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Pseudo-Riemannian structure of the noncommutative Standard Model

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In this overview we present a proposition for a definition of finite real pseudo-Riemannian spectral triples and show how the existence of such a structure for the spectral triple used for a description of the Standard Model of Particle Physics can predict the lack of leptoquarks, that is our main result from [6]. We discuss few examples of such structures, present main ideas and show general tendencies.

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1. Introduction

The approach to the Standard Model of Particle Physics based on the classical field-theoretical (both perturbative and non-perturbative) description is very efficient and allows for series of detailed computations. Nevertheless, there is a more elegant way to present this model, based on the proposal made by A. Connes [7],[8] that it is possible to describe all features of this theory in the geometrical language. For this reason one has to generalize the notion of geometry and use the so-called noncommutative spaces which are described in terms of the spectral geometry, especially using spectral triples. In [6] we have proposed an extension of this description for finite geometries in case of the incorporation of pseudo-Riemannian structure. We have shown that for a suitable choice of it we can restrict possible classes of models, especially there exists such a pseudo-Riemannian structure on the spectral triple originally introduced by A. Connes and A.H. Chamseddine (see e.g. [21]) that provide additional symmetry in our system which we interpreted as a source of the lack of the mixing between leptons and quarks. Moreover, we have classified all possible such structures on that triple and shown that the only physically interesting one is the aforementioned one. Furthermore, it allows for the extension of the Standard Model by adding sterile neutrinos.

2. Noncommutative Geometry for Particle Physics

We start this section with a brief description of the Connes' reconstruction theorem [9]. One can summarize it as follows. Let (M,g) be a closed, orientable Riemannian spin^c manifold. Then the metric and spin structure of (M,g) can be encoded in a system $(C^\infty(M), L^2(M), D_M)$ enlarged by two other elements: grading γ^5 in the associated Cliford algebra and the charge conjugation operator. Here $C^\infty(M)$ is the $*$-algebra of smooth complex-valued functions on M, $L^2(H)$ is a Hilbert space of square-integrable spinors and D_M is the Dirac operator acting on sections of the spinor bundle.

Therefore, there is sugestion to replace the standard notion of geometry by considering such systems that additionally satisfies few compatibility conditions. Moreover, the algebra $C^\infty(M)$ can be replaced by an arbitrary, even noncommutative, algebra and hence one can consider *noncommutative geometries*. To clarify this concept let us present the formal definition of a spectral triple. We say that a system (A,H,D,γ,J) is a spectral triple if A is a $*$-algebra represented, through a faithfull representation π, on the Hilbert space H, D is an essentially self-adjoint operator on H. It is assumed that D has a compact resolvent and satisfies few compatibility conditions (see e.g.[14]). Furthermore, it is assumed that we have a $\mathbb{Z}/2\mathbb{Z}$-grading $\gamma = \gamma^*$ that commutes with the representation of A, and there is an antilinear isometry J for which it is assumed that the 0th order condition is fulfilled, i.e. for every $a,b \in A$ we require $[J\pi(a^*)J^{-1}, \pi(b)] = 0$. It makes H the $A-A$-bimodule. Moreover, there is a series of conditions between these elements. First of all, the first-order condition is required to be satisfied: for every $a,b \in A$ $[[D,\pi(a)], J\pi(b^*)J^{-1}] = 0$. The behaviour of the system is mostly determined by the so-called KO-dimension that is a number from 0 to 8 defined by the choice of signs $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ that appear in the following compatibility conditions

$$DJ = \varepsilon JD, \qquad J^2 = \varepsilon' \mathrm{id}, \qquad J\gamma = \varepsilon'' \gamma J. \tag{2.1}$$

These objects have an application in Particle Physics when one consider the so-called almost-commutative spectral triples, i.e. a system with an algebra being of the form $C^\infty(M) \otimes A_F$, where

A_F is some finite-dimensional matrix algebra, which choice is related to the gauge group of a given physical model. The rest of ingredients of the product triple are also constructed from the usual spectral triple for M and a finite triple with A_F as an algebra. For a detailed discussion see e.g. [14], [4], [21] and [7], where one can also find the description of a method to obtain an effective action for these models using spectral geometry. We will not concentrate on these aspects here, but rather analyse the structure of the finite triple. We use here (and also in [6]) the standard description (in conventions from [21]) of the finite spectral triple for the Standard Model of Particle Physics (for the detailed discussion see e.g. [21] and [14]). As an algebra we take

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{2.2}$$

where $\mathbb{H}$ states for quaternions.

The Hilbert space for one particle generation is of the form

$$H_F = (H_l \oplus H_q) \oplus (H_{\bar{l}} \oplus H_{\bar{q}}). \tag{2.3}$$

Here H_l denotes the leptonic space with the basis $\{\nu_R, e_r, (\nu_L, e_L)\}$ and H_q is the quark space with a basis $\{u_R, d_R, (u_L, d_L)\}$ (in each quark color). $H_{\bar{l}}$ and $H_{\bar{q}}$ are antiparticle sectors. Notice that $\dim H_F = 96$.

The representation of A_F on H_F is given on each sector separately. We have

- on H_l and H_q (for each color) : $\pi(\lambda, h, m) = \lambda \oplus \overline{\lambda} \oplus m$,
- on $H_{\bar{l}}$: $\pi(\lambda, h, m) = \lambda$,
- on $H_{\bar{q}}$: $\pi(\lambda, h, m) = 1_4 \otimes m$.

For N generations o particles we enlarge the space by tensoring by $\mathbb{C}^N$.

The Dirac operator has a form

$$D_F = \begin{bmatrix} S & T^\dagger \\ T & \overline{S} \end{bmatrix}, \tag{2.4}$$

where S is expressed in terms of Yukawa mass matrices Y_ν, Y_e, Y_u and Y_d, and the operator T is given by $T\nu_R = Y_R \overline{\nu}_R$ with $Y_R \in M_N(\mathbb{C})$, and is zero on other fermions.

The graiding γ_F is acting as 1 on right-handed and -1 on left-handed particles. The real structure J_F acts by exchanging particles with antiparticles composed with the complex conjugation. It is a non-orientable (due to the existence of right neutrinos [18][6]) finite spectral triple of KO-dimension 6.

This spectral triple is the one usually considered in the case of the Standard Model, but notice that the choice of D_F in not unique. In principle there are other possibilities and the resulting system will still fulfilled required conditions for being a spectral triple. For example it is known (see [10],[15]) that there is a possibility of having leptoquark fields in this theory. Since leptoquarks are not observed in the Nature there were several attempts for introducing additional constraints that will allow for the elimination of these Dirac operators. There was an approach based on the K-theorethic arguments ([8],[13]). Later on, the so-called second order condition [10] and the Hodge duality condition [11] were introduced, but none of them was fully satisfactory and in general do not exclude all terms that can be the origin of leptoquark fields. In [6] we poposed an alternative explanation of the lack of leptoquarks based on the analysis of the pseudo-Riemannian structure on the spectral triple. We will briefly describe this approach in the forthcoming sections.

3. Pseudo-Riemannian structures

The standard notion of the spectral triple is dedicated to the description of Riemannian geometries. Nowadays, there is no analogue for the Connes' reconstruction theorem in case of the pseudo-Riemmanian geometry. It turns out that there is no canonical Hilbert space structure on the space $\Gamma_c(S)$ of compactly supported sections of the spinor bundle [2],[17]. The analytical conditions for this class of triple are the most requiring one. There are several different approaches to the incorporation of pseudo-Riemannian structures into the noncommutative geometry, e.g. one can find some of them in two ground-breaking articles: [1] and [19]. A lot of interesting results one can find in [20] and also in the PhD thesis written by an author of that article. For a reader interested in this topic we recommend the overview [12] and references therein. Notice that since we are interested in finite spectral triples we do not need to concentrate on these analytical aspects and consider only algebraic relations between some operators. We also analysed in [3] some algebraic aspects of non-finite triples. In [6] we proposed alternative definition of the finite pseudo-Riemannian spectral triples introduced as a generalization of that one presented in [16] in the following way.

The real pseudo-Riemannian spectral triple of signature (p,q) is a system $(A,\pi,H,D,J,\gamma,\beta)$ consisting of

(1) an involutive algebra A,

(2) an Hilbert space H,

(3) a faithful representation π of A on H,

(4) (possibly unbounded) densely defined operator D, called a Dirac operator,

(5) (for even $p+q$) a $\mathbb{Z}/2\mathbb{Z}$-grading γ that is selfadjoint and $\gamma^2 = 1$,

(6) an antilinear isometry J,

(7) a $\mathbb{Z}/2\mathbb{Z}$-grading β that is selfadjoint and $\beta^2 = 1$

such that the following conditions hold:

(A) γ commutes with the representation of A,

(B) for all $a,b \in A$ we have $[J\pi(a^*)J^{-1},\pi(b)] = 0$ (zeroth order condition),

(C) β commutes with the representation of A,

(D) D is β-selfadjoint, i.e. $D^\dagger = (-1)^p \beta D \beta$,

(E) for every $a \in A$ $[D,\pi(a)]$ is bounded,

(F) $D\gamma = -\gamma D$,

$p-q$ mod 8	0	1	2	3	4	5	6	7
ε	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ε'	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ε''	$+$		$-$		$+$		$-$	

Table 1: *KO*-dimensions for a pseudo-Riemannian spectral triple.

(G) $DJ = \varepsilon JD$, $J^2 = \varepsilon' \mathrm{id}$, $J\gamma = \varepsilon'' \gamma J$, where $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$ define *KO*-dimension $p-q \pmod 8$ according to the table below

(H) $\beta\gamma = (-1)^p \gamma\beta,\ \beta J = (-1)^{\frac{p(p-1)}{2}} \varepsilon^p J\beta,$

(I) for all $a, b \in A \quad \left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0$ (first order condition),

(J) $\langle D \rangle = \sqrt{\frac{1}{2}(DD^\dagger + D^\dagger D)}$ has a compact resolvent,

(K) for all $a \in A \quad [\langle D \rangle, [D, \pi(a)]]$ is bounded.

We also introduced the notion of the orientability and time-orientability using Hochschild homology. The real finite pseudo-Riemannian spectral triple is said to be orientable if there exists a Hochschild cycle of dimension $n = p+q$ (valued in $A^\circ \otimes A$) $c = \left(a^i, a^i_0, a^i_1, \ldots, a^i_n\right)_{i=1,\ldots,k}$ such that

$$\sum_{i=1}^{k} \left(J\pi(a^i)J^{-1}\right) \pi(a^i_0)[D, \pi(a^i_1)]\ldots[D, \pi(a^i_n)] = \begin{cases} \gamma, & n\text{-even} \\ 1, & n\text{-odd} \end{cases} \tag{3.1}$$

The time-orientation is defined in an analogous way. We say that an operator β is a time-orientation if it is a p-form valued in the opposite algebra, that is there exists a collection of elements from the algebra $(b^i, b^i_0, b^i_1, \ldots, b^i_p),\ i = 1, \ldots, k$ such that

$$\beta = \sum_{i=1}^{k} \left(J\pi(b^i)J^{-1}\right) \pi(b^i_0)[D, \pi(b^i_1)]\ldots[D, \pi(b^i_p)]. \tag{3.2}$$

Having a pseudo-Riemannian spectral triple $T_{pR} = \{A, \pi, H, D, J, \gamma, \beta\}$ of signature (p, q) we can introduce two operators

$$D_+ = \frac{1}{2}(D + D^\dagger), \qquad D_- = \frac{i}{2}(D - D^\dagger) \tag{3.3}$$

and get two Riemannian spectral triples, with D_{pm} as Dirac operators, that have an additional grading β satisfying

1. $\beta^2 = 1,\ \beta^\dagger = \beta,$
2. $\beta D_\pm = \pm(-1)^p D_\pm \beta,$
3. $\beta\gamma = (-1)^p \gamma\beta,$
4. $\beta J = (-1)^{\frac{p(p-1)}{2}} \varepsilon^p J\beta.$

In [6] we have shown that constructing the operator $D_E = D_+ + D_-$ and changing the real structure into $J_E = J\beta$ or $J_E = J\beta\gamma$ (depending on the value of p) results in the Riemannian spectral triple $T_R = (A, H, D_E, J_E, \gamma)$ with the *KO*-dimension dependent on the value $p \pmod 4$. This construction is an analogue of the Wick rotation, since the resulting signature is always $(0, -(p+q)) \pmod 8$.

We will now briefly present two examples. The first one will illustrate the construction of Riemannian triple and was considered in [3] and also depicted in our latest overview [5]. The second one will show how the existence of β reduces classes of possible Dirac operators. It is the motivation to use this structure for the spectral triple of the Standard Model to eliminate leptoquarks.

We start with the Lorentzian spectral triple of the noncommutative torus[16]. Notice that it is not a finite triple, but nevertheless we can discuss its algebraic relations. Let $\{|n,m,\pm\rangle\}_{n,m\in\mathbb{Z}}$ be the orthonormal basis of the Hilbert space $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2$ and take $\lambda = e^{2\pi i\theta} \in \mathbb{C}$. We define $A(\mathbb{T}^2_\theta)$ as the algebra generated by operators U, V of the form

$$U\,|n,m,\pm\rangle = |n+1,m,\pm\rangle\,, \qquad V\,|n,m,\pm\rangle = \lambda^{-n}\,|n,m+1,\pm\rangle\,. \tag{3.4}$$

Then there exists a time-orientable pseudo-Riemannian spectral triple of signature $(1,1)$ with $A(\mathbb{T}^2_\theta)$ as an algebra and $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2$ as an Hilbert space. The Dirac operator is given by

$$D\,|n,m,\pm\rangle = (n\pm m)\,|n,m,\mp\rangle\,, \tag{3.5}$$

and the time-orientation is specified by

$$\beta\,|n,m,\pm\rangle = \pm i\,|n,m,\mp\rangle\,. \tag{3.6}$$

The rest elements of the structure are also defined by their action on the basis [3], [5].

Using the general construction we obtain

$$D_E\,|n,m,\pm\rangle = (n\pm im)\,|n,m,\mp\rangle \tag{3.7}$$

and the resulting Riemannian spectral triple is the one known as the equivariant Riemannian spectral triple for the noncommutative torus[16].

The second example we already presented in [6] as the motivation for the application of this method in the Particle Physics. We considered a triple for two-point geometry [11] and observed that the existence of β reduces the number of parameters in the possible Dirac operators by a factor of 2.

4. Leptoquarks

We are now ready to present the main result from [6]. We observed that for the spectral triple for the Standard Model discussed in the section 2 there exists an operator

$$\beta = \pi(1,1,-1)J_F\pi(1,1,-1)J_F^{-1} \tag{4.1}$$

that makes the system a pseudo-Riemannian spectral triple of signature $(0,2)$.

Moreover, this β acts as 1 on leptonic sector and -1 on quark sector. Therefore this 0-cycle provides a natural grading that distinguishes between lepton and quarks.

We also found all such possible 0-cycles β that commutes with the most general Dirac operator D_F and find restrictions on D_F that arise from this requirement. We parametrized the Hilbert space in a slightly different way, namely we write a typical element $\begin{bmatrix} v \\ w \end{bmatrix} \in H_F$ with $v, w \in M_4(\mathbb{C})$ parametrized as follows

$$v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix}, \qquad w = \begin{bmatrix} \overline{\nu_R} & \overline{e_R} & \overline{\nu_L} & \overline{e_L} \\ \overline{u_R^1} & \overline{d_R^1} & \overline{u_L^1} & \overline{d_L^1} \\ \overline{u_R^2} & \overline{d_R^2} & \overline{u_L^2} & \overline{d_L^2} \\ \overline{u_R^3} & \overline{d_R^3} & \overline{u_L^3} & \overline{d_L^3} \end{bmatrix}. \tag{4.2}$$

Observing that $\mathrm{End}(H_F) \cong M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$ we rewrote every operator we had had in that language and looked for possible β which is sum a elements of a form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1} \tag{4.3}$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, 1_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$ and the Dirac operator of the form (which follows from the requirement about first-order condition)

$$D_F = \begin{bmatrix} & M \\ M^\dagger & \end{bmatrix} \otimes e_{11} \otimes e_{11} + \begin{bmatrix} & N \\ N^\dagger & \end{bmatrix} \otimes e_{11} \otimes (1 - e_{11}) + \begin{bmatrix} A & B \\ & \end{bmatrix} \otimes e_{12} \otimes e_{11} + \begin{bmatrix} A^\dagger & \\ B^\dagger & \end{bmatrix} \otimes e_{21} \otimes e_{11}, \tag{4.4}$$

where $M, N, A, B \in M_2(\mathbb{C})$ and e_{ij} is a matrix with 1 in position (i, j) and zeros everywhere else.

After some computations we deduced that the only physically accepted (i.e. with nonzero Yukawa parameters etc.) Dirac operator is of the form above with $B = 0$ and $A = A \cdot \mathrm{diag}(1, -1)$ and the corresponding β has to be equal $\pi(1, 1, -1) J_F \pi(1, 1, -1) J_F^{-1}$. The vanishing of B and some entries of A is equivalent with the lack of leptoquarks. Moreover, the only nonzero entries of A that are allowed are related to the sterile neutrinos.

5. Summary and outlook

We presented the approach to the Standard Model based on the consideration of pseudo-Riemannian structures on spectral triples. The new definition of pseudo-Riemannian spectral triple was described, together with our main result from [6]: The existence of such a structure can be interpreted as a source of the lack of leptoquarks.

We have to mention that the generalization of these ideas into the case of non-finite spectral triples can shed a new light on the structure of the Lorentzian formulation of the Standard Model. We recently started the program of the reverse engineering for the Standard Model and the preliminary results, that are contained in [3], suggest the necessity of the extension of the notion of a spectral triple. The brief overview of this problem one can find also in [5].

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