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Sedenions, the Clifford algebra $\mathbb{C}\ell(8)$, and three fermion generations

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Recently there has been renewed interest in using tensor products of division algebras, together with their associated Clifford algebras, to identify the structures of the Standard Model. One full generations of leptons and quarks transforming correctly under the electrocolor group $SU(3)_c \otimes U(1)_{em}$ can be described in terms of complex octonion algebra $\mathbb{C}\otimes\mathbb{O}$. By going beyond the division algebras, and considering the larger Cayley-Dickson algebra of sedenions $\mathbb{S}$, this one generation model is extended to exactly three generations. Each generation is contained in an $\mathbb{C}\otimes\mathbb{O}$ subalgebra of $\mathbb{C}\otimes\mathbb{S}$, however these three subalgebras are not independent of one another. This three generation model can be related to an alternative model of three generations based on the exceptional Jordan algebra $J_3(\mathbb{O})$. It is speculated that the shared $\mathbb{C}\otimes\mathbb{H}$ algebra common to all three generations might form a basis for CKM mixing.

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1. Introduction

The establishment of the Standard Model (SM) as a description of the structure of particles and their interactions has been one of the great achievements in theoretical physics. However, despite the model's overwhelming success, its underlying mathematical structure is not derived from more fundamental principles. Finding a theoretical basis for the mathematical structure that underlies the SM remains a prominent challenge in physics. Instead of the common approach of embedding the SM gauge group into some larger group, recently there has been an interest in using the fundamental and generative structures of the four normed division algebras as a simple mathematical framework for particle physics [1, 2, 3, 4, 5, 6, 7, 8].

There are only four normed division algebras; the reals $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. In [2] one full generations of leptons and quarks transforming correctly under the electrocolor group $SU(3)_c \otimes U(1)_{em}$ is found starting from the complex octonion algebra $\mathbb{C}\otimes\mathbb{O}$ (itself not a division algebra). That work builds on initial results from the 1970s relating the octonions to the symmetries of (one generation of) quarks 1970s [9].

One prominent open question in the division algebra program is how to extend the many results from a single generation to exactly three generations. In this short paper some recent results are reviewed which show that by going beyond the division algebras, and including the Cayley-Dickson algebra of sedenions $\mathbb{S}$, it is possible to extend the one generation results of [2] to exactly three generations in a very natural way [10]. Some ongoing research and speculations relating to this approach are then discussed.

2. Three generations from complex sedenions

Although the algebra of complex octonions $\mathbb{C}\otimes\mathbb{O}$ is nonassociative, its left adjoint algebra $(\mathbb{C}\otimes\mathbb{O})_L$ generated via the left adjoint actions of the algebra on itself is associative, and isomorphic to the complex Clifford algebra $\mathbb{C}\ell(6)$. In [2] it was shown that a Witt decomposition of $(\mathbb{C}\otimes\mathbb{O})_L \cong \mathbb{C}\ell(6)$ decomposes $(\mathbb{C}\otimes\mathbb{O})_L$ into minimal left ideals whose basis states transform as a single generation of leptons and quarks under the unbroken electrocolor $SU(3)_c \otimes U(1)_{em}$[1].

The octonions $\mathbb{O}$ are spanned by the identity $1 = e_0$ and seven anti-commuting square roots of minus one e_i satisfying

$$e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k, \quad \text{where} \quad e_i e_0 = e_0 e_i = e_i,\ e_0^2 = e_0, \tag{2.1}$$

and ε_{ijk} is a completely antisymmetric tensor with value +1 when $ijk = 124,\ 156,\ 137,\ 235,\ 267,\ 346,\ 457$. By selecting e_7 as a special direction, a split basis of nilpotents can be defined as[2]

$$\alpha_1 \equiv \frac{1}{2}(-e_5 + ie_4), \qquad \alpha_2 \equiv \frac{1}{2}(-e_3 + ie_1), \qquad \alpha_3 \equiv \frac{1}{2}(-e_6 + ie_2), \tag{2.2}$$

$$\alpha_1^\dagger \equiv \frac{1}{2}(e_5 + ie_4), \qquad \alpha_2^\dagger \equiv \frac{1}{2}(e_3 + ie_1), \qquad \alpha_3^\dagger \equiv \frac{1}{2}(e_6 + ie_2), \tag{2.3}$$

[1] It is worth mentioning that a similar approach, using $\mathbb{C}\ell(6)$ as a starting point, was concurrently developed in [11], which is also consistent with [1]. Those works include a copy of the quaternion algebra, recovering the full SM gauge group from the algebra $\mathbb{T} = \mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. The quaternionic factor in $\mathbb{T}$ is responsible for the broken $SU(2)$ chiral symmetry.

[2] following the convention of [2].

satisfying the anticommutator algebra of fermionic ladder operators

$$\left\{\alpha_i^\dagger, \alpha_j^\dagger\right\} = \left\{\alpha_i, \alpha_j\right\} = 0, \qquad \left\{\alpha_i^\dagger, \alpha_j\right\} = \delta_{ij}. \tag{2.4}$$

Considering the basis vectors as elements of the left adjoint algebra $(\mathbb{C}\otimes\mathbb{O})_L \cong \mathbb{C}\ell(6)$, allows for the construction of a primitive idempotent $\omega\omega^\dagger = \alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$ which defines a minimal left ideal $S^u \equiv \mathbb{C}\ell(6)\omega\omega^\dagger$. Explicitly:

$$\begin{aligned} S^u \equiv & \\ & \nu\omega\omega^\dagger + \\ & \bar{d}^r\alpha_1^\dagger\omega\omega^\dagger + \bar{d}^g\alpha_2^\dagger\omega\omega^\dagger + \bar{d}^b\alpha_3^\dagger\omega\omega^\dagger \\ & u^r\alpha_3^\dagger\alpha_2^\dagger\omega\omega^\dagger + u^g\alpha_1^\dagger\alpha_3^\dagger\omega\omega^\dagger + u^b\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger \\ & + e^+\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger, \end{aligned} \tag{2.5}$$

where ν, $\bar{d}^r$ etc. are suggestively labeled complex coefficients denoting the isospin-up elementary fermions. The conjugate system analogously gives a second linearly independent minimal left ideal of isospin-down elementary fermions. These representations of the minimal left ideals are invariant under the electrocolor symmetry $SU(3)_c \otimes U(1)_{em}$, with each basis state in the ideals transforming as a specific lepton or quark as indicated by their suggestively labeled complex coefficients.

One way of extending this model from a single generation to exactly three generations is to go beyond the division algebras, and consider the next Cayley-Dickson algebra after the octonions [10]. This is the algebra of sedenions $\mathbb{S}$. The sedenions $\mathbb{S}$ are spanned by the identity $1 = e_0$ and seven anti-commuting square roots of minus one e_i satisfying the multiplication table given in Appecndix A of [10]. This larger algebra also generates a larger left adjoint algebra $(\mathbb{C}\otimes\mathbb{S})_L \cong \mathbb{C}\ell(8)$. A decomposition of $(\mathbb{C}\otimes\mathbb{S})_L$ naturally extends the above construction of a single generation to exactly three generations.

Selecting e_{15} as a special imaginary unit (in the same way that e_7 is selected as a special imaginary unit in the octonion case), this time one can define a basis of seven nilpotents η_i, $i = 1,2,...,7$ of the form $\frac{1}{2}(-e_a + ie_{15-a})$ where $a \in 5,7,13,1,4,6,12$, together with seven nilpotent conjugates $\eta_i^\dagger$ of the form $\frac{1}{2}(e_a + ie_{15-a})$, satisfying

$$\left\{\eta_i^\dagger, \eta_j^\dagger\right\} = \left\{\eta_i, \eta_j\right\} = 0, \qquad \left\{\eta_i^\dagger, \eta_j\right\} = \delta_{ij}. \tag{2.6}$$

These η_i and $\eta_i^\dagger$ uniquely divide into three intersecting sets

$$\alpha_1 = \frac{1}{2}(-e_5 + ie_{10}), \qquad \beta_1 = \frac{1}{2}(-e_7 + ie_8), \qquad \gamma_1 = \frac{1}{2}(-e_{13} + ie_2), \tag{2.7}$$

$$\alpha_2 = \beta_2 = \gamma_2 = \frac{1}{2}(-e_1 + ie_{14}), \tag{2.8}$$

$$\alpha_3 = \frac{1}{2}(-e_4 + ie_{11}), \qquad \beta_3 = \frac{1}{2}(-e_6 + ie_9), \qquad \gamma_3 = \frac{1}{2}(-e_{12} + ie_3), \tag{2.9}$$

together with their conjugates. The sets $\{\alpha_i^\dagger, \alpha_i\}$, $\{\beta_i^\dagger, \beta_i\}$, and $\{\gamma_i^\dagger, \gamma_i\}$ each individually form a split basis for $\mathbb{C}\otimes\mathbb{O}$ and satisfy algebra (2.4). That is, selecting e_{15} as the special unit imaginary singles out three intersecting octonionic subalgebras. This allows us to describe three generations

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of leptons and quarks with unbroken $SU(3)_c \otimes U(1)_{em}$ symmetry. These three $\mathbb{C}\otimes\mathbb{O}$ subalgebras are not independent but share a common quaternionic subalgebra spanned by $\{1, e_1, e_{14}, e_{15}\}$. The three intersecting octonion subalgebras together with the multiplication rules of their base elements may be represented by the three Fano planes in Figure 1. Each of the three octonionic subalgebras

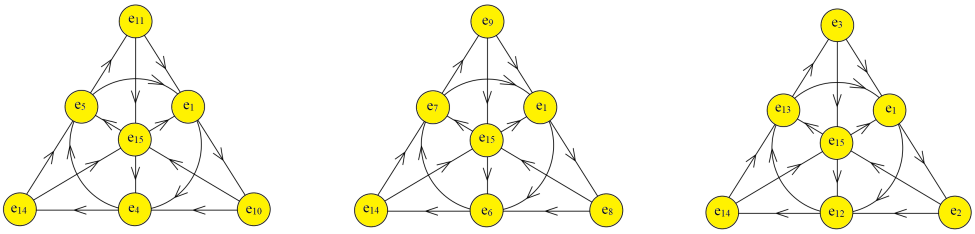

Figure 1: Fano planes of three octonionic subalgebras in the sedenions.

above generates via its adjoint left actions a copy of $\mathbb{C}\ell(6)$. From each of these algebras one constructs $SU(3)_c$ and $U(1)_{em}$ generators as above. That is, each generation has its own copy of $SU(3)_c$ and $U(1)_{em}$ associated with it.

2.1 Spinorial degrees of freedom

The spinorial degrees of freedom and the weak force are not captured by $\mathbb{C}\otimes\mathbb{O}$ nor by $\mathbb{C}\otimes\mathbb{S}$. There are several ways to include these additional degrees of freedom. In [2] they are added by including a copy of the complex quaternions $\mathbb{C}\otimes\mathbb{H}$, for an overall model based on $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ (as in [1]). The left adjoint algebra in this case is $(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O})_L = \mathbb{C}\ell(8)$, the same as the left adjoint algebra of $\mathbb{C}\otimes\mathbb{S}$. In [11] on the other hand, all the degrees of freedom are included by considering all eight minimal left ideals of $\mathbb{C}\ell(6)$, instead of just two as in [2, 10]. An alternative way of describing the full degrees of freedom is to therefore consider all 16 minimal left ideals of $\mathbb{C}\ell(8)$.

Including a copy of the quaternions into the sedenion model would enlarge the left adjoint algebra to $(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{S})_L \cong \mathbb{C}\ell 10$. Associated with this algebra is the group Spin(10), which forms the basis of the $SO(10)$ grand unified theory. It is however not clear why a copy of the octonions is missing in this case. Including a copy of the octonions, one gets $(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}\otimes\mathbb{S})_L \cong \mathbb{C}\ell(16)$, which contains the $e_8 \oplus e_8$ algebra of the anomaly free 10D heterotic string [12].

3. The intersectionality of three generations as a basis for CKM mixing

The three Fano planes in Figure 1 share a common complex quaternionic line of intersection. This reflects the fact that the three generations in the sedenion model are not independent of one another. This leads to unphysical transformations between generations not observed in nature such as

$$\left[\Lambda_1^{(1)}, \beta_1^\dagger\right] = i\gamma_3, \; \left[\Lambda_1^{(1)}, \beta_3^\dagger\right] = i\gamma_1, \; \left[\Lambda_1^{(1)}, \gamma_1^\dagger\right] = -i\beta_3, \; \left[\Lambda_1^{(1)}, \gamma_3^\dagger\right] = -i\beta_1 \tag{3.1}$$

where $\Lambda_1^{(1)} = -(\alpha_2^\dagger\alpha_1 + \alpha_1^\dagger\alpha_2)$ is the first $SU(3)$ generator constructed from the first $\mathbb{C}\otimes\mathbb{O}$ subalgebra associated with the first generation of fermions. To eleminate these unphysical transformations one may be able to consider linear combinations of fermion states, symmetry generators, or both.

This at the same time may provide a basis for the CKM mixing of quarks. This is currently being investigated.

4. Three generations from the triality of $\mathbb{C}\ell(8)$

It was mentioned earlier that $(\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O})_L \cong (\mathbb{C}\otimes\mathbb{S})_L \cong \mathbb{C}\ell(8)$. Unlike the smaller algebra $\mathbb{C}\ell(6)$ associated with $\mathbb{C}\otimes\mathbb{O}$, this larger algebra admits a triality automorphism associated with Spin(8). Triality is a non-linear outer automorphism of Spin(8) of order three. It would be interesting to interpret this triality physically in the $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ and $\mathbb{C}\otimes\mathbb{S}$ models. Triality has been associated with three generations in the context of the exceptional Jordan algebra [13, 14].

5. Discussion

A considerable amount of the SM structure for a single generation of fermions can be realised starting from $\mathbb{C}\otimes\mathbb{O}$ and its left adjoint algebra $(\mathbb{C}\otimes\mathbb{O})_L \cong C\ell(6)$. Since describing a single generation requires one copy of the octonions, it seems reasonable to expect a three generation model to require three copies of the octonions. Extending from $\mathbb{C}\otimes\mathbb{O}$ to $\mathbb{C}\otimes\mathbb{S}$ one can generalize the results from a single generation to exactly three, with each generation living in a $\mathbb{C}\otimes\mathbb{O}$ subalgebra $\mathbb{C}\otimes\mathbb{S}$. One finds that the three generations are not independent of one another but all three generations share a common $\mathbb{C}\otimes\mathbb{H}$ subalgebra. This intersectionality of three generations may provide a basis for including CKM quark mixing into the model.

The exceptional Jordan algebra $J_3(\mathbb{O})$ also contains three copies of the octonions, making this algebra another natural candidate to describe three generations [13, 14, 15]. There one likewise finds that the three generations are not truly independent of one another, but rather each generation corresponds to one of three canonical $J_2(\mathbb{O})$ subalgebras of $J_3(\mathbb{O})$ [14]. One expects a close relationship between the approach based on $(\mathbb{C}\otimes\mathbb{S})_L$ and three generation models based on $J_3(\mathbb{O})$. Indeed each Fano plane in Figure 1 gives the projective geometry of the octonionic projective plane $\mathbb{O}P^2$ [16], which is the quantum state space upon which the exceptional Jordan algebra $J_3(\mathbb{O})$ acts [17].

The automorphism group of the octonions is the exceptional Lie group G_2. For the sedenions one finds that $\mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{O})\times S_3$. The only difference between the octonion and sedenion automorphism groups is a factor of the permutation group S_3. This permutation group can be constructed from the triality automorphism of $\mathrm{Spin}(8)$ [18]. The fundamental symmetries of $\mathbb{S}$ are the same as those of $\mathbb{O}$, although the factor of S_3 suggests one obtains three copies. For higher Cayley-Dickson ($n>3$) algebras $\mathrm{Aut}(\mathbb{A}_n) = \mathrm{Aut}(\mathbb{O})\times(n-3)S_3$, indicating the higher Cayley-Dickson algebras only add additional trialities, and perhaps no new fundamental physics should be expected beyond $\mathbb{C}\otimes\mathbb{S}$.

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