PROCEEDINGS OF SCIENCE

Charm Quark Mass with Calibrated Uncertainty

Jens Erler
Departamento de Física Teórica, Instituto de Física Universidad Nacional Autónoma de México, Apartado Postal 20Ð364, CDMX 01000, México
E-mail: erler@fisica.unam.mx

Pere Masjuan*
Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Institut de Física d'Altes Energies (IFAE), The Barcelona Institute of Science and Technology, Campus UAB, 08193 Bellaterra (Barcelona).
E-mail: masjuan@ifae.es

Hubert Spiesberger
PRISMA Cluster of Excellence, Institut für Physik, Johannes Gutenberg-Universität, 55099 Mainz, Germany
E-mail: spiesberger@uni/mainz.de

We determine the charm quark mass $m_c(m_c)$ from QCD sum rules of moments of the vector current correlator calculated in perturbative QCD. Only experimental data for the charm resonances below the continuum threshold are needed in our approach, while the continuum contribution is determined by requiring self-consistency between various sum rules, including the one for the zeroth moment. Existing data from the continuum region can then be used to bound the theoretical error. Our result is $m_c(m_c) = 1272 \pm 8$ MeV for $\alpha_s(M_Z) = 0.1182$. Special attention is given to the question how to quantify and justify the uncertainty.

XIII Quark Confinement and the Hadron Spectrum - Confinement2018
31 July - 6 August 2018
Maynooth University, Ireland

*Speaker.

PoS(Confinement2018)126

1. Introduction

Based on Ref. [1], we propose to revisit the method of relativistic sum rules to extract the charm quark mass with emphasis on the evaluation of the uncertainty. Besides being a fundamental input parameter defining the Standard Model (SM), m_c enters many QCD and electroweak processes. The most important future application will be the test of the mass versus Yukawa coupling relation in the single Higgs SM, because at future lepton colliders it will be possible to measure the charm Yukawa coupling very precisely. If one — within the SM — converts the projected precision to a mass measurement this would correspond to an error of 8 MeV, which should be the benchmark of what one wants to achieve at least regarding the precision in m_c.

Among the most precise charm mass determinations, including lattice simulations [2–6] and Deep-Inelastic Scattering data [7,8], relativistic QCD sum rules play an important role on establishing the quark mass at the few-percent level [9–18]. The method, based on rigorous field theoretical principles, can be systematically improved. Nevertheless, the resulting uncertainties are dominated by theory errors which are notoriously difficult to estimate and often subject of vigorous debate.

We stress that the overall error may also be constrained within our approach to the QCD sum rules. To this end, we will adopt a strategy where the only exploited experimental information are the masses and electronic decay widths of the narrow resonances in the sub-continuum charm region, $J/\Psi(1S)$ and $\Psi(2S)$.

Consistency between two different QCD sum rules will be seen to suffice to constrain the continuum of charm pair production with good precision. For this procedure to work it is crucial to include alongside the first or second moment sum rules also the zeroth moment, as the latter exhibits enhanced sensitivity to the continuum. Comparison with existing data on the R-ratio for hadronic relative to leptonic final states in e^+e^- annihilation will then serve as a control, providing an independent error estimate which we interpret as the error on the method and (conservatively) add it as an *additional* error contribution. In this way, we can show that the overall precision in $\hat{m}_c$ from relativistic sum rules is at the sub-percent level.

2. Defining the zeroth sum rule

Let us consider the transverse part of the correlator $\Pi_q(t)$ of two heavy-quark vector currents. $\Pi_q(t)$ obeys the subtracted dispersion relation

$$12\pi^2(\Pi_q(0) - \Pi_q(-t)) = t \int_{s_0}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t} \,, \tag{2.1}$$

where we have defined $12\pi \mathrm{Im}[\Pi_q(t+i\varepsilon)] = R_q(t)$. By the optical theorem, $R_q(s)$ can be related to the measurable cross section for heavy-quark production in e^+e^- annihilation. The lower limit of the integral s_0 is fixed from the threshold for heavy quark production which is the unknown quantity we want to determine. Ultimately is the function $R_q(s)$ which will decide what exact value to use for s_0 since

$$R_q(s) = \begin{cases} R_q^{\mathrm{Res}}(s) & \text{if } 0 < s_0 < 4M^2 \\ R_q^{\mathrm{Cont}}(s) & \text{if } 4M^2 \le s_0 < \infty \end{cases} \tag{2.2}$$

where $R_q^{\rm Res}(s)$ contains a finite set of narrow resonances produced below the heavy-flavor production threshold, and $R_q^{\rm Cont}(s)$ describes the continuum production above that threshold. As soon as the resonance contribution $R_q^{\rm Res}(s)$ is separated from $R_q(s)$, s_0 is identified with the open charm threshold $s_0 = 4M^2$ with $M = M_{D^0} = 1864.84$MeV.

Assuming now global quark-hadron duality, we can write [1]:

$$\int_{s_0}^{\infty} \frac{{\rm d}s}{s(s+t)} R_q(s) = \int_{s_0}^{\infty} \frac{{\rm d}s}{s(s+t)} R_q^{\rm pQCD}(s) \tag{2.3}$$

where $R_q^{\rm pQCD}(s)$ corresponds to the $R_q(s)$ ratio calculated in perturbative QCD (pQCD) order by order in the $\alpha_s(s)$ expansion. Eq. (2.3) together with Eq. (2.1), implies:

$$\Pi_q(0) - \Pi_q(-t) = \hat{\Pi}_q^{\rm pQCD}(0) - \hat{\Pi}_q^{\rm pQCD}(-t) \tag{2.4}$$

where $\hat{\Pi}_q^{\rm pQCD}(t)$ is the correlator $\Pi_q(t)$ calculated in pQCD and the caret indicates the $\overline{\rm MS}$ scheme. $\hat{\Pi}_q^{\rm pQCD}(t)$ obeys a subtracted dispersion relation, then, given by

$$\int_{4\hat{m}_q^2}^{\infty} \frac{{\rm d}s}{s} \frac{R_q(s)}{s+t} = 12\pi^2 \frac{\hat{\Pi}_q^{\rm pQCD}(0) - \hat{\Pi}_q^{\rm pQCD}(-t)}{t} \,, \tag{2.5}$$

where $\hat{m}_q = \hat{m}_q(\hat{m}_q)$ is the mass of the heavy quark. Eq. (2.5) defines a set of sum rules that allow us to define theoretical $\mathscr{M}_n^{\rm th}$ and experimental $\mathscr{M}_n^{\rm exp}$ moments [9–11]:

$$\mathscr{M}_n^{\rm exp} = \int_{s_0}^{\infty} {\rm d}s \frac{R_q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \frac{{\rm d}^n}{{\rm d}t^n} \hat{\Pi}_q(t) \bigg|_{t=0} = \mathscr{M}_n^{\rm th} \,. \tag{2.6}$$

We can also define the zeroth moment $\mathscr{M}_0^{\rm th,exp}$ [1, 19] by taking the limit $\lim_{t\to\infty}$ in Eq. (2.5):

$$\mathscr{M}_0^{\rm exp} = \lim_{t\to\infty} \int_{4\hat{m}_q^2}^{\infty} \frac{{\rm d}s}{s} \frac{R_q(s)}{s+t} = 12\pi^2 \lim_{t\to\infty} \frac{\hat{\Pi}_q^{\rm pQCD}(0) - \hat{\Pi}_q^{\rm pQCD}(-t)}{t} = \mathscr{M}_0^{\rm th} \tag{2.7}$$

After taking the $\lim_{t\to\infty}$ and multiplying by t, Eq. (2.7) as it stands is not well defined neither for $\mathscr{M}_0^{\rm exp}$ nor for $\mathscr{M}_0^{\rm th}$ for which they must be regularized. At a given order in pQCD, the required regularization can be obtained by subtracting the zero-mass limit of $R_q(s)$, which we write as $3Q_q^2\lambda_1^q(s)$ with Q_q the quark charge. $\lambda_1^q(s)$ is known up to $\mathscr{O}(\hat{\alpha}_s^4)$, but we will only need the third-order expression [20],

$$\begin{aligned} \lambda_1^q(s) &= 1 + \frac{\hat{\alpha}_s}{\pi} + \frac{\hat{\alpha}_s^2}{\pi^2}\left[\frac{365}{24} - 11\zeta(3) + n_q\left(\frac{2}{3}\zeta(3) - \frac{11}{12}\right)\right] \\ &+ \frac{\hat{\alpha}_s^3}{\pi^3}\left[\frac{87029}{288} - \frac{121}{8}\zeta(2) - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right. \\ &+ \left. n_q\left(-\frac{7847}{216} + \frac{11}{6}\zeta(2) + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right) + n_q^2\left(\frac{151}{162} - \frac{\zeta(2)}{18} - \frac{19}{27}\zeta(3)\right)\right], \end{aligned} \tag{2.8}$$

where $\hat{\alpha}_s = \hat{\alpha}_s(s)$, $n_q = n_l + 1$ and n_l is the number of light flavors (taken as massless), *i.e.*, quarks with masses below the heavy quark under consideration.

Let us then define the function $H(t)$ to have exactly the same large t behavior as $\hat{\Pi}_q(t)$ (including leading divergence terms such as $\log(-t/\mu^2)$) up to $\mathcal{O}(\hat{\alpha}_s^3)$. This is, explicitely, $\lim_{t\to\infty}(\hat{\Pi}_q^{\rm pQCD}(t) - H(t)) = 0 + \mathcal{O}(\hat{\alpha}_s^4)$. $H(t)$ satisfies a subtracted dispersion relation:

$$H(0) - H(-t) = \frac{t}{\pi}\int_{\mu^2}^{\infty}\frac{{\rm d}s}{s}\frac{{\rm Im}[H(s)]}{s+t} \tag{2.9}$$

where, ${\rm Im}[H(s+i\varepsilon)] = \frac{3Q_q^2}{12\pi}\lambda_1^q(s)$ and the lower limit as such that after integrating over ${\rm Im}[H(s+i\varepsilon)]$, the leading logarithms from the large t behavior of $\hat{\Pi}_q(t)$, i.e., the $\log(-t/\mu^2)$'s, are recovered.

Finally, we can cancel the divergences in Eq. (2.7) by subtracting Eq. (2.9) from it [1]:

$$\begin{aligned} &12\pi^2\ \lim_{t\to\infty}\frac{\hat{\Pi}_q^{\rm pQCD}(0) - \hat{\Pi}_q^{\rm pQCD}(-t) - (H(0)-H(-t))}{t} \\ &= \lim_{t\to\infty}\int_{4\hat{m}^2}^{\infty}\frac{{\rm d}s}{s}\frac{R_q(s) - 12\pi^2{\rm Im}[H(s)]}{s+t} - \int_{\mu^2}^{4\hat{m}^2}\frac{{\rm d}s}{s}\frac{12\pi^2{\rm Im}[H(s)]}{s+t}\,. \end{aligned} \tag{2.10}$$

Eq. (2.10) defines the regularized zeroth moment. As we have said, the optical theorem relates $R_q(s)$ in Eq. (2.2) with the cross section for heavy-quark production in e^+e^- annihilation. Below the threshold for continuum heavy-flavor production, $R_q^{\rm Res}(s)$ is approximated by δ-functions [9],

$$R_q^{\rm Res}(s) = \frac{9\pi}{\alpha_{\rm em}^2(M_R)}M_R\Gamma_R^e\delta(s-M_R^2). \tag{2.11}$$

The masses M_R and electronic widths Γ_R^e of the resonances [21] are listed in Table 1 and $\alpha_{\rm em}(M_R)$ is the running fine structure constant at the resonance[1]. To parametrize $R_q^{\rm Cont}(s)$, we assume that continuum production can be described on average by the simple ansatz [1,19],

$$R_q^{\rm Cont}(s) = 3Q_q^2\lambda_1^q(s)\sqrt{1-\frac{4\hat{m}_q^2(2M)}{s'}}\left[1+\lambda_3^q\frac{2\hat{m}_q^2(2M)}{s'}\right] \tag{2.12}$$

where $s' := s + 4(\hat{m}_q^2(2M) - M^2)$, and M is taken as the mass of the lightest pseudoscalar heavy meson, *i.e.*, $M = M_{D^0} = 1864.84$ MeV for charm quarks [21]. λ_3^q is a constant to be determined. Eq. (2.12) interpolates smoothly between the threshold and the onset of open heavy-quark pair production and coincides asymptotically with the prediction of pQCD for massless quarks [33].

Performing the limit $\lim_{t\to\infty}$ in Eq. (2.10) will allow us to define the zeroth sum rule. Doing so, we need the results of Refs. [23,24] and $R_q(s) = \sum_{\rm resonances} R_q^{\rm Res}(s) + R_q^{\rm cont}(s)$. Multiplying then by $t/3Q_q^2$, and setting $\mu^2 = \hat{m}^2$, the zeroth sum rule reads [1]:

$$\begin{aligned} &\sum_{\rm resonances}\frac{9\pi\Gamma_R^e}{3Q_q^2M_R\hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty}\frac{{\rm d}s}{s}\left(\frac{R_q^{\rm Cont}(s)}{3Q_q^2} - \lambda_1^q(s)\right) - \int_{\hat{m}_q^2}^{4M^2}\frac{{\rm d}s}{s}\lambda_1^q(s) = \\ &= -\frac{5}{3} + \frac{\hat{\alpha}_s}{\pi}\left[4\zeta(3) - \frac{7}{2}\right] \\ &+ \left(\frac{\hat{\alpha}_s}{\pi}\right)^2\left[\frac{2429}{48}\zeta(3) - \frac{25}{3}\zeta(5) - \frac{2543}{48} + n_q\left(\frac{677}{216} - \frac{19}{9}\zeta(3)\right)\right] + \left(\frac{\hat{\alpha}_s}{\pi}\right)^3 A_3, \end{aligned} \tag{2.13}$$

[1] The values for $\alpha_{\rm em}(M_R)$ were determined with help of the program `hadr5n12` [22].

where $\hat{\alpha}_s = \hat{\alpha}_s(\hat{m}_q)$. The third-order coefficient A_3 is available in numerical form [25, 26],

$$A_3 = -9.863 + 0.399\, n_q - 0.010\, n_q^2 \,. \tag{2.14}$$

Notice that the continuum $R_q^{\rm Cont}(s)$ contributes with the lower integration limit $4M^2$, while the subtraction term $\lambda_1^q(s)$ is integrated starting from $\hat{m}_q^2$.

To perform the integral $\int\limits_{4M^2}^{\infty} \frac{{\rm d}s}{s}\left(\frac{R_q^{\rm Cont}}{3Q_q^2} - \lambda_1^q(s)\right)$ in Eq. (2.13), it is convenient to expand first $\lambda_1^q(s)$ in $\hat{\alpha}_s$ using the RGE for $\hat{\alpha}_s(s)$ selecting the reference scale as $\hat{m}_q^2$. In this way, the integral over s becomes well defined, i.e., with all the divergences in both $R_q^{\rm Cont}(s)$ and $\lambda_1^q(s)$ removed.

Eq. (2.13) contains two unknowns, the quark mass $\hat{m}_q(\hat{m}_q)$ and the parameter λ_3^q entering in our prescription for $R_q^{\rm Cont}(s)$. The zeroth sum rule is the most sensitive to the continuum region as is the one with less powers of s, and shall be used to determine λ_3^q. Self-consistency with another moment sum rule can then be used to determine the quark mass as soon as λ_3^c is determined.

Theory predictions for the higher moments in perturbative QCD can be cast into the form

$$\mathscr{M}_n^{\rm pQCD} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \hat{C}_n \tag{2.15}$$

with

$$\hat{C}_n = C_n^{(0)} + \left(\frac{\hat{\alpha}_s}{\pi}\right) C_n^{(1)} + \left(\frac{\hat{\alpha}_s}{\pi}\right)^2 C_n^{(2)} + \left(\frac{\hat{\alpha}_s}{\pi}\right)^3 C_n^{(3)} + \mathscr{O}(\hat{\alpha}_s^4)\,. \tag{2.16}$$

The $C_n^{(i)}$ are known up to $\mathscr{O}(\hat{\alpha}_s^3)$ for $n \leq 3$ [27–30], and up to $\mathscr{O}(\hat{\alpha}_s^2)$ for the rest [31, 32]. Since we need all the moments up to $\mathscr{O}(\hat{\alpha}_s^3)$ we use the predictions for $n > 3$ provided in Ref. [33]. Once $\mathscr{M}_n^{\rm pQCD}$ are available, the determination of both $\hat{m}_q(\hat{m}_q)$ and λ_3^q come from solving the system of the two equations. In this last step, the theoretical moments are equated with their corresponding experimental counterparts defined in Eq. (2.6), i.e., $\mathscr{M}_n^{\rm pQCD} = \mathscr{M}_n^{\rm exp}$ for $n > 0$.

In general, vacuum expectation values of higher-dimensional operators in the operator product expansion (OPE) contribute to the moments of the current correlator as well. These condensates may be important for a high-precision determination of heavy-quark masses, in particular in the case of the charm quark. The leading term involves the dimension-4 gluon condensate [9],

$$\mathscr{M}_n^{\rm cond} = \frac{12\pi^2 Q_c^2}{(4\hat{m}_c^2)^{n+2}} \langle \frac{\hat{\alpha}_s}{\pi} G^2 \rangle\, a_n \left(1 + \frac{\hat{\alpha}_s(\hat{m}_c^2)}{\pi} b_n\right)\,. \tag{2.17}$$

The coefficients a_n and b_n can be found in [34, 35]. In our fits we use the central value $\langle \frac{\hat{\alpha}_s}{\pi} G^2 \rangle^{\rm exp} = 0.005$ GeV^4 with an uncertainty of $\Delta\langle \frac{\hat{\alpha}_s}{\pi} G^2 \rangle = 0.005$ GeV^4, taken from the recent analysis [36].

R	M_R [GeV]	Γ_R^e [keV]
$J/\Psi(1S)$	3.096916	5.55(14)
$\Psi(2S)$	3.686109	2.36(4)

Table 1: Resonance data [21] used in the analysis. The uncertainties from the resonance masses are negligible for our purpose.

Including the condensate contribution when equating Eqs. (2.6) and (2.15), i.e., $\mathscr{M}_n^{\rm exp} = \mathscr{M}_n^{\rm pQCD}$ together with the zeroth sum rule, we determine values for the heavy quark mass $\hat{m}_c(\hat{m}_c)$ and the constant λ_3^c. The other moments are then fixed and can be used to check the consistency of our approach. No experimental data other than the resonance parameters in Table 1 are necessary. From the combination $0^{\rm th}+2^{\rm nd}$ sum rules, we obtain $\lambda_3^c = 1.23$ and $\hat{m}_c(\hat{m}_c) = 1.272$GeV without errors as they come from solving a system of two equations. The error estimation is discussed in the next section. Once both $\hat{m}_c(\hat{m}_c)$ and λ_3^c are determined, we can compare our prescription for $R_q^{\rm Cont}(s)$ with experimental data in the threshold region, Fig. 1. The full red curve shows $R_c^{\rm Cont}(s)$ with $\lambda_3^c = 1.23$ and $\hat{m}_c(\hat{m}_c) = 1.272$GeV and should be understood as an average determination of the cross section in the threshold region.

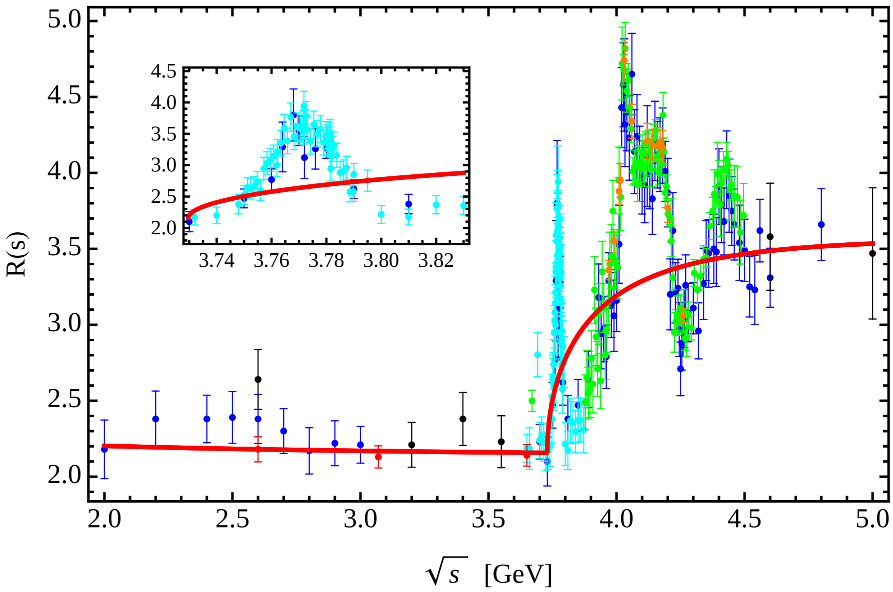

Figure 1: Data for the ratio R for $e^+e^- \to$ hadrons in the charm threshold region: Crystal Ball CB86 (green) [37]; BES00, 02, 06, 09 (black, blue, cyan, and red) [38–41], and CLEO09 (orange) [42]. The full (red) curve shows $R_c^{\rm Cont}(s)$ with $\lambda_3^c = 1.23$ and $\hat{m}_c(\hat{m}_c) = 1.272$GeV. The inner plot is a zoom-in into the energy range $2M_{D^0} \le \sqrt{s} \le 3.83$ GeV.

2.1 Uncertainty estimate

To determine an error for the continuum contributions we proceed in the following way [1]: instead of using Eqs. (2.13, 2.15), we can compare experimental data shown in Fig. 1 with the zeroth moment in the restricted energy range of the threshold region, $2M_{D^0} \le \sqrt{s} \le 4.8$ GeV to obtain an *experimental* value for λ_3^c, denoted $\lambda_3^{c,\rm exp}$. Here we fix $\hat{m}_c(\hat{m}_c)$ using Eq. (2.15) and proceed to solve Eq. (2.13) by comparing with $\mathscr{M}_0^{\rm exp}$. Then we can also determine an error, $\Delta\lambda_3^{c,\rm exp}$ from the experimental uncertainty of the data in this threshold region.

We calculate the experimental moments via numerical integrals over the available experimental data, cf. Fig. 1. Experimental data is classified in five different intervals, see Fig. 2, which allow us to fully take into account correlated and uncorrelated uncertainties among different

n	Data	$\lambda_3^c = 1.34(17)$	$\lambda_3^c = 1.23$
0	0.6367(195)	0.6367(195)	0.6239
1	0.3500(102)	0.3509(111)	0.3436
2	0.1957(54)	0.1970(65)	0.1928
3	0.1111(29)	0.1127(38)	0.1102
4	0.0641(16)	0.0657(23)	0.0642
5	0.0375(9)	0.0389(14)	0.0380

Table 2: Contributions to the charm moments ($\times 10^n\,\mathrm{GeV}^{2n}$) from the energy range $2M_{D^0} \le \sqrt{s} \le 4.8$ GeV. For the results in the column labeled 'Data', light-quark contributions have been subtracted using the pQCD prediction at order $\mathscr{O}(\hat{\alpha}_s^3)$. These entries are obtained from the data displayed in Fig. 2, taking into account the correlation of systematic errors within each experiment. The third column uses $\hat{m}_c = 1.272\,\mathrm{GeV}$ and $\lambda_3^{c,\exp}$ determined by the zeroth experimental moment (see text for details). The last column shows the theoretical prediction for the moments using $\hat{m}_c = 1.272\,\mathrm{GeV}$ and $\lambda_3^c = 1.23$, not including condensates.

collaborations and intervals. The results for the experimental moments in the threshold region, $2M_{D^0} \le \sqrt{s} \le 4.8$ GeV are given in Table 2. For the results in the columns labeled 'Data', light-quark contributions have been subtracted using the pQCD prediction at order $\mathscr{O}(\hat{\alpha}_s^3)$, see Ref. [1]. Its second column shows the required value for $\lambda_3^{c,\exp}$ such that the zeroth moment sum rule is experimentally satisfied after fixing $\hat{m}_c(\hat{m}_c) = 1.272$GeV. The rest of the moments in this column are reported to show the consistency of the approach. Even for the highest moments, the consistency is very good. The last column collects, for comparison, the value for the moments in the same energy region using $\lambda_3^c = 1.23$ extracted from the theoretical determination.

The shift in the moments resulting from the different values for λ_3^c (either from two moments combined with resonance data only, or from the comparison of the 0th moment with continuum data in the threshold region) turns out to be small. Strictly speaking this shift is a one-sided error, but to be conservative we include it as an additional double-sided error in the results of Table 3. A graphical account of this shift is shown in Fig. 3 as a cyan band for the result of the $0^{th}+2^{nd}$ moments pair for $\hat{m}_c(\hat{m}_c) = 1.272$GeV. In this case, $\lambda_3^{c,\exp} = 1.34(17)$, c.f. Table 3. The red solid curve corresponds to the same pair of moments and the same quark mass with $\lambda_3^c = 1.23$, and well overlaps with the cyan band.

Finally, we assign a truncation error to the theory prediction of the moments following the method proposed in Ref. [19] which considers the largest group theoretical factor in the next uncalculated perturbative order as a way to estimate errors,

$$\Delta\mathscr{M}_n^{(i)} = \pm Q_q^2 N_C C_F C_A^{i-1} \left(\frac{\hat{\alpha}_s(\hat{m}_q)}{\pi}\right)^i \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \tag{2.18}$$

($N_C = C_A = 3$, $C_F = 4/3$). At order $\mathscr{O}(\hat{\alpha}_s^4)$, this corresponds to an uncertainty of $\pm 48(\hat{\alpha}_s/\pi)^4$ for $\hat{C}_n^{(4)}$ in Eq. (2.15).

For the moments with $n > 3$ taken from Ref. [33] we have to include additional uncertainties specific to the method used to obtain predictions for $\mathscr{M}_n$. These errors are very small, but included

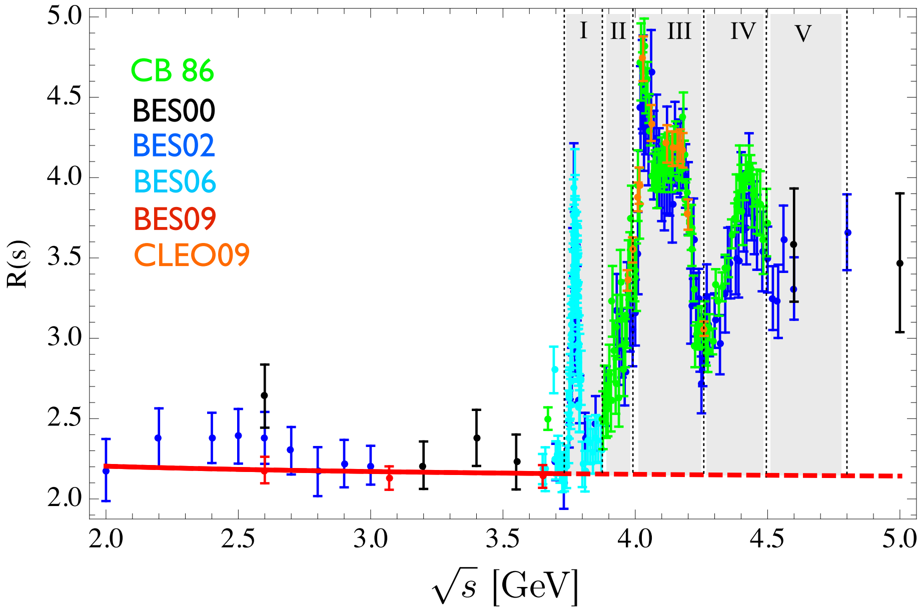

Figure 2: Data for the ratio R for $e^+e^- \to$ hadrons in the charm threshold region: Crystal Ball CB86 (green) [37]; BES00, 02, 06, 09 (black, blue, cyan, and red) [38–41], and CLEO09 (orange) [42]. The gray bands indicate the five intervals considered for evaluating the experimental moments.

for completeness.

The charm mass and the continuum parameter λ_3^c can, in principle, be determined from any combination of two moments, not only $0^{th}+2^{nd}$. The zeroth moment, however, is expected to provide the highest sensitivity. The results for combinations of the zeroth with one higher moment are summarized in Table 3 and visualized in Fig. 5. We include the difference between the two possibilities to determine λ_3^c as described above as an additional error.

As an example on how to understand Table 3, select the $0^{th}+2^{nd}$ moments pair as the result for the quark mass, we would combine the total error ± 8.0 MeV with ± 4.2 MeV from $\hat{\alpha}_s(M_z) = 0.1182(16)$ [21]. Then, $\hat{m}_c(\hat{m}_c) = 1272(9)$ MeV. Doing so for each pair of moments collected in the table, we notice that the combination $0^{th}+2^{nd}$ provides the smallest total error for the heavy quark mass. Let us remark that for the highest moments, the truncation of the OPE series, i.e., condensates of higher dimension not considered in our approach can be important [1]. While difficult to assert, we belief that these higher dimension condensates are well included in our condensate error estimate. However, to be on the safe side, charm quark mass determination using the $4^{\rm th}$ and $5^{\rm th}$ moment sum rules may have an underestimated error. They should not be considered. On the contrary, the 0^{th} and 1^{st} moments are the ones less sensitive to the OPE truncation with the combination $0^{th}+1^{st}$ being a most favorable choice. However, this combination is the one most sensitive to the continuum region, with largest shift in $\lambda_3^{c,\exp}$, cf. Table 3, and for the same reason most sensitive to $\Delta\alpha_s$. The pair of the 0^{th} and 2^{nd} moments is our optimal choice since balance well between reduced effects of the OPE series truncation and good description of the continuum

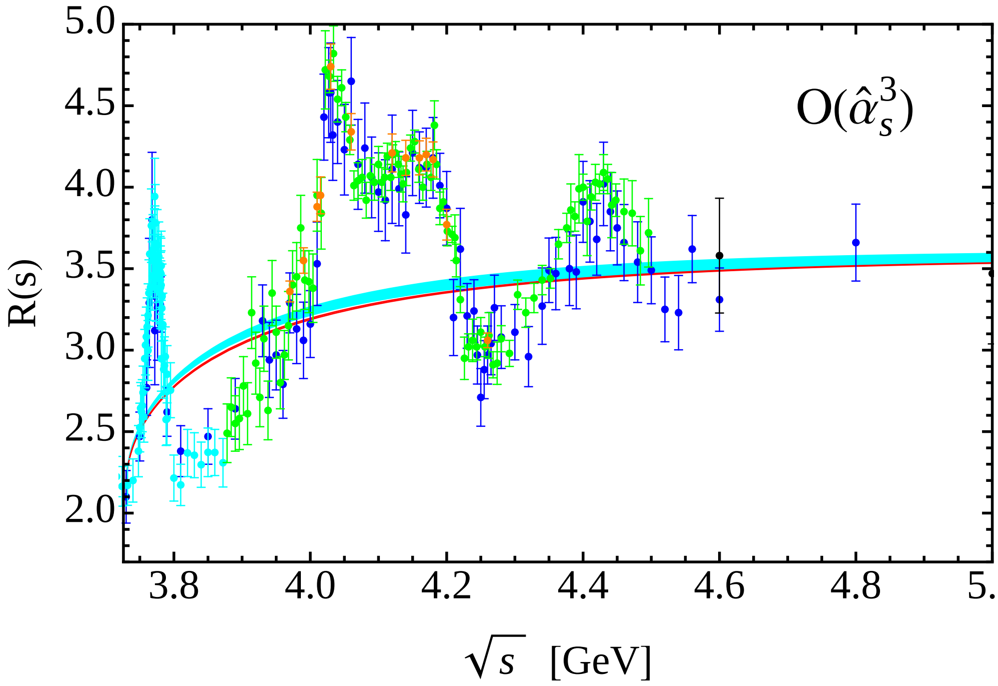

Figure 3: Data for the ratio R for $e^+e^- \to$ hadrons in the charm threshold region: Crystal Ball CB86 (green) [37]; BES00, 02, 06, 09 (black, blue, cyan, and red) [38–41], and CLEO09 (orange) [42]. The full (red) curve shows $R_c^{\rm Cont}(s)$ with $\lambda_3^c = 1.19$ and $\hat{m}_c(\hat{m}_c) = 1.279\text{GeV}$ and the cyan band shows $R_c^{\rm Cont}(s)$ with $\hat{m}_c(\hat{m}_c) = 1.279\text{GeV}$ and $\lambda_3^{c,\rm exp} = 1.35(17)$.

region. This pair has also the smallest total uncertainty in the charm mass determination.

3. Conclusions

In this talk we presented a determination of the charm quark mass based on the work of Ref. [1]. We revisit there the method of relativistic sum rules with emphasis on the evaluation of the uncertainty. By invoking the zeroth sum rule and requiring self-consistency with higher-moment sum rules, we can show that the overall error may be constrained within the approach.

After considering the combination of two different sum rules, the only experimental information required are the masses and electronic decay widths of the narrow resonances in the sub-continuum charm region, $J/\Psi(1S)$ and $\Psi(2S)$. Comparison with experimental data in this region is later on used to check the results and determine an experimental error.

Acknowledgments

This work has been supported by DFG through the Collaborative Research Center ÒThe Low-Energy Frontier of the Standard ModelÓ (SFB 1044), by CONACyT (Mexico) project 252167-F and The work of P.M. is supported by the Beatriu de Pinós postdoctoral programme of the Government of Catalonia's Secretariat for Universities and Research of the Ministry of Economy and Knowledge of Spain and by the Generalitat de Catalunya 2017 SGR 1069.

$\Delta\hat{m}_c(\hat{m}_c)$	$(\mathscr{M}_0,\mathscr{M}_1)$	$(\mathscr{M}_0,\mathscr{M}_2)$	$(\mathscr{M}_0,\mathscr{M}_3)$	$(\mathscr{M}_0,\mathscr{M}_4)$	$(\mathscr{M}_0,\mathscr{M}_5)$
$\hat{m}_c(\hat{m}_c)$	1280.9	**1272.4**	1269.1	1265.8	1262.2
λ_3^c	1.154	**1.230**	1.262	1.291	1.323
$\lambda_3^{c,\text{exp}}$	1.35(17)	**1.34(17)**	1.34(17)	1.33(17)	1.32(17)
Resonances	5.8	**4.5**	3.9	3.3	2.8
Truncation	6.3	**5.9**	7.2	8.9	10.5
$\lambda_3^c - \lambda_3^{c,\text{exp}}$	+6.4	**+1.5**	+0.3	+0.1	+0.1
$\Delta\lambda_3^{c,\text{exp}}$	4.7	**1.7**	0.7	0.3	0.2
$10^3\times\Delta_G$	$-0.25\Delta_G$	$-0.37\Delta_G$	$-0.54\Delta_G$	$-0.73\Delta_G$	$-0.88\Delta_G$
	(-1.3)	$(-\mathbf{1.9})$	(-2.7)	(-3.7)	(-4.4)
Total	± 11.7	± 8.0	± 8.7	± 10.2	± 11.7
$10^3\times\Delta\hat{\alpha}_s(M_z)$	$+3.6\Delta\hat{\alpha}_s$	$+2.6\Delta\hat{\alpha}_s$	$+1.6\Delta\hat{\alpha}_s$	$+0.6\Delta\hat{\alpha}_s$	$-0.4\Delta\hat{\alpha}_s$
Electroweak fit	(+5.8)	**(+4.2)**	(+2.6)	(+1.0)	(−0.6)
Lattice	(+4.3)	**(+3.1)**	(+1.9)	(+0.7)	(−0.5)

Table 3: Values and breakdown of the uncertainties of $\hat{m}_c(\hat{m}_c)$ (in MeV) and λ_3^c determined from different pairs of moments. The line denoted 'Total' gives the quadratic sum of the errors from λ_3^c, the resonances, the gluon condensate and the pQCD truncation. Numerical values for the uncertainties from $\hat{\alpha}_s$ and $C_G = \langle \frac{\hat{\alpha}_s}{\pi} G^2 \rangle$ (in units of GeV^4) are shown in separate lines. In the line labeled 'Electroweak fit' we use $\Delta\hat{\alpha}_s(M_z) = 0.0016$ [21], in the last line denoted 'Lattice', we use $\Delta\hat{\alpha}_s(M_z) = 0.0012$ [21]. See Figure 5 for a graphical representation.

References

[1] J. Erler, P. Masjuan and H. Spiesberger, Eur. Phys. J. C **77**, no. 2, 99 (2017) [arXiv:1610.08531 [hep-ph]].

[2] A. Laschka, N. Kaiser, and W. Weise. *Phys. Rev. D*, **83**, 094002, 2011.

[3] Y. Namekawa et al. *Phys. Rev. D*, **84**, 074505, 2011.

[4] J. Heitger, G. M. von Hippel, S. Schaefer, and F. Virotta. *PoS*, LATTICE2013, 475, 2014.

[5] N. Carrasco et al. *Nucl. Phys. B*, **887**, 19, 2014.

[6] B. Chakraborty, C. T. H. Davies, B. Galloway, P. Knecht, J. Koponen, G.C. Donald, R.J. Dowdall, G.P. Lepage, and C. McNeile. *Phys. Rev. D*, bf 91, 054508 , 2015.

[7] S. Alekhin, J. Blumlein, and S. Moch. *Phys. Rev. D*, **89**, 054028, 2014.

[8] H. Abramowicz et al. *Eur. Phys. J. C*, **73**, 2311, 2013.

[9] V. A. Novikov, L. B. Okun, Mikhail A. Shifman, A. I. Vainshtein, M. B. Voloshin, and Valentin I. Zakharov. *Phys. Rept.*, **41**, 1, 1978.

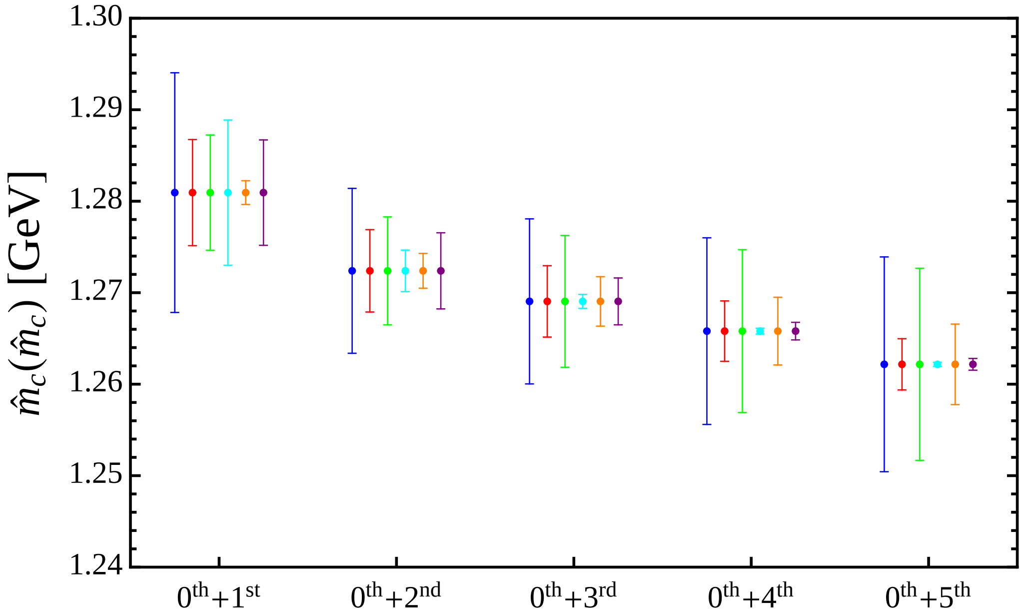

Figure 4: $\hat{m}_c(\hat{m}_c)$ using different combinations of moments and the error budget for each combination. Blue is the full error, red is the one from the resonance region, green from the truncation errors of the theoretical moments, cyan from the error of λ_3^c (which is the combination of the shift and the experimental error on λ_3^c), orange from the gluon condensate and purple from the uncertainty of $\Delta\hat{\alpha}_s(M_z)$.

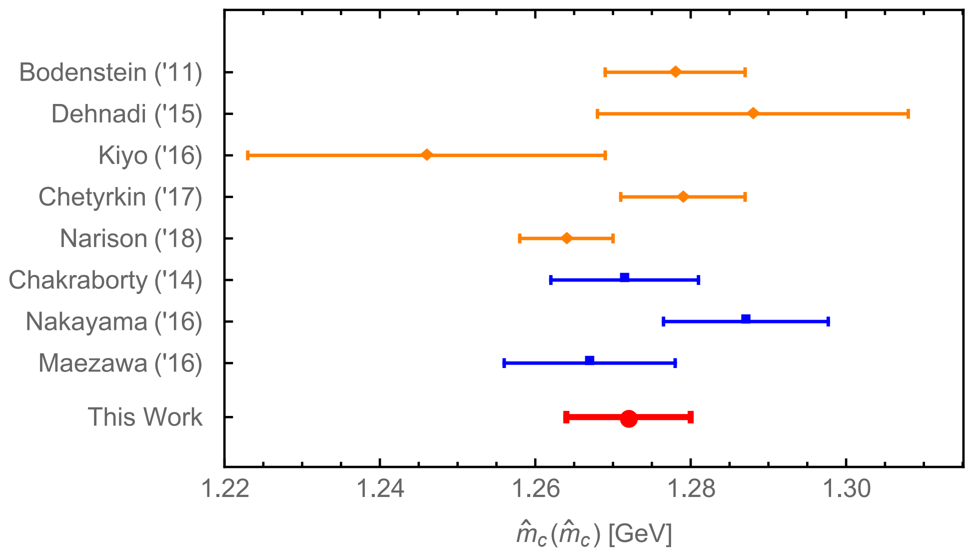

Figure 5: Recent charm quark mass determinations, cf. references.

[10] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov. *Nucl. Phys. B*, **147**, 385, 1979.

[11] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov. *Nucl. Phys. B*, **147**, 448, 1979.

[12] B. Dehnadi, A. H. Hoang, V. Mateu, and S. M. Zebarjad. *JHEP*, **09**, 103, 2013.

[13] S. Bodenstein, J. Bordes, C. A. Dominguez, J. Penarrocha, and K. Schilcher. *Phys. Rev. D*, **83**, 074014 , 2011.

[14] Y. Kiyo, G. Mishima and Y. Sumino, Phys. Lett. B **752** (2016) 122 Erratum: [Phys. Lett. B **772** (2017) 878] [arXiv:1510.07072 [hep-ph]].

[15] K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, Phys. Rev. D **96** (2017) no.11, 116007 [arXiv:1710.04249 [hep-ph]].

[16] V. Mateu and P. G. Ortega, JHEP **1801** (2018) 122 [arXiv:1711.05755 [hep-ph]].

[17] C. Peset, A. Pineda and J. Segovia, JHEP **1809** (2018) 167 [arXiv:1806.05197 [hep-ph]].

[18] S. Narison, Phys. Lett. B **784** (2018) 261 [arXiv:1808.01969 [hep-ph]].

[19] J. Erler and M.-X. Luo. *Phys. Lett. B*, **558**, 125, 2003.

[20] K. G. Chetyrkin, R. V. Harlander, and J. H. Kuhn. *Nucl. Phys. B*, **586**, 56, 2000.

[21] K. A. Olive et al. *Chin. Phys. C*, **38**, 090001, 2014.

[22] F. Jegerlehner. `http://www-com.physik.hu-berlin.de/{~}fjeger/`.

[23] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser. *Nucl. Phys. B*, **482**, 213, 1996.

[24] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser. *Nucl. Phys. B*, **510**, 61, 1998.

[25] Y. Kiyo, A. Maier, P. Maierhofer, and P. Marquard. *Nucl. Phys. B*, **823**, 269, 2009.

[26] A. H. Hoang, V. Mateu, and S. M. Zebarjad. *Nucl. Phys. B*, **813**, 349, 2009.

[27] K. G. Chetyrkin, J. H. Kuhn, and C. Sturm. *Eur. Phys. J. C*, **48**, 107, 2006.

[28] R. Boughezal, M. Czakon, and T. Schutzmeier. *Phys. Rev. D*, **74**, 074006 , 2006.

[29] A. Maier, P. Maierhofer, and P. Marqaurd. *Phys. Lett. B*, **669**, 88, 2008.

[30] A. Maier, P. Maierhofer, P. Marquard, and A. V. Smirnov. *Nucl. Phys. B*, **824**, 1, 2010.

[31] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser. *Nucl. Phys. B*, **505**, 40, 1997.

[32] A. Maier, P. Maierhofer, and P. Marquard. *Nucl. Phys. B*, **797**, 218, 2008.

[33] D. Greynat, P. Masjuan, and S. Peris. *Phys. Rev. D*, **85**, 054008, 2012.

[34] D. J. Broadhurst, P. A. Baikov, V. A. Ilyin, J. Fleischer, O. V. Tarasov, and V. A. Smirnov. *Phys. Lett. B*, **329**, 103, 1994.

[35] J. H. Kuhn, M. Steinhauser, and C. Sturm. *Nucl. Phys. B*, **778**, 192, 2007.

[36] C. A. Dominguez, L. A. Hernandez, K. Schilcher, and H. Spiesberger. *JHEP*, **03**, 053, 2015.

[37] A. Osterheld, R. Hofstadter, R. Horisberger, I. Kirkbride, H. Kolanoski, et al. *Submitted to: Phys. Rev. D*, 1986.

[38] J. Z. Bai et al. *Phys. Rev. Lett.*, **84**, 594, 2000.

[39] J. Z. Bai et al. *Phys. Rev. Lett.*, **88**, 101802, 2002.

[40] M. Ablikim, J. Z. Bai, Y. Ban, J. G. Bian, X. Cai, et al. *Phys. Rev. Lett.*, **97**, 262001, 2006.

[41] M. Ablikim et al. *Phys. Lett. B*, **677**, 239, 2009.

[42] D. Cronin-Hennessy et al. *Phys. Rev. D*, **80**, 072001, 2009.

PoS(Confinement2018)126