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Killing spinors from classical r-matrices

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In this article we summarize the interplay between supersymmetry and integrable deformations of superstring theory backgrounds. We start from a brief review of a recipe for the preserved Killing spinors in TsT deformed backgrounds. Then a few examples in TsT deformations of flat space, $\mathrm{AdS}_5 \times S^5$, and $\mathrm{AdS}_7 \times S^4$ are presented. We also comment on the general Killing spinor formula expressed only in terms of the bi-vector Θ, obtained by the Seiberg-Witten map.

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1. Introduction

Ever since the discovery of integrable structures behind the AdS/CFT correspondence [3], the recent study has focused on various applications of integrability techniques (for a comprehensive review, see [4]). In the context of $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality, the Green-Schwarz superstring in $\mathrm{AdS}_5 \times S^5$ can formulated as a sigma model for the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [5] , whose associated superalgebras admit a $\mathbb{Z}_4$ grading [6]. With this $\mathbb{Z}_4$ structure, the classical integrability on the string theory side is ensured in the sense of the existence of Lax pairs in the sigma model the above coset.

In order to extend and further confirm the AdS/CFT paradigm, one of the significant research directions is to deform the $\mathrm{AdS}_5 \times S^5$ background while preserving integrability, leading to deformed variations of the AdS/CFT correspondence. The so-called Yang-Baxter (YB) deformation [7, 8, 9, 10] is a systematic way of performing integrable deformations [11, 12]. It is simply characterised by classical r-matrices r^{ij} satisfying the classical Yang-Baxter equation (CYBE). Nowadays, once we have encoded a certain r-matrix into the Yang-Baxter deformed sigma model with the supercoset construction [13, 16], we are in a position to systematically generate the corresponding data of the integrable deformed string theory backgrounds. This systematised machinery has led to a large amount of new integrable solutions of superstring theory. Also, various examples of the AdS/CFT duality [14, 15, 50] were subsequently revisited and interpreted as YB deformations.

Integrability is so powerful that it allows exact calculations even at the finite coupling and without relying on supersymmetries. Therefore, the existence of supersymmetry is irrelevant from the viewpoint of integrability. On the other hand, when we aim to realize deformed supersymmetric gauge theories via branes in the integrable deformed string theory backgrounds, such as the Ω-background [17], it is essential to explicitly obtain the Killing spinors. For example, see [18].

Thus, in the spirit of realizing supersymmetric field theories from integrable deformed backgrounds, we focused on the interplay between supersymmetry and YB deformations. In [1], we built a recipe for an explicit form of Killing spinors in a simple class of YB deformed backgrounds, and thereby studied various examples. These backgrounds are labelled by unimodular r-matrices and realized by the so-called TsT transformation. It utilizes the two-torus and consists of two T-dualities and an angular shift [19, 20, 21, 22, 23, 24]. Motivated by the notion of M-theory T-duality [25, 26, 27], we also explicitly computed the Killing spinors in the M-theory equivalence of TsT transformations of the $\mathrm{AdS}_7 \times S^4$ background.

Moreover, analyzing TsT examples on a case-by-case basis, we also empirically deduced a general Killing spinor formula given by

$$\varepsilon_-^{(\mathrm{fin})} = \Pi^{\mathrm{Proj}}\varepsilon_-^{(\mathrm{in})}\,, \qquad \varepsilon_+^{(\mathrm{fin})} = e^{\omega(\Theta)\frac{1}{2}\Theta^{mn}\Gamma_{mn}}\Pi^{\mathrm{Proj}}\varepsilon_+^{(\mathrm{in})}\,, \tag{1.1}$$

where $\varepsilon^{(\mathrm{in/fin})}$ are the Killing spinors in the initial/final backgrounds and $\omega(\Theta)$ is a normalization factor satisfying

$$\tan\left(\omega(\theta)\sqrt{\frac{1}{2}\Theta^{mn}\Theta_{mn}}\right) = \sqrt{\frac{1}{2}\Theta^{mn}\Theta_{mn}}\,. \tag{1.2}$$

This conjectured formula (1.1) has two features. First, it is expressed only in terms of an anti-symmetric bi-vector Θ, which is also called a β-field [36, 37, 38]. It is referred to as non-commutativity on the dual gauge theory side as originally pointed out in [20] and developed in

[41, 42, 43, 44, 45]. The Θ-parameter is obtained by applying the (generalized) Seiberg-Witten map [28, 29]:

$$\begin{aligned} G_{mn} &= (g - Bg^{-1}B)_{mn}\,, \\ \Theta^{mn} &- \left((g+B)^{-1}B(g-B)^{-1}\right)^{mn}\,, \\ G_s &= g_s\left(\frac{\det(g+B)}{\det g}\right)^{1/2}\,, \end{aligned} \tag{1.3}$$

where g_{mn}, B_{mn} and g_s are the closed string metric, B_2-field, and string coupling, respectively. Also, G_{mn} and G_s are the open string metric and coupling.

Second, the formula (1.1) has a certain projection matrix Π^{Proj}, which removes the dependence of T-duality directions from spinors, and then breaks supersymmetries by parts [30]. In fact, the projectors can be derived by demanding vanishing Kosmann Lie derivatives along the Killing vectors in the T-duality directions [31, 32, 33]:

$$\mathscr{L}_K\varepsilon \equiv K^m\nabla_m\varepsilon + \frac{1}{4}(\nabla K)_{mn}\Gamma^{mn}\varepsilon = 0\,, \tag{1.4}$$

where the Killing vectors are denoted by $K = K^m\partial_m$. Note that if the Killing vector is $K = \partial_z$ and the background allows for a $U(1)$ isometry in the z-direction, then the Kosmann Lie derivative along ∂_z simply becomes $\mathscr{L}_{\partial_z}\varepsilon = \partial_z\varepsilon$.

To extend the availability of our conjecture (1.1), we focused on YB deformations beyond TsT transformations in our subsequent work [2]. The examples discussed there are, for example, characterized by classical r-matrices of higher ranks [39] or non-unimodular r-matrices [40]. In particular, the non-unimodular cases lead to deformed backgrounds, which do not satisfy the equations of motion in the conventional supergravity, but the generalized supergravity [34]. For the cases beyond TsT deformations, we tested the general Killing spinor formula (1.1) by directly substituting it into supersymmetry variations in the standard and generalized supergravities [35]. As remarked in conclusions, we corroborated the validity of the Killing spinor formula (1.1) beyond TsT examples.

The rest of the article is organized as follows. In section 2, I review how to construct a Killing spinor in the TsT deformed backgrounds. In section 3, I present a couple of representative examples to show how our construction of Killing spinors can be applied. The examples range over TsT transformations of flat space and $\mathrm{AdS}_5\times S^5$, and $\mathrm{AdS}_7\times S^4$.

2. Recipe for Killing spinors in TsT deformed backgrounds

First, let us review the effect of T-duality on Killing spinors. We consider two configurations in type II superstring theories, which are T-dual to each other. Suppose that the initial background is supersymmetric. To preserve supersymmetry after T-duality along z-direction, the Killing spinors in the doublet notation $\varepsilon = (\varepsilon_+, \varepsilon_-)^{\mathrm{T}}$ have to transform as

$$\tilde{\varepsilon}_+ = -(g_{zz})^{-1/2}\Gamma_z\varepsilon_+\,, \qquad \tilde{\varepsilon}_- = \varepsilon_-\,. \tag{2.1}$$

provided that the Kosmann Lie derivative along ∂_z vanishes:

$$\mathscr{L}_{\partial_z}\varepsilon = \partial_z\varepsilon = 0\,, \tag{2.2}$$

where Γ_z has a curved index and g_{zz} is the (z,z)-component of the metric. The factor $(g_{zz})^{-1/2}\Gamma_z$ is invariant under T-duality along z-direction. The condition (2.2) literally states that the Killing spinor has to be independent of isometry direction [30].

Next, we review the TsT transformation, comprising T-duality, angular shift, and T-duality. Suppose that the type II string theory background is compactified on a two-torus generated by two isometry directions u, v. Then the TsT transformation, denoted by $(u,v)_\lambda$, is given as follows:

$$(u,v)_\lambda \equiv \begin{cases} 1.\ \text{T-duality on } u & : u \to \tilde{u} \\ 2.\ \text{shift } v \text{ by } \lambda\tilde{u} & : v \to v + \lambda\tilde{u} \\ 3.\ \text{T-duality on } \tilde{u} & : \tilde{u} \to u\,, \end{cases} \tag{2.3}$$

where λ is a constant parameter. As remarked earlier, this solution generating technique corresponds to unimodular r-matrices in the context of Yang-Baxter deformations.

Keeping the above rule (2.3) in mind, we can construct the following relation between Killing spinors in the initial and final configurations via $(u,v)_\lambda$:

$$\varepsilon_+^{(\text{fin})} = (g_{uu}^{(\text{fin})} g_{uu}^{(\text{in})})^{-1/2}\Gamma_u^{(\text{fin})}\Gamma_u^{(\text{in})}\Pi^{\text{Proj}}\varepsilon_+\,, \qquad \varepsilon_-^{(\text{fin})} = \Pi^{\text{Proj}}\varepsilon_- \tag{2.4}$$

Note that the projection matrix Π^{Proj} has to be inserted such that the dependence of T-duality and shift directions has to be removed from the Killing spinors. The projector can be obtained just by reading explicitly the initial Killing spinor, or by combining both Kosmann Lie derivative along T-dual/shift directions and supersymmetry variations for fermions.

The TsT transformation can be uplifted to M-theory using a three-torus generated by u, v, and w directions [46, 47]. The M-theory TsT transformation, denoted by $(u,v,w)_\lambda$, consists of TsT transformation and dimensional reduction/oxidation between type IIA and M-theory backgrounds:

$$(u,v,w)_\lambda \equiv \begin{cases} 1.\ \text{reduction on } w \\ 2.\ \text{TsT } (u,v)_\lambda \\ 3.\ \text{oxidation with } w\,. \end{cases} \tag{2.5}$$

Recall a relation between the Killing spinors in M-theory and type IIA ([48], for example):

$$\varepsilon_{\text{M}} = e^{-\Phi_{\text{IIA}}/6}\varepsilon_{\text{IIA}} \tag{2.6}$$

where Φ is a dilaton. The combination of (2.4) with (2.6) leads to the transformation rule for Killing spinors under $(u,v,w)_\lambda$:

$$\varepsilon_{\text{M}}^{(\text{fin})} = e^{\left(\Phi_{\text{IIA}}^{(\text{in})} - \Phi_{\text{IIA}}^{(\text{fin})}\right)/6}\left[\Pi_- + (g_{uu,\text{IIA}}^{(\text{fin})} g_{uu,\text{IIA}}^{(\text{in})})^{-1/2}\Gamma_{u,\text{IIA}}^{(\text{fin})}\Gamma_{u,\text{IIA}}^{(\text{in})}\Pi_+\right]\varepsilon_{\text{M}}^{(\text{in})}\,, \tag{2.7}$$

where $\Pi_\pm$ are projectors for the chirality defined in type IIA.

3. Applications

Let us apply the recipe (2.4) and (2.7) to concrete examples. These representative examples range over TsT deformations of flat space, $\text{AdS}_5 \times S^5$, and $\text{AdS}_7 \times S^4$.

3.1 Ω-deformation

We start from a ten-dimensional flat spacetime, where two 2-planes are expressed by two sets of polar coordinates (ρ_1, ϕ_1) and (ρ_2, ϕ_2):

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \sum_{i=1}^{2} \left(d\rho_i^2 + \rho_i^2 d\phi_i^2\right) + \sum_{k=6}^{9} (dx^k)^2 . \tag{3.1}$$

The TsT transformation $(x^1, \frac{\phi_1+\phi_2}{2})_\lambda$ leads to the following supersymmetric configuration:

$$\begin{aligned} ds^2 &= -(dx^0)^2 + \Delta^{-2} \left[(dx^1)^2 + \sum_{i=1}^{2} \rho_i^2 d\phi_i^2 + \lambda^2 \rho_1^2 \rho_2^2 (d\phi_1 - d\phi_2)^2 \right] + \sum_{i=1}^{2} d\rho_i^2 + \sum_{k=6}^{9} (dx^k)^2 , \\ e^{-2\Phi} &= \Delta^2 , \\ B_2 &= \lambda \Delta^{-2} dx^1 \wedge \left(\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2\right) , \\ \Delta &= 1 + \lambda^2 \left(\rho_1^2 + \rho_2^2\right) . \end{aligned} \tag{3.2}$$

This background was studied in [17] for realizing the string theory realization of Ω-deformation [49] and studying deformed supersymmetric gauge theories in various dimensions via probe branes. Following the recipe (2.4), one finds the explicit form of the Killing spinor as follows:

$$\varepsilon_+^{(\mathrm{fin})} = \Delta^{-1}(1 - \lambda(\rho_1 \Gamma_{\underline{x^1 \phi_1}} + \rho_2 \Gamma_{\underline{x^1 \phi_2}}))\varepsilon_+ , \qquad \varepsilon_-^{(\mathrm{fin})} = \varepsilon_- . \tag{3.3}$$

where

$$\varepsilon_+ + i\varepsilon_- = e^{\frac{\phi_1-\phi_2}{2}\Gamma_{\underline{\rho_1 \phi_1}}} \Pi^{\mathrm{Proj}} \varepsilon_0 \qquad \text{with} \quad \varepsilon_0 : \text{constant} . \tag{3.4}$$

In the above formula, we inserted one projector to remove the $\frac{\phi_+ + \phi_-}{2}$-dependence, since the $U(1)$ isometry acts on x^1 freely and so the initial Killing spinor is independent of x^1:

$$\Pi^{\mathrm{Proj}} = \frac{1}{2}(1 + \Gamma_{\underline{\rho_1 \phi_1 \rho_2 \phi_2}}) , \tag{3.5}$$

where the Gamma matrices have flat indices. As a result, the supersymmetry is reduced by half due to one projector.

Applying the Seiberg-Witten map to the background (3.2), we obtain the bi-vector

$$\Theta = -\lambda \partial_{x^1} \wedge (\partial_{\phi_1} + \partial_{\phi_2}) . \tag{3.6}$$

It is remarkable to note that we can reconstruct the Killing spinor using the conjectured formula (1.1) as well as the concrete bi-vector Θ (3.6).

3.2 Lunin-Maldacena

The next example is obtained by TsT transformation of $\mathrm{AdS}_5 \times S^5$ [50]. The deformation acts only on S^5 . The metric of undeformed $\mathrm{AdS}_5 \times S^5$ background is written as

$$\begin{aligned} ds^2 &= ds^2_{\mathrm{AdS}_5} + \sum_{i=1}^{3} \left(d\rho_i^2 + \rho_i^2 d\phi_i^2\right) \\ &= ds^2_{\mathrm{AdS}_5} + d\alpha^2 + \sin^2\alpha\, d\theta^2 + \cos^2\alpha\, (d\psi - d\varphi_2)^2 \\ &\qquad + \sin^2\alpha \cos^2\theta\, (d\psi + d\varphi_1 + d\varphi_2)^2 + \sin^2\alpha \sin^2\theta\, (d\psi - d\varphi_1)^2 , \end{aligned} \tag{3.7}$$

where we performed the following identifications of angular variables:

$$\phi_1 = \psi - \varphi_1\,, \qquad \phi_2 = \psi + \varphi_1 + \varphi_2\,, \qquad \phi_3 = \psi - \varphi_2\,. \tag{3.8}$$

Moreover, the ρ_i's, $i = 1,2,3$, are rewritten as

$$\rho_1 = \sin\alpha\,\cos\theta\,, \qquad \rho_2 = \sin\alpha\,\sin\theta\,, \qquad \rho_3 = \cos\alpha\,. \tag{3.9}$$

One of the possible supersymmetric TsT transformations is $(\varphi_1,\varphi_2)_\lambda$. This leads to the following deformed background:

$$\begin{aligned}
ds^2 &= dr^2 + e^{2r}\eta_{\mu\nu}dx^\mu dx^\nu + d\alpha^2 + \sin^2\alpha\,d\theta^2 \\
&\qquad + \Delta^{-2}\left(\sum_{i=1}^{3}\rho_i^2 d\phi_i^2 + \lambda^2\rho_1^2\rho_2^2\rho_3^2(d\phi_1 + d\phi_2 + d\phi_3)^2\right), \\
B_2 &= -\lambda\Delta^{-2}(\rho_1^2\rho_2^2 d\phi_1\wedge d\phi_2 + \rho_2^2\rho_3^2 d\phi_2\wedge d\phi_3 + \rho_3^2\rho_1^2 d\phi_3\wedge d\phi_1)\,, \\
e^{-2\Phi} &= \Delta^2 \\
C_2 &= \lambda\sin\theta\,\cos\theta\,\sin^4\alpha\,d\theta\wedge(d\phi_1 + d\phi_2 + d\phi_3)\,, \\
C_4 &= e^{4r}dx^0\wedge dx^1\wedge dx^2\wedge dx^3 + \sin\theta\,\cos\theta\,\sin^4\alpha\,d\theta\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3\,. \\
\Delta^2 &= 1 + \lambda^2(\rho_1^2\rho_2^2 + \rho_2^2\rho_3^2 + \rho_3^2\rho_1^2)\,,
\end{aligned} \tag{3.10}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)\,,\mu,\nu = 0,1,2,3$. In this case, we need two projectors to get rid of φ_1- and φ_2-dependences from the Killing spinors. Indeed, the product of these projectors is given by

$$\Pi^{\mathrm{Proj}} = \frac{1}{2}(1 - \Gamma_{\underline{\theta_1\phi_2\theta_2\phi_1}})\cdot\frac{1}{2}(1 + \gamma\Gamma_{\underline{\phi_3\theta_1\phi_2}})\,, \tag{3.11}$$

where γ is a ten-dimensional chirality matrix $\gamma = \Gamma_{\underline{0123r\theta_1\theta_2\phi_1\phi_2\phi_3}}$. Therefore, one preserves the $\frac{1}{4}$-supersymmetry after the deformation. The resulting Killing spinors are of the form

$$\varepsilon_+^{(\mathrm{fin})} = \Delta^{-1}\left[1 + \lambda(\rho_1\rho_2\Gamma_{\underline{\phi_1\phi_2}} + \rho_2\rho_3\Gamma_{\underline{\phi_2\phi_3}} + \rho_3\rho_1\Gamma_{\underline{\phi_3\phi_1}})\right]\Pi^{\mathrm{Proj}}\varepsilon_+^{(\mathrm{in})}\,, \qquad \varepsilon_-^{(\mathrm{fin})} = \Pi^{\mathrm{Proj}}\varepsilon_-^{(\mathrm{in})}\,, \tag{3.12}$$

where the undeformed Killing spinor is given by

$$\begin{aligned}
\varepsilon_+^{(\mathrm{in})} + i\varepsilon_-^{(\mathrm{in})} &= e^{\frac{i}{2}r\gamma\Gamma_{\underline{r}}}\left[1 + \frac{x^\mu}{2}\left(i\gamma\Gamma_{\underline{\mu}} + \Gamma_{\underline{r}}\Gamma_{\underline{\mu}}\right)\right] \\
&\qquad\times e^{-\theta_2\gamma\Gamma_{\underline{\theta_2}}/2}e^{-\phi_3\gamma\Gamma_{\underline{\phi_3}}/2}e^{\theta_1\Gamma_{\underline{\theta_2\theta_1}}/2}e^{\phi_1\Gamma_{\underline{\theta_2\phi_1}}/2}e^{\phi_2\Gamma_{\underline{\theta_1\phi_2}}/2}\varepsilon_0\,,
\end{aligned} \tag{3.13}$$

where ε_0 is a constant spinor.

As in the previous example, (3.12) can be rewritten using the general formula (1.1) and the following bi-vector:

$$\Theta = -\lambda\partial_{\varphi_1}\wedge\partial_{\varphi_2} = \lambda(\partial_{\phi_1}\wedge\partial_{\phi_2} + \partial_{\phi_2}\wedge\partial_{\phi_3} + \partial_{\phi_3}\wedge\partial_{\phi_1})\,. \tag{3.14}$$

3.3 M-theory TsT

Finally, we present an example of M-theory TsT transformations on $\mathrm{AdS}_7\times S^4$. We act on the deformation only on AdS_7 by introducing an extra boundary condition. The starting background is given by

$$\begin{aligned} ds^2 &= dr^2 + e^r\left(-(dx^0)^2+(dx^1)^2+\sum_{i=1}^{2}(d\rho_i^2+\rho_i^2 d\varphi_i^2)\right) \\ &\qquad + d\theta_2^2 + \sin^2\theta_2\,(d\theta_1^2+\cos^2\theta_1\,d\phi_1^2+\sin^2\theta_1\,d\phi_2^2)\,. \\ C_3 &= -\frac{3}{4}\cos 2\theta_1\,\sin^3\theta_2\,d\theta_2\wedge d\phi_1\wedge d\phi_2\,. \end{aligned} \tag{3.15}$$

Then we assume that x^1 is periodic. Taking the three-torus generated by x^1, $\phi_+=\frac{\phi_1+\phi_2}{2}$, and $\varphi_+=\frac{\varphi_1+\varphi_2}{2}$ directions, we perform the M-theory TsT transformation $(\phi_+,x^1,\varphi_+)_\lambda$ to obtain

$$\begin{aligned} ds^2 &= \Delta^{2/3}\left[dr^2+e^r\left(-(dx^0)^2+(dx^1)^2+\sum_{i=1}^{2}(d\rho_i^2+\rho_i^2 d\varphi_i^2+\frac{(\rho_i^2 d\varphi_i)^2}{\rho_1^2+\rho_2^2})\right)+\sin^2 2\theta_1\,\sin^2\theta_2\,d\phi_-^2\right], \\ C_3 &= -\frac{3}{4}\cos 2\theta_1\,\sin^3\theta_2\,d\theta_2\wedge d\phi_1\wedge d\phi_2 \\ &\qquad +\frac{1-\Delta^2}{\lambda\Delta^2}dx^1\wedge\left(\cos 2\theta_1\,d\phi_-\wedge d\varphi_+ + 2\frac{\rho_1^2-\rho_2^2}{\rho_1^2+\rho_2^2}(d\phi_+ + 3\cos 2\theta_1\,d\phi_-)\wedge d\varphi_-\right), \\ \Delta^2 &= 1+\lambda^2 e^{2r}\sin^2\theta_2\,(\rho_1^2+\rho_2^2)\,. \end{aligned} \tag{3.16}$$

In this example, none of $U(1)$ isometries act freely on any direction of the three-torus. Therefore, we need to insert three projection matrices into Killing spinors. The whole projector is given by

$$\Pi^{\mathrm{Proj}} = \frac{1}{2}(1+\Gamma_{\underline{\theta_2\phi_1\theta_1\phi_2}})\cdot\frac{1}{2}(1+\gamma\Gamma_{\underline{r}})\cdot\frac{1}{2}(1+\Gamma_{\underline{\rho_1\varphi_1\rho_2\varphi_2}})\,, \tag{3.17}$$

with which we preserve the $\frac{1}{8}$-supersymmetry under $(\phi_+,x^1,\varphi_+)_\lambda$. Following the recipe (2.7), we can explicitly write down the following Killing spinor:

$$\varepsilon_{\mathrm{M}}^{(\mathrm{fin})} = \Delta^{1/6}\left[\Pi_- + \Delta^{-1}\left(1-\lambda e^r\sin\theta_2\,(\rho_1^2+\rho_2^2)^{1/2}(\cos\theta_1\,\Gamma_{\underline{\phi_1 x^1}}+\sin\theta_1\,\Gamma_{\underline{\phi_2 x^1}})\right)\Pi_+\right]\varepsilon_{\mathrm{M}} \tag{3.18}$$

with

$$\varepsilon_{\mathrm{M}} = e^{\frac{r}{4}\gamma\Gamma_{\underline{r}}}e^{\varphi_-\Gamma_{\underline{\rho_1\varphi_1}}}e^{\frac{\theta_2}{2}\gamma\Gamma_{\underline{\theta_2}}}e^{-\frac{\theta_1}{2}\Gamma_{\underline{\theta_1\theta_2}}}e^{\phi_-\Gamma_{\underline{\theta_2\phi_1}}}\Pi^{\mathrm{Proj}}\varepsilon_0\,, \tag{3.19}$$

where $\gamma=\Gamma_{\underline{\theta_1\theta_2\phi_1\phi_2}}$ and ε_0 is a constant spinor.

4. Comments and conclusions

In this article, we reviewed our recent work [1] on the interplay between supersymmetry and integrable deformations of superstring theory backgrounds. Motivated by our interests in constructing deformed supersymmetric gauge theories realized from the integrable deformed backgrounds, we focused on the amount of preserved supersymmetries, or Killing spinors. The deformations we discussed can be characterized by unimodular classical r-matrices, which satisfy the homogenous CYBE. They can be realized as TsT transformations on the two-torus in the string theory backgrounds.

Our main result is the explicit formula of Killing spinors in the TsT deformed backgrounds. Following the T-duality rule for Killing spinors, we kept track of Killing spinors set by step in the process of TsT deformations. As such we constructed a concise recipe for Killing spinors under TsT transformations. Here we reviewed a couple of representative examples ranging over the TsT deformation of flat space, $\mathrm{AdS}_5 \times S^5$, and $\mathrm{AdS}_7 \times S^4$. Remarkably, we found that each result can be reconstructed in some general formula (1.1). This general Killing spinor formula is expressed only in terms of the anti-symmetric bi-vector Θ, obtained by formally applying the Seiberg-Witten map. Moreover, it has a projector matrix to remove the dependence of isometry directions. This projector can be derived by Kosmann Lie derivatives along T-duality directions in a frame independent way.

Let us further comment on the general Killing spinor formula (1.1). This formula was empirically found through various concrete examples of TsT deformed backgrounds. To confirm its applicability beyond TsT transformations, we further investigated more complicated Yang-Baxter deformations, characterized by classical r-matrices listed in [39, 40]. In particular, we applied our general formula for checking the preservation of supersymmetries in the non-abelian unimodular rank-four cases in [39]. We evaluated supersymmetry variations for fermions using our genera formula combined with concrete Θ-parameters. Consequently, we corroborated that our conjectured formula applied even to non-TsT examples.

Finally, we mention a few directions of further investigations. We hope to address them soon.

- The first example in this arcitle, Ω-deformation, was interpreted as a TsT transformation. Therefore it turned out to be integrable. It would be interesting to construct the gravity dual of the Ω-deformed gauge theory. It will be interesting to relate our construction to [53] .
- As for M-theory TsT transformation, it would be interesting to pursue a Killing spinor formula written only in terms of a tri-vector. This might be related to the notion of non-commutativity in M-theory. It might be instructive to revisit [47] using the so-called generalized Θ-parameter [51].
- In [2], we derived an algebraic equation to extract a projector matrix Π^{Proj} using only the Θ-parameter:

$$[\Theta^{mn}\Gamma_m \mathscr{S} \Gamma_n + \nabla_m \Theta^{np} \Gamma^m{}_{np} - 4\nabla_m \Theta^{mn}\Gamma_n]\,\Pi^{\mathrm{Proj}} = 0\,, \tag{4.1}$$

  where $\mathscr{S}$ are contributions from Ramond-Ramond fluxes. For the derivation we assumed that Kosmann Lie derivatives along all the Killing vectors contained in the bi-Killing structure of the bi-vector Θ vanish in the undeformed background. We believe that this equation can be more formally derived by analyzing the supersymmetry variations in the deformed background using the Θ-expansion as performed in [52]. Or the supersymmetry variation in the β-supergravity [54] would be useful.

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PoS(CORFU2018)118

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PoS(CORFU2018)118