Two-pion scattering amplitude from Bethe-Salpeter wave function at the interaction boundary

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We observe that the ratio of the on-shell scattering amplitude to the Bethe-Salpeter (BS) wave function outside the interaction range is almost independent of time in our quenched calculation of the $I = 2$ two-pion scattering with almost zero momentum. In order to discuss the time independence, we present a relation between the two-pion scattering amplitude and the surface term of the BS wave function at the boundary. Using the relation under some assumptions, we show that the ratio is independent of time if the two-pion four-point function in early time is dominated by scattering states with almost zero momentum in addition to the ground state of the two-pion scattering.

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1. Introduction

The finite volume method derived by Lüscher [1] is widely employed for calculations of the scattering phase shift $\delta(k)$ in lattice QCD. This method is based on a relation between $\delta(k)$ and the two-particle energy on finite volume of L^3. The relation is originally derived in quantum mechanics [1], and then the same relation is obtained in quantum field theory using the BS wave function [2, 3]. While in the derivation of the relation the BS wave function outside the interaction range R is discussed, the one inside R in the infinite volume is also related to $\delta(k)$ through the on-shell scattering amplitude [3]. The half-off-shell scattering amplitude can be defined in a similar way [4].

We extend the relation of the BS wave function inside R in the infinite volume to the one on finite volume, and perform an exploratory study using the extended relation in the $I = 2$ two-pion scattering with a small relative on-shell momentum k at a heavy pion mass m_π in the quenched QCD [5]. It is found that the results of $\delta(k)$ obtained from the two methods, the finite volume method and the extended relation, completely agree with each other, and the half-off-shell amplitude can be calculated in a wide range of the momentum with reasonable statistical error. Furthermore, we confirm that similar results for the on-shell and half-off-shell amplitudes are obtained in smaller m_π [6].

In the study, we find that a ratio of the time dependent on-shell amplitude on the lattice $H_L(t,k;k)$ to the four-point function $C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ at a reference position $\mathbf{x}_{\text{ref}}$ with $x_{\text{ref}} = |\mathbf{x}_{\text{ref}}| > R$ is independent of time t as shown in Fig. 1. This behavior is interesting, because the numerator and denominator have significant t dependences in small t region. Figure 2 presents the t dependences for the numerator and denominator normalized by the trivial exponential t dependence of the ground state. In this report, we discuss conditions for this t independence through a definition of the scattering amplitude on the lattice under some assumptions. The results in this report have already been presented in our paper [6].
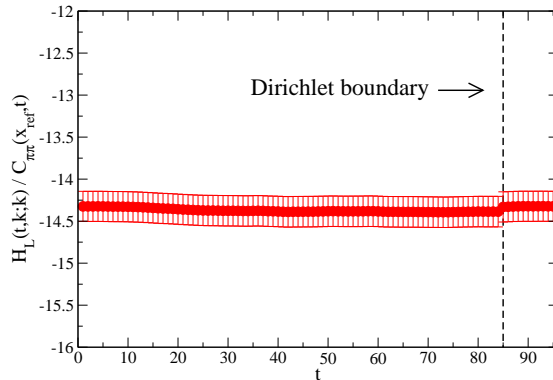

Figure 1: Ratio of time dependent on-shell amplitude on the lattice $H_L(t,k;k)$ to the two-pion four-point function $C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ at a reference point $\mathbf{x}_{\text{ref}}$ ($x_{\text{ref}} > R$) as a function of t. The vertical dashed line represents the time slice of the Dirichlet boundary condition.
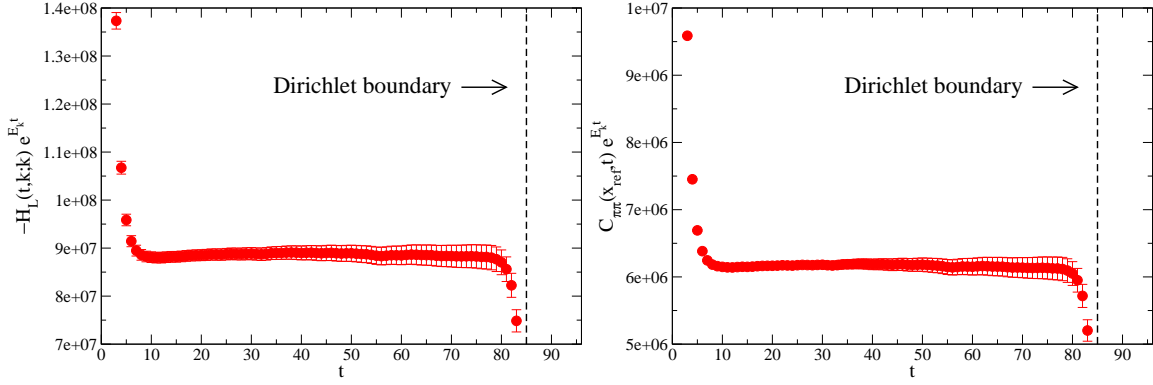
Figure 2: Time dependences for $H_L(t,k;k)$ (Left) and $C_{\pi\pi}(\mathbf{x}_{\rm ref},t)$ (Right) normalized by $e^{E_k t}$ with the energy of the two-pion ground state E_k. The vertical dashed line represents the time slice of the Dirichlet boundary condition.

2. Definitions

In this report the S-wave two-pion scattering in the center of mass frame is considered. We assume that the interaction range is smaller than half of the spatial extent, $R < L/2$, and effects of inelastic scatterings are negligible. The half-off-shell amplitude $H_L(p;k)$ is defined by the BS wave function of the two-pion ground state on the lattice $\phi(\mathbf{x};k)$ [5],

$$H_L(p;k) = -\sum_{\mathbf{x}} j_0(px)(\Delta+k^2)\phi(\mathbf{x};k), \tag{2.1}$$

where $k^2 = (E_k^2 - 4m_\pi^2)/4$ with the energy of two-pion ground state E_k, and Δ is the symmetric Laplacian on the lattice. $j_0(px)$ is the spherical Bessel function. From a ratio of the on-shell amplitude $H_L(k;k)$ to $\phi(\mathbf{x}_{\rm ref};k)$ in $x_{\rm ref} > R$, $\delta(k)$ is obtained through the following relation,

$$\frac{H_L(k;k)}{\phi(\mathbf{x}_{\rm ref};k)} = \frac{4\pi x_{\rm ref}\sin\delta(k)}{\sin(kx_{\rm ref}+\delta(k))}. \tag{2.2}$$

It is assumed that contributions for higher angular momenta of $l \geq 4$ in $\phi(\mathbf{x}_{\rm ref};k)$ are negligible in the equation.

$H_L(p;k)$ can be written by a surface term using the partial integration of Eq. (2.1),

$$H_L(p;k) = -\sum_{\mathbf{x}}[(\Delta+k^2)j_0(px)]\phi(\mathbf{x};k) + \mathrm{surf}(p;k) \tag{2.3}$$

with

$$\mathrm{surf}(p;k) = -3\sum_{x_{1,2}=-L_{\min}}^{L_{\max}} [j_0(pX'(L_{\min})) - j_0(pX'(L_{\max}+1))]\phi(\mathbf{X}'(L_{\max});k), \tag{2.4}$$

where $\mathbf{X}'(a) = (x_1, x_2, a)$, $L_{\max} = L/2$, and $L_{\min} = L/2 - 1$. Using this expression $H_L(k;k)$ is essentially given by the surface term, in other words, $\phi(\mathbf{x};k)$ at the boundary on the lattice, because $(\Delta+k^2)j_0(kx) \approx 0$ in the small k^2 as in our calculation. It is noted that since the summation in Eq. (2.1) can be replaced by the one inside R because of $(\Delta+k^2)\phi(\mathbf{x};k) = 0$ in $x > R$, $H_L(k;k)$ is expressed by $\phi(\mathbf{x};k)$ at the interaction boundary with a different form of $\mathrm{surf}(k;k)$ [6]. A similar relation to Eq. (2.1) is obtained in the infinite volume as discussed in Ref. [7].

The time dependent amplitude on the lattice $H_L(t,k;k)$ is defined by the two-pion four-point function $C_{\pi\pi}(\mathbf{x},t)$ as

$$H_L(t,k;k) = -\sum_{\mathbf{x}} j_0(kx)(\Delta + k^2)C_{\pi\pi}(\mathbf{x},t), \tag{2.5}$$

where

$$C_{\pi\pi}(\mathbf{x},t) = \langle 0|\Phi(\mathbf{x},t)\Omega^{\dagger}(0)|0\rangle. \tag{2.6}$$

$\Omega^{\dagger}(0)$ is the two-pion source operator at $t = 0$, where each pion operator is projected to the zero momentum. $\Phi(\mathbf{x},t)$ is an A_1^+ projected two-pion operator,

$$\Phi(\mathbf{x},t) = \sum_{\mathbf{r}} \pi^+(R_{A_1^+}[\mathbf{x}] + \mathbf{r},t)\pi^+(\mathbf{r},t), \tag{2.7}$$

where $R_{A_1^+}[\mathbf{x}]$ denotes the A_1^+ projection.

Since we discuss the t dependences for $H_L(t,k;k)$ and $C_{\pi\pi}(\mathbf{x},t)$ in small t region, we consider $\pi\pi'$ scattering states as well as the $\pi\pi$ scatterings in $C_{\pi\pi}(\mathbf{x},t)$, where π' is the first radial excitation of π. Then, $C_{\pi\pi}(\mathbf{x},t)$ is written by those states as,

$$C_{\pi\pi}(\mathbf{x},t) = \sum_q A_q(t)\phi(\mathbf{x};q) + \sum_{q'} A'_{q'}(t)\phi(\mathbf{x};q') = A_k(t)\phi(\mathbf{x};k)(1 + \delta C_{\pi\pi}(\mathbf{x},t)), \tag{2.8}$$

where $A_q(t) = C_q e^{-E_q t}$ and $A'_{q'}(t) = C'_{q'} e^{-E'_{q'}t}$ with $E_q = 2\sqrt{m_\pi^2 + q^2}$ and $E'_q = \sqrt{m_\pi^2 + q^2} + \sqrt{m_{\pi'}^2 + q^2}$. C_q and $C'_{q'}$ are overall constants, and $\delta C_{\pi\pi}(\mathbf{x},t)$ is all the excited state contributions divided by the ground state contribution of $\pi\pi$ scattering, $A_k(t)\phi(\mathbf{x};k)$. In the $t \gg 1$ region, where the ground state dominates in $C_{\pi\pi}(\mathbf{x},t)$, the ratio $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ is reduced to the ratio in Eq. (2.2). Note that it is straightforward to include $\pi'\pi'$ scattering states in this discussion [6]. Using a relation of $\text{surf}(k;q)$,

$$-\sum_{\mathbf{x}} j_0(kx)(\Delta + k^2)\phi(\mathbf{x};q) = \text{surf}(k;q), \tag{2.9}$$

which is obtained using integration by parts under the assumption $(\Delta + k^2)j_0(kx) = 0$, $H_L(t,k;k)$ is written in a similar form to Eq. (2.8), $H_L(t,k;k) = A_k(t)H_L(k;k)(1 + \delta H_L(t,k;k))$, where $\delta H_L(t,k;k)$ is given by the sum of the surface terms for the excited states,

$$\delta H_L(t,k;k) = \frac{\sum_{q \neq k} A_q(t)\text{surf}(k;q) + \sum_{q'} A'_{q'}(t)\text{surf}(k;q')}{A_k(t)H_L(k;k)}. \tag{2.10}$$

3. Time independence of $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$

The t dependence of the ratio $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ is explained by excited state contributions in Eqs. (2.8) and (2.10),

$$\frac{H_L(t,k;k)}{C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)} = \frac{H_L(k;k)}{\phi(\mathbf{x}_{\text{ref}};k)}\frac{1 + \delta H_L(t,k;k)}{1 + \delta C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)}. \tag{3.1}$$

As shown in Fig. 1, this ratio behaves as a constant in t. In the following we discuss sufficient conditions for the t independence of the ratio using the surface term $\text{surf}(k;q)$.

In a large t region, the excited state parts decrease exponentially compared to unity and the ground state dominates in both numerator and denominator, so that the flat behavior in Fig. 1 is easy to understand. On the other hand, in a small t region, it requires a nontrivial cancellation of the t dependences in the excited state parts of Eq. (3.1), *i.e*, $\delta H_L(t,k;k) = \delta C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$. In order to make the discussion simple, we assume that the t dependences reasonably coincide in each state, in other words,

$$\frac{\text{surf}(k;q)}{\text{surf}(k;k)} \sim \frac{\phi(\mathbf{x}_{\text{ref}};q)}{\phi(\mathbf{x}_{\text{ref}};k)}, \tag{3.2}$$

for each state with the momentum q, where we use $H_L(k;k) = \text{surf}(k;k)$.

$\phi(\mathbf{x};p)$ in $x > R$ is proportional to the solution of the Helmholtz equation $G(\mathbf{x};p)$ on finite volume [1],

$$G(\mathbf{x};p) = \frac{1}{L^3} \sum_{\mathbf{q} \in \Gamma} \frac{e^{i\mathbf{x} \cdot \mathbf{q}}}{q^2 - p^2}, \tag{3.3}$$

with

$$\Gamma = \{\mathbf{q} | \mathbf{q} = \frac{2\pi}{L}\mathbf{n}, \mathbf{n} \in \mathbf{Z}^3\}. \tag{3.4}$$

Furthermore, $\text{surf}(k;p)$ can be evaluated with $G(\mathbf{x};p)$, because $\text{surf}(k;p)$ is given by $\phi(\mathbf{x};p)$ at the boundary, where $x > R$ is satisfied, as shown in Eq. (2.4). Thus, the condition in Eq. (3.2) is rewritten in terms of $G(\mathbf{x};p)$,

$$\frac{R_{\text{surf}}(k;p)}{R_{\text{surf}}(k;k)} \sim 1, \tag{3.5}$$

where

$$R_{\text{surf}}(k;p) = \frac{\text{surf}_G(k;p)}{G(\mathbf{x}_{\text{ref}};p)}. \tag{3.6}$$

$\text{surf}_G(k;p)$ is the same as the surface term in Eq. (2.4), while the BS wave function is replaced by $G(\mathbf{X}'(L_{\text{max}});p)$.

The ratio $R_{\text{surf}}(k;p)/R_{\text{surf}}(k;k)$ is plotted as a function of p^2 in Fig. 3 with a reference position $\mathbf{x}_{\text{ref}} = (12,7,2)$. The left panel presents that the ratio agrees with unity within 3% in the small momentum region of $k^2 \leq p^2 \leq 10k^2$. This ratio decreases rapidly and significantly differs from unity near the non-zero smallest momentum on finite volume, $p = 2\pi/L$, denoted by the vertical dashed line in the right panel.

From the evaluation of $R_{\text{surf}}(k;p)$ we conclude that the condition in Eq. (3.2) is reasonably satisfied for states with almost zero momentum, whose value is similar to k. In $C_{\pi\pi}(\mathbf{x},t)$ such a state is identified as the lowest $\pi\pi'$ scattering state, which is expected to have almost zero momentum. Therefore, the sufficient condition for the t independence of $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ is that in small t region the lowest $\pi\pi'$ scattering state has a large contribution as well as the ground $\pi\pi$ state, and other excited states are negligible in $H_L(t,k;k)$ and $C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$. The conditions are reasonable, because $C_{\pi\pi}(\mathbf{x},t)$ is calculated using the zero momentum projected π operator, so that scattering states with finite momentum are largely suppressed compared to the almost zero momentum scattering states. Furthermore, since we observe significant effect of the π' state in the effective mass of the single π correlator in $t < 10$, it is expected that the lowest $\pi\pi'$ state also has a large contribution in $C_{\pi\pi}(\mathbf{x},t)$ in the t region.
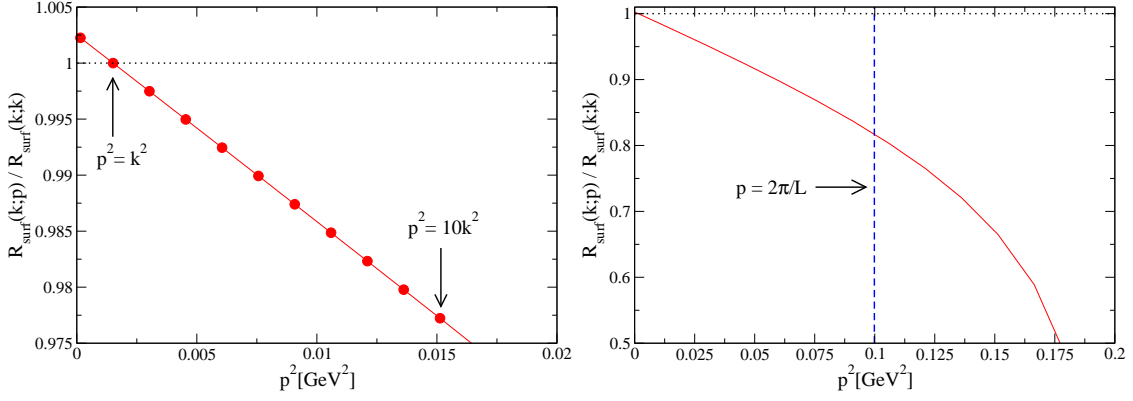
Figure 3: p^2 dependence of the ratio of $R_{\text{surf}}(k;p)/R_{\text{surf}}(k;k)$ with $R_{\text{surf}}(k;p)$ in Eq. (3.6) at a reference position $\mathbf{x}_{\text{ref}} = (12,7,2)$. The left and right panels are the ratio in small and large momentum regions, respectively. The vertical dashed line in the right panel expresses $p = 2\pi/L$.

4. Summary

We have observed an interesting behavior, that the ratio $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ is independent of t, in the calculation of $\delta(k)$ from the BS wave function $\phi(\mathbf{x};k)$ inside R. To discuss sufficient conditions for the t independence of the ratio, we have derived another expression of the half-off-shell amplitude on the lattice, which is written by the surface term, in other words, the summation of $\phi(\mathbf{x};k)$ at the boundary. Not only $\pi\pi$ but also $\pi\pi'$ scattering states in $C_{\pi\pi}(\mathbf{x},t)$ are considered, because we discuss $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ in the small t region. Under an assumption that the t dependences for $\delta H_L(t,k;k)$ and $\delta C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ reasonably agree in each state, we have obtained a condition given by the surface term and the BS wave function. The condition is examined by evaluating these quantities using the solution of the Helmholtz equation on finite volume. From the evaluation, we have found that the condition is satisfied when excited states have almost zero momentum. Thus, the sufficient condition for the t independence of $H_L(t,k;k)/C_{\pi\pi}(\mathbf{x}_{\text{ref}},t)$ is that $C_{\pi\pi}(\mathbf{x},t)$ is dominated by the lowest $\pi\pi$ and $\pi\pi'$ scattering states with almost zero momentum in small t region. In order to confirm this condition analysis of $C_{\pi\pi}(\mathbf{x},t)$ with more sophisticated method is required, such as the variational method [8].

Acknowledgments

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