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α_s from the ALPHA collaboration (part I)

Stefan Sint*
School of Mathematics and Hamilton Mathematics Institute, Hamilton building, Trinity College Dublin, Dublin 2, Ireland
E-mail: sint@maths.tcd.i

The recent determination of $\alpha_s(m_Z) = 0.11852(84)$ by the ALPHA collaboration [1] distinguishes itself by the very good control of perturbative truncation and other systematic errors. A variety of tools and methods had to be deployed to enable this result. In this contribution I will give a short account of the step-scaling method and its application to QCD couplings in finite volume renormalization schemes. Tracing the running couplings non-perturbatively between scales μ_0 and $32\mu_0$ (corresponding roughly to the range 4–128 GeV) leads to the intermediate result $\Lambda^{(3)}_{\overline{\rm MS}}/\mu_0 = 0.0791(19)$ in 3-flavour QCD. By computing this ratio in variety of ways, using perturbation theory in different schemes and at different energy scales at intermediate stages, gives us confidence in the error estimate and also enables a number of useful tests of perturbation theory. The remaining steps required for $\alpha_s(m_Z)$ will be discussed by Mattia Dalla Brida in these proceedings [2].

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*Speaker.

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Introduction

The recent result for α_s by the ALPHA collaboration relies on the combination of various tools and techniques that have been developed and improved over the last 20–30 years. A crucial ingredient is the recursive step-scaling method [3] applied to QCD couplings renormalized in a finite Euclidean space-time volume. This allows us to overcome the typical limitation of lattice QCD, whereby large scale differences cannot be resolved on a single lattice without incurring large computational costs [4]. As will become clear in this and in Mattia Dalla Brida's companion contribution [2], we have covered a range of energy scales differing by 2–3 orders of magnitude, thus connecting hadronic scales of $\mathscr{O}(100)$ MeV with electroweak scales of $\mathscr{O}(100)$ GeV. The scale evolution of QCD couplings in so-called Schrödinger functional (SF) schemes is obtained non-perturbatively and in the continuum limit. Given the good perturbative knowledge for the SF schemes one may assess at which scale perturbative behaviour sets in and extract the Λ parameter. In this way, the systematic error due to the truncation of the perturbative series can be well-controlled and kept at a level that remains subdominant compared to current statistical errors. This is in contrast to many other lattice determinations of α_s where perturbative uncertainties arise at much lower energy scales and are thus much harder to quantify.

We remark that all our simulations are carried out for 3-flavour QCD. Therefore the result for $\alpha_s(m_Z)$ in 5-flavour QCD also relies on decoupling relations across charm and bottom quark thresholds; I refer to [2] for references and a discussion. The ALPHA collaboration's strategy involves two different finite volume renormalization schemes for the 3-flavour QCD coupling. At low energies, a coupling based on the gradient flow (GF) has advantageous properties (cf. [2]). The high energy regime is covered using a 1-parameter family of SF couplings, for which the 2-loop matching to the $\overline{\text{MS}}$-coupling and the 3-loop β-functions are known [5]–[8]. Our strategy then requires a matching between the GF and SF couplings at an intermediate scale, μ_0, which is implicitly defined by the SF coupling and turns out to be around 4 GeV in physical units.

In the following, I will briefly review the step-scaling method and illustrate it with our results for the SF coupling. The exactly known scheme dependence of the Λ parameter makes it a useful reference quantity, which enables various tests of perturbation theory. The main intermediate outcome of this first part is $\Lambda^{(3)}_{\overline{\text{MS}}}/\mu_0 = 0.0791(19)$, which defines the starting point for Mattia Dalla Brida's contribution [2].

Non-perturbatively defined QCD couplings and the Λ parameter

Let us assume we have an observable[1], $\langle O\rangle$, with a finite continuum limit and also possessing a perturbative expansion starting with g^2. We will assume throughout that all three light quark masses are set to zero. If the Euclidean time and space extents are given by L and all dimensionful parameters, such as momenta, distances, or background fields are taken in a fixed proportion to L then the observable depends on a single scale $\mu = 1/L$ and we may define[2] $\bar{g}^2(L) = \langle O\rangle$. Examples for such finite volume couplings are the GF coupling discussed in [2] and the family of SF couplings

[1] In this context, an observable is given as a correlation function of gauge invariant fields defined with the Euclidean (lattice) QCD path integral. These are the quantities estimated in a numerical simulation of lattice QCD.

[2] To denote the scale dependence we use the convention $\bar{g}^2 = \bar{g}^2(L)$ and $\alpha(\mu = 1/L) = \bar{g}^2(L)/(4\pi)$.

introduced in [9]–[11], which derive from the QCD SF [12, 13]. For details we refer to [14]. Physically, the SF$_\nu$ couplings are response coefficients to the variation of an Abelian colour electric background field. The dependence on the parameter ν takes the simple form

$$\frac{1}{\bar{g}^2_\nu} = \frac{1}{\bar{g}^2} - \nu\bar{v}, \tag{1}$$

in terms of two correlation functions $1/\bar{g}^2 = \langle O_1\rangle$ and $\bar{v} = \langle O_2\rangle$, measured in a simulation at $\nu = 0$.

Given such a coupling, its β-function $\beta(\bar{g}) = -L\partial\bar{g}/\partial L$ is non-perturbatively defined too. Yet it has the usual weak coupling expansion $\beta(g) = -b_0g^3 - b_1g^5 + \ldots$ with the universal coefficients $b_0 = 9/16\pi^2$ and $b_1 = 1/4\pi^4$ (for 3-flavour QCD). Hence also the associated Λ parameter, given as an exact solution of the Callan–Symanzik equation, is non-perturbatively defined. Indicating the dependence on the scheme 'x' by a subscript, it takes the form

$$\Lambda_{\rm x} \;=\; L^{-1}\varphi_{\rm x}(\bar{g}_{\rm x}(L))\,, \tag{2}$$

with

$$\varphi_{\rm x}(\bar{g}) = (b_0\bar{g}^2)^{-b_1/(2b_0^2)}{\rm e}^{-1/(2b_0\bar{g}^2)} \times \exp\left\{-\int_0^{\bar{g}} {\rm d}g\,\left[\frac{1}{\beta_{\rm x}(g)} + \frac{1}{b_0g^3} - \frac{b_1}{b_0^2g}\right]\right\}. \tag{3}$$

Its behaviour under a change from scheme x to y is *exactly* determined by the one-loop coefficient relating the respective couplings, i.e. if $g^2_{\rm x} = g^2_{\rm y} + c_{\rm xy}g^4_{\rm y} + \ldots$ then $\Lambda_{\rm x}/\Lambda_{\rm y} = \exp\left(c_{\rm xy}/2b_0\right)$. Thus the relations between Λ parameters for all SF$_\nu$ schemes and the $\overline{\rm MS}$ scheme are known. Note that $\Lambda_{\overline{\rm MS}}$ is thus indirectly defined beyond perturbation theory, even though the $\overline{\rm MS}$ scheme is otherwise only perturbative. Furthermore, also the 2-loop relations between the respective couplings are known and thus the 3-loop coefficients $b_{\nu,2}$ for SF$_\nu$ schemes can be inferred. Numerical values with parameter $\nu = {\rm O}(1)$ seem reasonable from a perturbative viewpoint [14].

Step-scaling

Given a QCD coupling in a mass-independent finite volume renormalization scheme, its step-scaling function (SSF) is defined by,

$$\sigma(u) = \bar{g}^2(2L)|_{u=\bar{g}^2(L)}, \tag{4}$$

and thus yields the coupling at $2L$ given the coupling at L. In other words it determines the coupling if the scale is changed by a step factor 2 and is related to an integral of the β-function,

$$\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{{\rm d}g}{\beta(g)} = -\ln 2\,, \tag{5}$$

For a fixed argument u, the SSF can be obtained as the continuum limit of lattice approximants,

$$\sigma(u) = \lim_{a\to 0}\Sigma(u, a/L), \tag{6}$$

where a lattice approximant $\Sigma(u, a/L)$ requires the measurements on pairs of lattices with linear extents L/a and $2L/a$. To keep the lattice spacing a fixed, one uses the same bare lattice coupling,

g_0^2, for each pair. In principle, keeping u fixed is achieved by tuning the bare coupling g_0^2 such that $\bar{g}^2(L) = u$ on an L/a-lattice. In practice, however, it is more convenient to produce data for the function $\Sigma(u, a/L)$ at various values of its arguments and then perform a global fit of the form

$$\Sigma(u, a/L) = \sigma(u) + \rho(u)\,(a/L)^2\,, \tag{7}$$

where both $\sigma(u)$ and $\rho(u)$ are polynomials in u [14]. A typical parameterization for $\sigma(u)$ is given by

$$\sigma(u) = u + s_0 u^2 + s_1 u^3 + s_2 u^4 + c_1 u^5, \tag{8}$$

where c_1 is a fit parameter and $s_{0,1,2}$ are fixed to their perturbative values in terms of $b_{0,1,2}$. The non-perturbatively defined function $\sigma(u)$ is then represented by the fit function for u in some interval $[u_{\rm min}, u_{\rm max}]$, cf. Fig. 1.

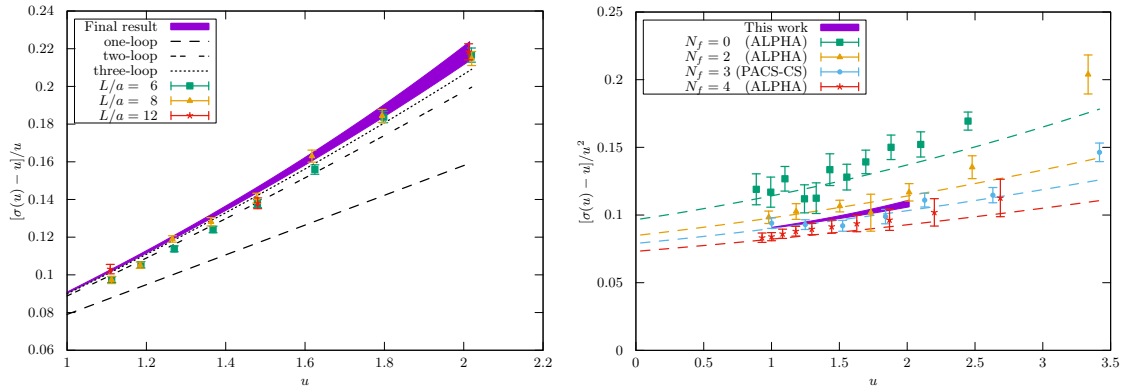

Figure 1: Left: Continuum extrapolation of the lattice data for $\Sigma(u, a/L)$ yielding $\sigma(u)$ for a range of u-values with errors indicated by the blue band. Right: Comparison with earlier studies in QCD with $N_{\rm f} = 0, 2, 3, 4$ flavours [15]–[18] illustrating the reduced errors in our new $N_{\rm f} = 3$ data [14].

Given $\sigma(u)$ one may define the largest coupling $u_{\rm max} = u_0 = \bar{g}^2(L_0)$ and then recursively step up the energy scale by factors of 2, i.e.

$$u_n = \sigma(u_{n+1}), \qquad u_n = \bar{g}^2(L_n), \qquad L_n = L_0/2^n, \tag{9}$$

until one reaches the smallest coupling still covered by the data[3]. In our case we set $\bar{g}^2(L_0) = 2.012$ for the SF scheme at our default choice $\nu = 0$, and this implicitly defines the scale $\mu_0 = 1/L_0$. The data shown in Fig. 1 then allows us to make up to $n = 5$ steps from L_0, reaching energy scales $\mu_n = 1/L_n$ up to $\mu_5 = 32\mu_0$. In order to do the same steps for any other value of ν one needs $\bar{\nu}(L_0) = 0.1199(10)$ to define the start value, $\bar{g}_\nu^2(L_0)$, for the recursion (cf. [14] for details).

Tests of perturbation theory and extraction of $\Lambda_{\overline{\rm MS}}$

Taking the Λ parameter in the SF scheme with $\nu = 0$ as our reference quantity we can now obtain it in a variety of ways

$$\Lambda L_0 \equiv \Lambda_{{\rm SF}_{\nu=0}} L_0 = (\Lambda/\Lambda_{{\rm SF}_\nu}) \times 2^n \varphi_{{\rm SF}_\nu}\left(\bar{g}_\nu(L_n)\right)\,. \tag{10}$$

[3]Note that evolving towards higher energies requires to invert the step-scaling function. This poses no practical problems.

Obviously, the LHS of this equation must always be the same up to the perturbative approximation to the integral in the exponent of Eq. (3), which reads

$$\int_0^{\bar{g}} \mathrm{d}g \left[\frac{1}{\beta_{\mathrm{x}}(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right] = \frac{b_{\mathrm{x},2} b_0 - b_1^2}{2b_0^3}\bar{g}^2 + \mathrm{O}(\bar{g}^4). \tag{11}$$

Hence, given that the 3-loop coefficient, $b_{\mathrm{x},2}$, is known for all the SF$_\nu$ schemes, we have a parametric uncertainty of $\mathscr{O}(\alpha^2)$ (with $\alpha = \bar{g}^2/4\pi$) for this integral and thus for Λ. Obviously, the higher the scale $\mu_n = 1/L_n$, the smaller this uncertainty should become. We test this by evaluating the RHS of Eq. (10) for different values of ν and n, cf. Fig. 2. As expected all points come together as α decreases. We also observe a roughly linear behaviour in α^2, as expected from Eq. (11). However, the slope for $\nu = -0.5$ seems rather large, whereas it almost vanishes for $\nu = 0.3$. Our final result, shown as grey band in Fig. 2, is extracted at scales reached after $n = 4$ steps (i.e. around 70 GeV),

$$\Lambda/\mu_0 = 0.0303(7) \quad \Rightarrow \quad \Lambda_{\overline{\mathrm{MS}}}/\mu_0 = 0.0791(19)\,. \tag{12}$$

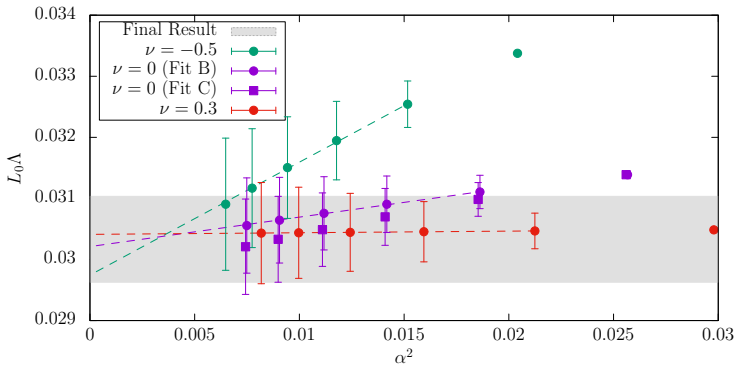

Figure 2: Estimates for ΛL_0 as a function of the parametric uncertainty α^2. "Fit A" and "Fit B" correspond to two different fit functions for $\sigma(u)$, cf. [14] for details.

A further test can be performed by first converting the couplings to the $\overline{\mathrm{MS}}$ coupling at 2-loop order, and then extracting the Λ-parameter within the $\overline{\mathrm{MS}}$ scheme using the β-function up to 5-loop order [19]–[23]. In the conversion between the couplings we allow for a scale factor, s,

$$\begin{aligned}\Lambda_{\overline{\mathrm{MS}}} L_0 &= s\frac{L_0}{L_n}\varphi_{\overline{\mathrm{MS}}}\left(\bar{g}_{\overline{\mathrm{MS}}}(L_n/s)\right) \\ &= s\,2^n \varphi_{\overline{\mathrm{MS}}}\left(\sqrt{\bar{g}_\nu^2(L_n) + p_1^\nu(s)\bar{g}_\nu^4(L_n) + p_2^\nu(s)\bar{g}_\nu^6(L_n) + \mathrm{O}\left[\bar{g}_\nu^8(L_n)\right]}\right),\end{aligned} \tag{13}$$

and the result must be independent of s, ν and n. As our best value of s we choose $s = s^*$ such that the one loop coefficient $p_1^\nu(s) \approx 0$, which determines s^* as the ratio of the corresponding Λ parameters. We then vary s in the interval $[s^*/2, 2s^*]$, in order to obtain a measure for the uncertainty from neglected higher order terms. This estimate can then be compared with the true deviation from $\Lambda_{\overline{\mathrm{MS}}}/\mu_0$, Eq. (12).

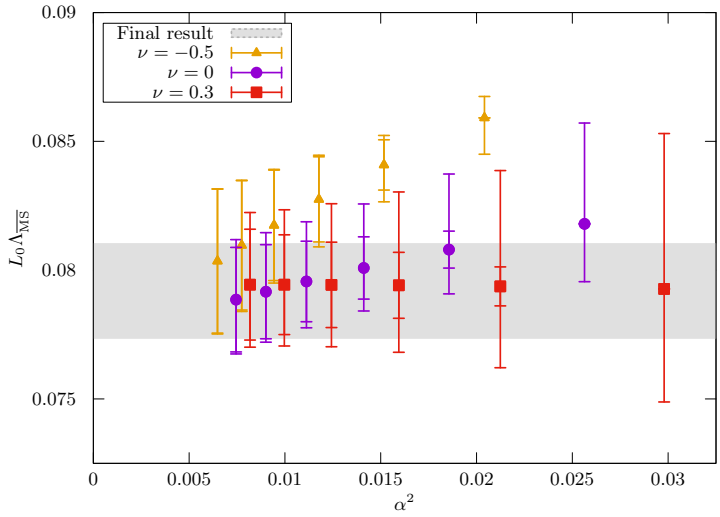

Figure 3: Statistical (interior error band) and total (exterior error band) uncertainties in the determination of $L_0\Lambda_{\overline{\rm MS}}$. The total error is the combination in quadrature of the statistical and systematic error, where the latter is obtained by varying s between $s^*/2$ and $2s^*$. The grey band is our final estimate, Eq. (12).

Conclusion

We have studied the non-perturbative scale evolution of for a 1-parameter family of SF couplings for energies between roughly 4 and 128 GeV. We conclude that one needs to reach $\alpha \approx 0.1$ in order to confidently extract the Λ parameter with an error below 3%. In a further consistency check we first converted the SF to the $\overline{\rm MS}$ coupling and then varied the relative scale within a factor of two either way around a preferred choice. We note that this common recipe may nor may not capture the true perturbative uncertainty. This reinforces the general warning that perturbative truncation errors are easily underestimated.

References

[1] M. Bruno, M. Dalla Brida *et al.* [ALPHA Collaboration], Phys. Rev. Lett. **119** (2017) 102001.

[2] M. Dalla Brida, α_s *from the ALPHA collaboration (part II)*, these proceedings.

[3] M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B **359** (1991) 221.

[4] K. Jansen, C. Liu, M. Lüscher, H. Simma, S. Sint, R. Sommer, P. Weisz, and U. Wolff, Phys. Lett. B **372** (1996) 275.

[5] A. Bode, U. Wolff, and P. Weisz [ALPHA Collaboration], Nucl. Phys. B **540** (1999) 491.

[6] A. Bode, P. Weisz, and U. Wolff [ALPHA Collaboration], Nucl. Phys. B **576** (2000) 517 [Erratum: Nucl. Phys. B608 (2001) 481].

[7] C. Christou, H. Panagopoulos, A. Feo, and E. Vicari, Phys. Lett. B **426** (1998) 121.

[8] C. Christou, A. Feo, H. Panagopoulos, and E. Vicari, Nucl. Phys. B **525** (1998) 387 [Erratum: Nucl. Phys. B 608 (2001) 479].

[9] M. Lüscher, R. Sommer, P. Weisz, and U. Wolff, Nucl. Phys. B **413** (1994) 481.

[10] S. Sint and R. Sommer, Nucl. Phys. B **465** (1996) 71.

[11] M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint, and R. Sommer [ALPHA Collaboration], Phys. Rev. Lett. **117** (2016) 182001.

[12] M. Lüscher, R. Narayanan, P. Weisz, and U. Wolff, Nucl. Phys. B **384** (1992) 168.

[13] S. Sint, Nucl. Phys. B **421** (1994) 135.

[14] M. Dalla Brida *et al.* [ALPHA Collaboration], Eur. Phys. J. C **78** (2018) 372.

[15] S. Capitani, M. Lüscher, R. Sommer, and H. Wittig [ALPHA Collaboration], Nucl. Phys. B **544** (1999) 669.

[16] M. Della Morte *et al.* [ALPHA Collaboration], Nucl. Phys. B **713** (2005) 378.

[17] S. Aoki *et al.* [PACS-CS collaboration], JHEP **10** (2009) 053.

[18] F. Tekin, R. Sommer, and U. Wolff [ALPHA Collaboration], Nucl. Phys. B **840** (2010) 114.

[19] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B **400** (1997) 379.

[20] M. Czakon, Nucl. Phys. B **710** (2005) 485.

[21] K. G. Chetyrkin, G. Falcioni, F. Herzog, and J. A. M. Vermaseren, JHEP **10** (2017) 179 [Addendum: JHEP 12 (2017) 006].

[22] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **118** (2017) 082002,

[23] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, JHEP **03** (2017) 020.