gauge condition for the Weyl transformation such that $\phi = \phi_0$ where ϕ_0 is a certain constant [14, 15, 21, 25, 27]. However, ϕ_0 is a free parameter which is not fixed from the stability argument of the potential either so it is not clear why we choose a specific value $\phi_0 \sim M_{Pl}$.

In this article, we would like to look for an alternative possibility by considering a conformally invariant gravitational theory where the scalar field ϕ acquires a vacuum expectation value (VEV) as a result of instabilities in the full quantum theory including quantum corrections from gravity. It is natural to conjecture that quantum gravity plays a role in generating the Planck mass scale dynamically since effects of quantum gravity are more dominant than the other interactions around the Planck mass. Technically speaking, what we expect is that after quantum corrections of gravitational fields are taken into consideration the effective potential has a form favoring the specific VEV, $\phi_0 \sim M_{Pl}$ [1, 26, 28].

To this aim, let us first expand the scalar field and the metric around a classical field ϕ_c and a flat Minkowski metric $\eta_{\mu\nu}$ like [1, 26, 28]

$$\phi = \phi_c + \varphi, \quad g_{\mu\nu} = \eta_{\mu\nu} + \xi h_{\mu\nu}, \tag{5.1}$$

where we take ϕ_c to be a constant since we are interested in the effective potential depending on the constant ϕ_c. Next, since we wish to calculate the one-loop effective potential, we will derive only quadratic terms in quantum fields from the classical Lagrangian density (4.4). Then, the Lagrangian density corresponding to the conformal tensor squared takes the form

$$\mathscr{L}_C \equiv -\frac{1}{2\xi^2}\sqrt{-g}\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = -\frac{1}{4}h^{\mu\nu}P^{(2)}_{\mu\nu,\rho\sigma}\Box^2 h^{\rho\sigma}, \tag{5.2}$$

where $P^{(2)}_{\mu\nu,\rho\sigma}$ is the projection operator for spin-2 modes[6] and $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. In a similar manner, the Lagrangian density corresponding to the scalar-tensor gravity in Eq. (4.4) reads

$$\begin{aligned}\mathscr{L}_{ST} &\equiv \sqrt{-g}\,\frac{1}{12}\phi^2 R \\ &= \frac{1}{48}\xi^2\phi_c^2 h^{\mu\nu}\left(P^{(2)}_{\mu\nu,\rho\sigma} - 2P^{(0,s)}_{\mu\nu,\rho\sigma}\right)\Box h^{\rho\sigma} - \frac{1}{6}\xi\phi_c\varphi\left(\eta_{\mu\nu} - \frac{1}{\Box}\partial_\mu\partial_\nu\right)\Box h^{\mu\nu}.\end{aligned} \tag{5.3}$$

[6]We follow the definition of projection operators in [35, 36].

The remaining Lagrangian density can be evaluated in a similar way and consequently all the quadratic terms in (4.4) are summarized to

$$\begin{aligned}\mathscr{L}_{QG} &= \frac{1}{4}h^{\mu\nu}\left[\left(-\Box+\frac{1}{12}\xi^2\phi_c^2\right)P^{(2)}_{\mu\nu,\rho\sigma}-\frac{1}{6}\xi^2\phi_c^2P^{(0,s)}_{\mu\nu,\rho\sigma}\right]\Box h^{\rho\sigma}\\ &- \frac{1}{6}\xi\phi_c\varphi\left(\eta_{\mu\nu}-\frac{1}{\Box}\partial_\mu\partial_\nu\right)\Box h^{\mu\nu}-\frac{\lambda_\phi}{4}\phi_c^2\varphi^2-\frac{\lambda_\phi}{12}\xi\phi_c^3h\varphi+\frac{1}{96}\lambda_\phi\xi^2\phi_c^4h_{\mu\nu}^2\\ &- \frac{1}{192}\lambda_\phi\xi^2\phi_c^4h^2-\frac{1}{4}H'^{\,2}_{\mu\nu}-\frac{1}{24}\xi^2\phi_c^2S'_\mu S'^\mu-\frac{1}{2}\varphi\Box\varphi, \end{aligned} \tag{5.4}$$

where we have defined $h=\eta^{\mu\nu}h_{\mu\nu}$ and set $S'_\mu=\frac{2\sqrt{3}f}{\xi}(S_\mu-\frac{1}{f\phi_c}\partial_\mu\varphi)$ and $H'_{\mu\nu}=\partial_\mu S'_\nu-\partial_\nu S'_\mu$. In what follows, we will assume that

$$\lambda_\phi \propto \xi^4 \ll 1, \tag{5.5}$$

and drop all the terms involving λ_ϕ. We will prove later that our assumption (5.5) is self-consistent and there are no large logarithms.

At this point, it is convenient to use the York decomposition for the metric fluctuation field $h_{\mu\nu}$ [37]:

$$\begin{aligned}h_{\mu\nu} &= h^{TT}_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+\partial_\nu\partial_\nu\sigma-\frac{1}{4}\eta_{\mu\nu}\Box\sigma+\frac{1}{4}\eta_{\mu\nu}h\\ &= h^{TT}_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+\partial_\nu\partial_\nu\sigma+\frac{1}{4}\theta_{\mu\nu}s+\frac{1}{4}\omega_{\mu\nu}w,\end{aligned} \tag{5.6}$$

where $h^{TT}_{\mu\nu}$ is both transverse and traceless, and ξ_μ is transverse:

$$\partial^\mu h^{TT}_{\mu\nu}=\eta^{\mu\nu}h^{TT}_{\mu\nu}=\partial^\mu\xi=0. \tag{5.7}$$

Moreover, we have defined

$$s=h-\Box\sigma, \quad w=h+3\Box\sigma, \quad \theta_{\mu\nu}=\eta_{\mu\nu}-\frac{1}{\Box}\partial_\mu\partial_\nu, \quad \omega_{\mu\nu}=\frac{1}{\Box}\partial_\mu\partial_\nu. \tag{5.8}$$

One advantage of the York decomposition (5.6) is that each term corresponds to the degree of freedom with the definite spin as seen in the following relations:

$$\begin{aligned}P^{(2)\rho\sigma}_{\mu\nu}h_{\rho\sigma} &= h^{TT}_{\mu\nu}, \qquad P^{(1)\rho\sigma}_{\mu\nu}h_{\rho\sigma}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu,\\ P^{(0,s)\rho\sigma}_{\mu\nu}h_{\rho\sigma} &= \frac{1}{4}\theta_{\mu\nu}s, \qquad P^{(0,w)\rho\sigma}_{\mu\nu}h_{\rho\sigma}=\frac{1}{4}\omega_{\mu\nu}w.\end{aligned} \tag{5.9}$$

Using these relations and our assumption (5.5), the Lagrangian density (5.4) reads

$$\mathscr{L}_{QG}=\frac{1}{4}h^{TT\mu\nu}(-\Box+m^2)\Box h^{TT}_{\mu\nu}-\frac{1}{2}\varphi'\Box\varphi'-\frac{1}{4}H'^{\,2}_{\mu\nu}-\frac{m^2}{2}S'_\mu S'^\mu, \tag{5.10}$$

where we have put $m^2=\frac{1}{12}\xi^2\phi_c^2$ and $\varphi'=\varphi-\frac{\sqrt{3}m}{4}s$.

Now let us calculate the functional Jacobian associated with the change of variables, $h_{\mu\nu}\to(h^{TT}_{\mu\nu},\xi_\mu,s,w)$. To do that, we will use the relation [38]

$$1=\int\mathscr{D}h_{\mu\nu}e^{-\mathscr{G}(h,h)}, \tag{5.11}$$

where $\mathscr{G}(h,h)$ is an inner product in the space of symmetric rank-2 tensors:

$$\begin{aligned} \mathscr{G}(h,h) &= \int d^4x (h_{\mu\nu}h^{\mu\nu} + \frac{a}{2}h^2) \\ &= \int d^4x \left[(h^{TT}_{\mu\nu})^2 - 2\xi_\mu \Box \xi^\mu + \frac{3(3a+2)}{32} s'^2 + \frac{2a+1}{8(3a+2)} w^2 \right], \end{aligned} \tag{5.12}$$

where a is an arbitrary constant and we have defined $s' = s + \frac{3}{3a+2} w$. Thus, the functional Jacobian J which is defined as

$$\mathscr{D}h_{\mu\nu} = J \mathscr{D}h^{TT}_{\mu\nu} \mathscr{D}\xi_\mu \mathscr{D}s' \mathscr{D}w, \tag{5.13}$$

is given by

$$J = (\det{}_\xi \Box)^{\frac{1}{2}}. \tag{5.14}$$

Next let us set up the gauge-fixing conditions. For diffeomorphisms and the Weyl transformation, we adopt gauge conditions, respectively

$$\partial^\nu h_{\mu\nu} = \Box \xi_\mu + \frac{1}{4}\partial_\mu w = 0, \qquad \partial_\mu S'^\mu = 0. \tag{5.15}$$

The corresponding FP ghost terms are respectively calculated to

$$\det \Delta^{(GCT)}_{FP} = \det(\Box \delta^\nu_\mu + \partial_\mu \partial^\nu), \qquad \det \Delta^{(Weyl)}_{FP} = \det(\Box). \tag{5.16}$$

Then, the partition function of the present theory is given by

$$\begin{aligned} Z[\phi_c] &= \int \mathscr{D}g_{\mu\nu} \mathscr{D}\phi \mathscr{D}S_\mu \det \Delta^{(GCT)}_{FP} \det \Delta^{(Weyl)}_{FP} \delta(\partial^\nu h_{\mu\nu}) \delta(\partial_\mu S'^\mu) \\ &\times \; exp \; i \int d^4x \left[\frac{1}{4} h^{TT\mu\nu}(-\Box + m^2)\Box h^{TT}_{\mu\nu} - \frac{1}{2}\varphi' \Box \varphi' - \frac{1}{4} H'^{\,2}_{\mu\nu} - \frac{m^2}{2} S'_\mu S'^\mu \right] \\ &= \int \mathscr{D}h^{TT}_{\mu\nu} \mathscr{D}\xi_\mu \mathscr{D}s' \mathscr{D}w \mathscr{D}\varphi' \mathscr{D}S'_\mu (\det{}_\xi \Box)^{\frac{1}{2}} \det(\Box \delta^\nu_\mu + \partial_\mu \partial^\nu) \det(\Box) \\ &\times \; \delta(\Box \xi_\mu + \frac{1}{4}\partial_\mu w) \delta(\partial_\mu S'^\mu) exp \; i \int d^4x \left[\frac{1}{4} h^{TT\mu\nu}(-\Box + m^2)\Box h^{TT}_{\mu\nu} \right. \\ &- \left. \frac{1}{2}\varphi' \Box \varphi' - \frac{1}{2} S'^\mu(-\Box + m^2) S'_\mu + \frac{1}{2}\partial_\mu S'^\mu)^2 \right] \\ &= \frac{\det(\Box \delta^\nu_\mu + \partial_\mu \partial^\nu) \det(\Box)}{(\det{}_\xi \Box)^{\frac{1}{2}} (\det{}_{\varphi'} \Box)^{\frac{1}{2}} (\det{}_{h^{TT}}(-\Box + m^2)\Box)^{\frac{1}{2}} (\det{}_{S'}(-\Box + m^2))^{\frac{1}{2}}}. \end{aligned} \tag{5.17}$$

Using the partition function (5.17), we can evaluate the one-loop effective action by integrating out quantum fluctuations. Then, up to a classical potential, recalling the definition $m^2 = \frac{1}{12}\xi^2 \phi_c^2$, the effective action $\Gamma[\phi_c]$ reads

$$\Gamma[\phi_c] = -i \log Z[\phi_c] = i \frac{5+3}{2} \log \det \left(-\Box + \frac{1}{12}\xi^2 \phi_c^2 \right). \tag{5.18}$$

Here some remarks are in order. First, in this expression, the factors 5 and 3 come from the fact that a massive spin-2 state and a massive spin-1 Weyl gauge field possess five and three physical degrees of freedom, respectively. Second, let us note that we have ignored the part of the effective action which is independent of ϕ_c since it never gives us the effective potential for ϕ_c.

To calculate $\Gamma[\phi_c]$, we will proceed step by step: First, let us note that $\Gamma[\phi_c]$ can be rewritten as follows:

$$
\begin{aligned}
\Gamma[\phi_c] &= 4i \mathrm{Tr} \log\left(-\Box + \frac{1}{12}\xi^2\phi_c^2\right) \\
&= 4i \int d^4x \, \langle x| \log\left(-\Box + \frac{1}{12}\xi^2\phi_c^2\right)|x\rangle \\
&= 4i \int d^4x \int \frac{d^4k}{(2\pi)^4} \langle x| \log\left(-\Box + \frac{1}{12}\xi^2\phi_c^2\right)|k\rangle\langle k|x\rangle \\
&= 4i(VT) \int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + \frac{1}{12}\xi^2\phi_c^2\right) \\
&= 4(VT)\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\left(\frac{1}{12}\xi^2\phi_c^2\right)^{\frac{d}{2}},
\end{aligned} \tag{5.19}
$$

where (VT) denotes the space-time volume and in the last equality we have used the Wick rotation and the dimensional regularization.

Next, let us evaluate the $\Gamma[\phi_c]$ in terms of the modified minimal subtraction scheme. In this scheme, the $\frac{1}{\varepsilon}$ poles (where $\varepsilon \equiv 4-d$) together with the Euler-Mascheroni constant γ and $\log(4\pi)$ are subtracted and then replaced with $\log M^2$ where M is an arbitrary mass parameter which is introduced to make the final equation dimensionally correct [39]. By subtracting the $\frac{1}{\varepsilon}$ pole, (5.19) is reduced to the form

$$
\begin{aligned}
-\frac{1}{VT}\Gamma[\phi_c] &= -4\frac{\Gamma(2-\frac{d}{2})}{\frac{d}{2}(\frac{d}{2}-1)}\frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{1}{12}\xi^2\phi_c^2\right)^{\frac{d}{2}} \\
&= -\frac{4}{2(4\pi)^2}\left(\frac{1}{12}\xi^2\phi_c^2\right)^2\left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left(\frac{1}{12}\xi^2\phi_c^2\right) + \frac{3}{2}\right] \\
&\to \frac{2}{(4\pi)^2}\left(\frac{1}{12}\xi^2\phi_c^2\right)^2\left[\log\left(\frac{\xi^2\phi_c^2}{12M^2}\right) - \frac{3}{2}\right].
\end{aligned} \tag{5.20}
$$

Then, the one-loop effective potential will be of form[7]

$$
V_{eff}^{(1)}(\phi_c) = c_1 + c_2\phi^2 + \frac{1}{1152\pi^2}\xi^4\phi_c^4 \log\left(\frac{\phi_c^2}{c_3}\right), \tag{5.21}
$$

where $c_i (i = 1,2,3)$ are constants to be determined by the renormalization conditions:

$$
V_{eff}^{(1)}\Big|_{\phi_c=0} = \frac{d^2V_{eff}^{(1)}}{d\phi_c^2}\Bigg|_{\phi_c=0} = \frac{d^4V_{eff}^{(1)}}{d\phi_c^4}\Bigg|_{\phi_c=\mu} = 0, \tag{5.22}
$$

[7] At first sight, the existence of the $c_2\phi^2$ might appear to be strange, but this term in fact emerges in the cutoff regularization. Note that the only logarithmically divergent term, but not quadratic divergent one, arises in the dimensional regularization.

where μ is the renormalization mass. As a result, we have the one-loop effective potential

$$V^{(1)}_{eff}(\phi_c) = \frac{1}{1152\pi^2}\xi^4\phi_c^4\left(\log\frac{\phi_c^2}{\mu^2} - \frac{25}{6}\right). \tag{5.23}$$

Finally, by adding the classical potential we can arrive at the effective potential in the one-loop approximation

$$V_{eff}(\phi_c) = \frac{\lambda_\phi}{4!}\phi_c^4 + \frac{1}{1152\pi^2}\xi^4\phi_c^4\left(\log\frac{\phi_c^2}{\mu^2} - \frac{25}{6}\right). \tag{5.24}$$

It is easy to see that this effective potential has a minimum at $\phi_c = \langle\phi\rangle$ away from the origin where the effective potential, $V_{eff}(\langle\phi\rangle)$, is negative. Since the renormalization mass μ is arbitrary, we will choose it to be the actual location of the minimum, $\mu = \langle\phi\rangle$ [33]:

$$V_{eff}(\phi_c) = \frac{\lambda_\phi}{4!}\phi_c^4 + \frac{1}{1152\pi^2}\xi^4\phi_c^4\left(\log\frac{\phi_c^2}{\langle\phi\rangle^2} - \frac{25}{6}\right). \tag{5.25}$$

Since $\phi_c = \langle\phi\rangle$ is defined to be the minimum of V_{eff}, we deduce

$$\begin{aligned} 0 &= \left.\frac{dV_{eff}}{d\phi_c}\right|_{\phi_c=\langle\phi\rangle} \\ &= \left(\frac{\lambda_\phi}{6} - \frac{11}{864\pi^2}\xi^4\right)\langle\phi\rangle^3, \end{aligned} \tag{5.26}$$

or equivalently,

$$\lambda_\phi = \frac{11}{144\pi^2}\xi^4. \tag{5.27}$$

This relation is similar to $\lambda = \frac{33}{8\pi^2}e^4$ in case of the scalar QED in Ref. [33], so as in that paper, the perturbation theory holds for very small ξ as well.

The substitution of Eq. (5.27) into V_{eff} in (5.25) leads to

$$V_{eff}(\phi_c) = \frac{1}{1152\pi^2}\xi^4\phi_c^4\left(\log\frac{\phi_c^2}{\langle\phi\rangle^2} - \frac{1}{2}\right). \tag{5.28}$$

Thus, the effective potential is now parametrized in terms of ξ and $\langle\phi\rangle$ instead of ξ and λ_ϕ; it is nothing but the well-known "dimensional transmutation" , i.e., a dimensionless coupling constant λ_ϕ is traded for a dimensional quantity $\langle\phi\rangle$ via symmetry breakdown of the *local* Weyl symmetry.

Hence, from the classical Lagrangian density (4.4) of quadratic gravity, via dimensional transmutation, the Einstein-Hilbert term for GR is induced in such a way that the Planck mass M_{Pl} is given by

$$M_{Pl}^2 = \frac{1}{6}\langle\phi\rangle^2. \tag{5.29}$$

At the same time, the Weyl gauge field becomes massive by 'eating' the scalar graviton s and a part of the dilaton φ whose magnitude of mass is given

$$m_S^2 = \frac{1}{12}\xi^2\langle\phi\rangle^2 = \frac{1}{2}\xi^2 M_{Pl}^2. \tag{5.30}$$

As long as the perturbation theory is concerned, the coupling constant ξ must take a small value, $\xi \ll 1$. At the low energy region satisfying $E \ll m_S$, we can integrate over the massive Weyl gauge field, and consequently not only we would have GR with the SM but also the second clock effect has no physical effects at low energies.

6. Conclusions

Shortly after Einstein constructed general relativity (GR) in 1915, Weyl has advocated a generalization in that the very notion of length becomes path-dependent. In Weyl's theory, even if the lightcones retain the fundamental role as in GR, there is no absolute meaning of scales for space-time, so the metric is defined only up to proportionality. It is this property that we have a scale symmetry prohibiting the appearance of any dimensionful parameters and coupling constants in the Weyl theory. The main complaint against the Weyl's idea is that it inevitably leads to the so-called "second clock effect": The rate where any clock measures would depend on its history. Since the second clock effect has not been observed by experiments, the Weyl theory might make no sense as a classical theory.[8]

However, viewed as a quantum field theory, the Weyl theory is a physically consistent theory and provides us with a natural playground for constructing conformally invariant quantum field theories as shown in this article.[9] Requiring the invariance under Weyl transformation is so strong that only quadratic curvature terms are allowed to exist in a classical action, which should be contrasted with the situation of GR where any number of curvature terms could be in principle the candidate of a classical action only ifs we require the action to be invariant under diffeomorphisms.

Of course, we have a serious problem to be solved in future; the problem of unitarity. The lack of perturbative unitarity is a common problem in the higher derivative gravity like the Weyl theory [43, 44, 45]. However, it is expected that the Weyl gravity, whose Lagrangian density is of form, $\sqrt{-g}C_{\mu\nu\rho\sigma}^2$, is asymptotically free, and the issue of the perturbative unitarity is closely relevant to infrared dynamics of asymptotic fields, so this problem becomes to be quite nontrivial. Provided that we can confine the ghosts in the Weyl theory like in QCD, we would be free of the perturbative unitarity.

References

[1] I. Oda, "Planck and Electroweak Scales Emerging from Conformal Gravity", Eur. Phys. J. **C 78** (2018) 798.

[2] C. W. Misner, K. S. Thorne and J. A. Wheeler, "Gravitation", W H Freeman and Co (Sd), 1973.

[3] Y. Fujii and K. Maeda, "The Scalar-Tensor Theory of Gravitation", Cambridge University Press, 2003.

[4] H. Weyl, "Gravitation und Elekrizität", Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 1918, pp. 465-480.

[5] L. O'Raifeartaigh, "The Dawning of Gauge Theory", Princeton University Press, 1997.

[6] E. Scholz, "The Unexpected Resurgence of Weyl Geometry in late 20th-Century Physics", Einstein Stud. **14** (2018) 261.

[7] M. Omote, "Scale Transformations of the Second Kind and the Weyl Space-Time", Lett. Nuovo. Cim. **2** (1971) 58.

[8] P. A. M. Dirac, "Long Range Forces and Broken Symmetries", Proc. Roy. Soc. Lond. **A 333** (1973) 403.

[8] The second clock problem and its resolution have been recently discussed in Ref. [30].

[9] We have already contructed the other scale invariant gravitational models [40, 41, 42].

[9] R. Utiyama, "On Weyl's Gauge Field", Prog. Theor. Phys. **50** (1973) 2080.

[10] P. G. O. Freund, "Local Scale Invariance and Gravitation", Ann. of Phys. **84** (1974) 440.

[11] R. Utiyama, "On Weyl's Gauge Field II", Prog. Theor. Phys. **53** (1975) 565.

[12] R. Adler, M. Bazin and M. Schiffer, "Introduction to General Relativity", McGraw-Hill Inc., 1975.

[13] K. Hayashi, M. Kasuya and T. Shirafuji, "Elementary Particles and Weyl's Gauge Field", Prog. Theor. Phys. **57** (1977) 431; Erratum: Prog. Theor. Phys. **59** (1978) 681.

[14] K. Hayashi and T. Kugo, "Remarks on Weyl's Gauge Field", Prog. Theor. Phys. **61** (1979) 334.

[15] L. Smolin, "Towards a Theory of Spacetime Structure at Very Short Distances", Nucl. Phys. **B 160** (1979) 253.

[16] H. Cheng, "Possible Existence of Weyl's Vector Meson", Phys. Rev. Lett. **61** (1988) 2182.

[17] W. Drechsler and H. Tann, "Broken Weyl Invariance and the Origin of Mass", Found. Phys. **29** (1999) 1023.

[18] H. Nishino and S. Rajpoot, "Implication of Compensator Field and Local Scale Invariance in the Standard Model", Phys. Rev. **D 79** (2009) 125025.

[19] C. Pagani and R. Percacci, "Quantization and Fixed Points of Non-Integrable Weyl Theory", Class. Quant. Grav. **31** (2014) 115005.

[20] H. C. Ohanian, "Weyl Gauge-vector and Complex Dilaton Scalar for Conformal Symmetry and Its Breaking", Gen. Rel. Grav. **48** (2016) 25.

[21] M. de Cesare, J. W. Moffat and M. Sakellariadou, "Local Conformal Symmetry in Non-Riemannian Geometry and the Origin of Physical Scales", Eur. Phys. J. **C 77** (2017) 605.

[22] C. Barceló, R. Carballo-Rubio and L. J. Garay, "Weyl Relativity: A Novel Approach to Weyl's Ideas", JCAP **1706** (2017) 014.

[23] S. Diles, "The Role of Weyl Symmetry in Hydrodynamics", Phys. Lett. **B 779** (2018) 331.

[24] L. Ciambelli and R. G. Leigh, "Weyl Connections and Their Role in Holography", arXiv:1905.04339 [hep-th].

[25] D. M. Ghilencea, "Spontaneous Breaking of Weyl Quadratic Gravity to Einstein Action and Higgs Potential", JHEP **1903** (2019) 049.

[26] I. Oda, "Planck and Electroweak Scales Emerging from Weyl Conformal Gravity", arXiv:1903.09309 [hep-th], To appear PoS, The Corfu Summer Institute 2018, Workshop on the Standard Model and Beyond.

[27] D. M. Ghilencea, "Stueckelberg Breaking of Weyl Conformal Geometry and Applications to Gravity", Phys. Rev. **D 101** (2020) 045010.

[28] I. Oda, "Planck Scale from Broken Local Conformal Invariance in Weyl Geometry", Adv. Studies in Theor. Phys. **14** (2020) 9.

[29] T. A. T. Sanomiya, I. P. Lobo, J. B. Formiga, F. Dahia and C. Romero, "An Invariant Approach to Weyl's Unified Field Theory", arXiv:2002.00285 [gr-qc].

[30] I. Oda, "Emergence of Einstein Gravity from Weyl Gravity", arXiv:2003.01437 [hep-th].

[31] I. Oda, "Spontaneous Symmetry Breakdown of Scale Symmetries", arXiv:2003.07947 [hep-th].

[32] R. Penrose, "The Road to Reality", Vintage Books, New York, 2007.

[33] S. R. Coleman and E. J. Weinberg, "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking", Phys. Rev. **D 7** (1973) 1888.

[34] D. M. Ghilencea, “Manifestly Scale-invariant Regularization and Quantum Effective Operators”, Phys. Rev. **D 93** (2016) 105006.

[35] M. Nakasone and I. Oda, "On Unitarity of Massive Gravity in Three Dimensions", Prog. Theor. Phys. **121** (2009) 1389.

[36] M. Nakasone and I. Oda, "Massive Gravity with Mass Term in Three Dimensions", Phys. Rev. **D 79** (2009) 104012.

[37] J. W. York, "Conformatlly invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial value problem of general relativity", J. Math. Phys. **14** (1973) 456.

[38] R. Percacci, "An Introduction to Covariant Quantum Gravity and Asymtotic Safety", World Scientific Publishing Co. Pte. Ltd., 2017.

[39] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory", Westview Press, 1995.

[40] I. Oda, "Classically Scale-invariant B-L Model and Dilaton Gravity", Phys. Rev. **D 87** (2013) 065025.

[41] I. Oda, "Classically Scale-invariant B-L Model and Conformal Gravity", Phys. Lett. **B 724** (2013) 160.

[42] I. Oda, "Higgs Mechanism in Scale-Invariant Gravity", Adv. Studies in Theor. Phys. **8** (2014) 215.

[43] K. S. Stelle, "Renormalization of Higher Derivative Quantum Gravity", Phys. Rev. **D 16** (1977) 953.

[44] J. Julve and M. Tonin, "Quantum Gravity with Higher Derivative Terms", Nuovo Cim. **B 46** (1978) 137.

[45] E. S. Fradkin and A. A. Tseytlin, "Renormalizable asymptotically free quantum theory of gravity", Nucl. Phys. **B 201** (1982) 469.