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Recent developments in AdS/CFT
-Emergent Space from Quantum Information-

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In this talk we would like to review developments of AdS/CFT correspondence, which plays significant roles in the developments of string theory. In particular, we will focus on the recent active subjects on the connection of AdS/CFT and quantum information, which leads to the idea of emergence space from quantum entanglement.

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1. Introduction

The most important goal of string theory is to complete a theory of quantum gravity. Many string theorists expect that a very important clue to this goal is the discovery of the AdS/CFT correspondence [1]. This provides a concrete example of holography [2, 3], which argues that a gravitational theory in $d+2$ dimension is equivalent to non-gravitational theory (such as quantum field theories) in $d+1$ dimension. Even though we now have vastly many evidences of AdS/CFT, we still do not know the basic mechanism how and why the AdS/CFT works so well. This is one of the most important problem in string theory at present. There have been remarkable developments on how the AdS/CFT works and how the spacetime emerges from holography during these few years, which we will review in this talk.

The simplest but non-trivial laboratory for thought experiments of gravitational aspects of string theory is black holes. One of the most important properties of black holes is the fact that they have entropy given by the Bekenstein-Hawking formula [4, 5]

$$S_{BH} = \frac{A(\Sigma_{BH})}{4G_N}. \tag{1.1}$$

This is a surprising formula because its shows that even classical theory can have non-zero entropy in gravity, as opposed to the scalar or fermion field theories. Moreover, it has a mysterious feature that the entropy is proportional not the volume but the area of a black hole, as opposed to usual thermal entropies.

These novel properties lead to the idea of holographic principle [2, 3] as the black hole entropy formula shows that degrees of freedom in gravity actually look like those live in lower dimension. The former fact that there is non-zero entropy in classical gravity has been the strong motivation for recent studies of AdS/CFT and black hole information paradox. The explanation of microscopic origin of black hole entropy for a class of black holes in string theory was first provided by the paper [6]. From modern understandings, we can regard this as an application of AdS_3/CFT_2 correspondence. Nevertheless, honestly we have to say that the origin of black hole entropy is still not well-understood directly from the viewpoint of quantum gravity. At least this suggests us that the structure of Hilbert space for quantum gravity is so different from what we expect for local quantum field theories as is important in the arguments of firewall paradox [7, 8]. The problem of mysterious origin of non-zero entropy in gravity gets sharper in the context of holographic entanglement entropy [9, 10], which can be regarded as a generalization of black hole entropy to generic spacetimes even without black holes.

2. AdS/CFT and Its Developments

The AdS/CFT correspondence [1] argues that a gravitational theory on a $d+2$ dimensional Anti-de Sitter space (AdS_{d+2}) is equivalent to a conformal field theory (CFT) on the $d+1$ dimensional spacetime, which is the boundary of AdS_{d+2}. This correspondence is often expressed as AdS_{d+2}/CFT_{d+1}.

In the Poincare coordinate, the metric of AdS_{d+2} is given by

$$ds^2 = R^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2}, \tag{2.1}$$

where $\mu = 0, 1, \cdots, d$ and R parameterizes the radius of AdS. In this coordinate, the boundary of AdS_{d+2} is $R^{1,d}$ described by the coordinate $(x^0, x^1, \cdots, x^d)$ at $z = 0$. As the metric at $z = 0$ gets divergent, we need to introduce the cut off as $z > \varepsilon$. Here an infinitesimally small constant ε, which is the cut off in the AdS space, is equivalent to the UV cut off (or lattice spacing) ε in CFT. The important fact implies that this new coordinate z correspond to the length scale (or inverse of energy scale) of the dual CFT in the sense of its RG-flow. The basic principle of AdS/CFT is the bulk to boundary relation [11, 12], which argues that the partition function in CFT is equal to that in gravity.

The AdS/CFT has been applied to various strong coupled quantum systems which appear in QCD and condensed matter physics [14, 15, 16]. Such applied AdS/CFT direct was pioneered by the calculation of shear viscosity η, which showed that the ratio of shear viscosity versus entropy density takes a universal value [14]. More recently, another important aspect of holographic CFTs has been uncovered: holographic CFTs are maximally chaotic in the sense of quantum chaos. A useful quantitative measure of chaos is the Lyapunov exponent and it was shown that in holographic CFTs, Lyapunov exponent takes the maximal value in [17].

3. Quantum Entanglement and Holography

To understand the fundamental mechanism of AdS/CFT, it is very important to study how the information in the CFT is encoded in that in the gravity theory. For this let us consider the amount of information included in a certain region A in the CFT and ask what is dual to it in the AdS gravity. The amount of information included in the region A can be measured by the quantity called entanglement entropy A. Thsu we consider an AdS counterpart of the entanglement entropy in a CFT.

3.1 Entanglement Entropy

A state in quantum mechanics is specified by a vector $|\Psi\rangle$ in a Hilbert space $\mathscr{H}$, which evolves in time by the Hamiltonian H. Consider a quantum system which has multiple degrees of freedom so that we can decompose the total system into two subsystems A and B. Then the total Hilbert space $\mathscr{H}$ becomes factorized as $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$.

In quantum mechanics, physical quantities are calculated as expectation values of operators: $O\rangle = \langle\Psi|O|\Psi\rangle = \text{Tr}[\rho \cdot O]$, where we defined the density matrix $\rho = |\Psi\rangle\langle\Psi|$. Such a system is called a pure state when it is described by a unique wave function $|\Psi\rangle$. In more general cases, called mixed states, the system is described by a density matrix ρ, normalized such that $\text{Tr}\rho = 1$.

Next we introduce so called reduced density matrix ρ_A for the subsystem A by tracing out with respect to $\mathscr{H}_B$ as $\rho_A = \text{Tr}_B[\rho]$. Finally, the entanglement entropy is defined as the von-Neumann entropy for the reduced density matrix ρ_A

$$S_A = -\text{Tr}[\rho_A \log \rho_A]. \tag{3.1}$$

We are interested in the entanglement entropy in quantum field theories (QFTs). Since we can view a QFT as an infinite copies of quantum mechanics, its Hilbert space $\mathscr{H}$ is given by all possible field configurations of QFT at a fixed time. Since the choice of A is uniquely defined in

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terms of the boundary ∂A, there are infinitely different definitions of the entanglement entropy S_A depending on the choices of A.

The basic properties of the entanglement entropy are summarized as follows. If the total system is pure, the simple identity $S_A = S_B$ always holds. Also, for any systems, when we decompose the system into four subsystems A, B, C and D so that there are no overlaps between each of them i.e. $\mathscr{H} = \mathscr{H}_{\mathscr{A}} \otimes \mathscr{H}_{\mathscr{B}} \otimes \mathscr{H}_{\mathscr{C}} \otimes \mathscr{H}_{\mathscr{D}}$, the following inequality, called strong subadditivity, is always satisfied:

$$S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_B. \tag{3.2}$$

One more important property is the area law for QFTs. Since QFTs have infinitely many degrees of freedom, the entanglement entropy S_A gets divergent. The area law states that the leading divergent term is proportional to the area of the boundary ∂A:

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{\varepsilon^{d-1}} + O(\varepsilon^{-(d-2)}), \tag{3.3}$$

where γ is a numerical constant; ε is the ultra-violet(UV) cut off or equally the lattice constant. We should note that there is an important exception of area law (3.3): in two dimensional CFTs, the area law is violated in a logarithmic way. When A is an length L interval, S_A is universally given by

$$S_A = \frac{c}{3}\log\frac{l}{a}, \tag{3.4}$$

where c is the central charge of the CFT.

3.2 Holographic Entanglement Entropy

Now we proceed to the holographic calculations of entanglement entropy. As first argued in [9], the entanglement entropy S_A for a holographic CFT is computed from the area of minimal area surface

$$S_A = \text{Min}_{\Sigma_A}\left[\frac{\text{Area}(\Sigma_A)}{4G_N}\right], \tag{3.5}$$

where Σ_A is a codimension two surface which satisfies $\partial\Sigma_A = \partial A$; in addition we require that Σ_A is homologous to A. The minimum in (3.5) should be taken for all surfaces Σ_A which satisfy this homology condition. This formula (3.5) can be applied to any static setups. The minimal area surface is well-defined in the static case because we can equivalently consider an Euclidean AdS space.

For time-dependent backgrounds, we need to use instead the covariant holographic entanglement entropy [10], which is given by replacing Σ_A with the extremal surface in the Lorentzian asymptotic AdS space with the previous condition.

The derivation of holographic entanglement entropy was given in [18, 19], based on the bulk-boundary relation of AdS/CFT [11, 12]. It is also intriguing that we can derive quantum information theoretical inequalities from geometric arguments using the holographic entanglement entropy. such as the proof of strong subadditivity (3.2) [20].

The entanglement entropy enjoys a useful dynamical property called the first law, which is analogous to the first law of thermodynamics. It is the property that when we slightly excite a quantum system, the entanglement entropy is increased such that it is proportional to a quantity

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called modular energy. This was first found in [22] for holographic systems. The energy is called modular Hamiltonian, given by $H_A = -\log \rho_A$ and the first law is written as $\Delta S_A \simeq \Delta H_A$ [22]. Interestingly it have been shown that this first law is equivalent to the linearized part of the Einstein equation [23, 24]. This suggests that the dynamics of quantum entanglement is equivalent to that of gravity. Generalizations of this argument to higher orders have recently been performed [25, 26].

Now let us turn to the original question in AdS/CFT: which part in AdS is dual to which part in CFT ? The holographic entanglement entropy formula implies that the region surrounded by Σ_A and A, which is called entanglement wedge, is dual to the reduced density matrix ρ_A. This expectation has been justified with more rigorous arguments in [27, 28] using a quantity called relative entropy.

3.3 Holograhic Entanglement of Purification

The entanglement wedge provided us a new interesting geometric quantity in AdS/CFT. Consider a entanglement wedge for the subsystem $AB(=A\cup B)$, where A and B are disconnected. If A and B are enough closed to each other, the entanglement wedge is connected and the mutual information $I(A:B) = S_A + S_B - S_{AB}$ gets positive. On the other other hand, if they are not enough close, the entanglement wedge gets disconnected and we have $I(A:B) = 0$. In the first case, there is a quite interesting geometrical object, i.e. the minimal area surface which separates the entanglement wedge into two regions which end on either A or B. We call this surface Σ_{AB}. This motivates us to introduce a new quantity in AdS:

$$E_W = \frac{A(\Sigma_{AB})}{4G_N}, \tag{3.6}$$

which is called the entanglement wedge cross section. Recently, it was conjectured that the entanglement wedge cross section E_W is dual to a quantity called entanglement of purification, which is a useful measure of correlations between two systems [29, 30].

To define the entanglement of purification for any given (reduced) density matrix ρ_{AB}, we introduce an extended Hilbert space $H_A \times H_B \times H_{A'} \times H_{B'}$ to purify ρ_{AB} into a pure stat e $\Psi\rangle_{ABA'B'}$ such that $\rho_{AB} = \mathrm{Tr}_{A'B'}[|\Psi\rangle\langle\Psi|]$. The entanglement of purification E_P for the state ρ_{AB} is defined by minimizing the entanglement entropy $S_{AA'}$ against all possible purifications

$$E_P(\rho_{AB}) = \min_{|\Psi\rangle_{ABA'B'}} [S_{AA'}]. \tag{3.7}$$

The conjecture in [29, 30] argues the simple relation $E_W = E_P$. In [31], this conjecture was confirmed in several examples in two dimensional CFTs.

4. Emergent Spacetime from Quantum Entanglement

Since the amount of quantum entanglement is related to the minimal surface area via the holographic entanglement entropy, this suggests the spacetime in gravity may emerge from quantum entanglement. One concrete possibility to realize this idea is to employ so called tensor networks, which is a network description of a quantum state in a many-body system as conjectured by [32]. One well-known class of tensor networks which well describe CFTs is called the MERA (Multiscale Entanglement Renormalization Ansatz) [33]. Recently, it was pointed out the quantum error

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correcting codes, which is a very useful method to protect the data against noises in actual quantum computations, is closely related to the reconstruction mechanism of the bulk AdS from a CFT [34]. By using this idea, a new tensor network called perfect tensor networks, have been constructed [35], which provides us with a helpful toy model of AdS/CFT.

In tensor networks, the quantum state is described by a network of tensors, where indices are contracted whenever two tensors are connected by a link. This network describes the way how the system is entangled and naturally adds an extra dimension, eventually leads to a time slice of AdS spacetime. Also this qualitatively reproduces the holographic entanglement entropy formula (3.5). However, the presence of time direction is not manifest in such a tensor network description. Nevertheless, we can describe how the network geometry changes under the time evolution, which is indeed similar to the propagation of gravitational waves [36].

A systematical formulation of this AdS/tensor network duality can be summarized into two principles: (i) a codimension two space-like surface corresponds to a quantum state in a holographic CFT (called surface/state correspondence) [37], and (ii) a codimension one (both time-like and space-like slice) corresponds to a quantum circuit defined by a path-integration on the slice with the Lagrangian given by an appropriate coarse-graining of the CFT [38]. The latter property is explained by the fact that the wave functional in CFTs is invariant under the Weyl transformation of the background metric, called path-integral optimization [39]. Indeed, after the path-integral optimization, we obtain the hyperbolic space metric, which coincides with the time sice of AdS.

5. Conclusions

Even though we still do not have complete understandings of the basic mechanism of AdS/CFT, we have made important progresses recently owing to quantum information theoretic viewpoints. To go further, we need to understand the connection between the AdS/CFT and tensor networks in a more concrete way in the continuum limit. This connection to tensor networks might be related to the calculation of computational complexity. This quantity is defined by the minimal number of quantum gates which produce a state we are interested in from a simple disentangled state. Recently an interesting holographic formula which estimates the complexity for CFT states was conjectured in [40]. In tensor network descriptions, we can estimate the complexity by counting the number of tensors. Therefore, we may expect that we may have a direct connection between gravity and tensor networks through studies of complexity. The first step for this approach was taken in the recent formulation of the path-integral optimization [39]. It will be important to extract special features for holographic CFTs which explain the basic mechanism of AdS/CFT. Such a fundamental understanding might allow us to understand possibility of holography also for de-Sitter spaces.

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