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Inclusive decay structure function for $B \to X_c \ell \nu$: a comparison of a lattice calculation with the heavy quark expansion

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A lattice calculation of inclusive decay structure functions for $B \to X_c \ell \nu$ is compared with the corresponding estimates based on the heavy quark expansion. Both methods are applicable in the region away from the resonances/cuts due to final charmed states, and one can test the theoretical methods employed on both sides.

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1. Introduction

In this work we consider the "inclusive" decay of the B meson. This means summing over all possible final states with a particular weight factor. To be specific, we consider the semileptonic decay of a B meson to a charm quark c and a lepton pair $\ell\nu$. The optical theorem can be schematically written as

$$\sum_{X_c = D,\, D^*,\, \cdots} |A(B \to X_c \ell \nu)|^2 \;=\; \mathrm{Im} A(B \to X_c \ell \nu \to B), \tag{1.1}$$

where the right-hand side really refers to the forward-scattering matrix element

$$T_{\mu\nu}(v \cdot q, q^2) = i \int d^4x e^{-iqx} \frac{1}{2M_B} \langle B(v)| T \left\{ J_\mu^\dagger(x) J_\nu(0) \right\} |B(v)\rangle. \tag{1.2}$$

Here, the initial B meson momentum is specified by $M_B v^\mu$, and q^μ is the momentum transfer to the lepton pair. J_μ stands for the weak current that induces the decay. The matrix element is a function of two invariant kinematical variables $v \cdot q$ and q^2. The inclusive decay rate is described by this matrix element, and the question is whether one can calculate this quantity using the techniques of lattice QCD.

It is not straightforward to apply lattice methods, however, especially in the kinematical region where the decay actually occurs. This is because the lattice theory is formulated in the Euclidean space and no imaginary part of the amplitude appears in the kinematical region accessible on the lattice. The strategy proposed by [1] is to use analytic continuation, or equivalently Cauchy's integral

$$T(v \cdot q) = \int_{-\infty}^{(v \cdot q)_{\max}} \frac{d(v \cdot q')}{\pi} \frac{\mathrm{Im}\, T(v \cdot q')}{v \cdot q' - v \cdot q}. \tag{1.3}$$

This relates the imaginary part corresponding to the physical decay amplitude to a value of $T(v \cdot q)$ at an unphysical kinematical region where the real decay does not occur, and thus no imaginary part develops.

At fixed q^2, the upper limit of the physical cut on $v \cdot q$ is given by $(v \cdot q)_{\max} = (M_B^2 + q^2 - M_D^2)/2M_B$. The cut continues down to negative infinity, while the real decay occurs only in the region $v \cdot q > 0$. There is another cut due to an unphysical process $b \to bb\bar{c}$ starting from $((2M_B +$

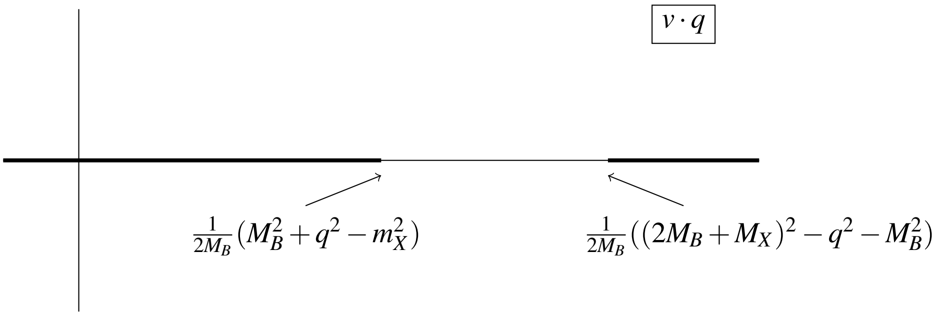


Figure 1: Analytic structure of the structure function $T(v \cdot q, q^2)$ in the complex plane of $v \cdot q$. The cuts are shown by thick lines. The cut on the left corresponds to the physical decay of $b \to c$, while the other represents an unphysical process $b \to \bar{c}bb$.

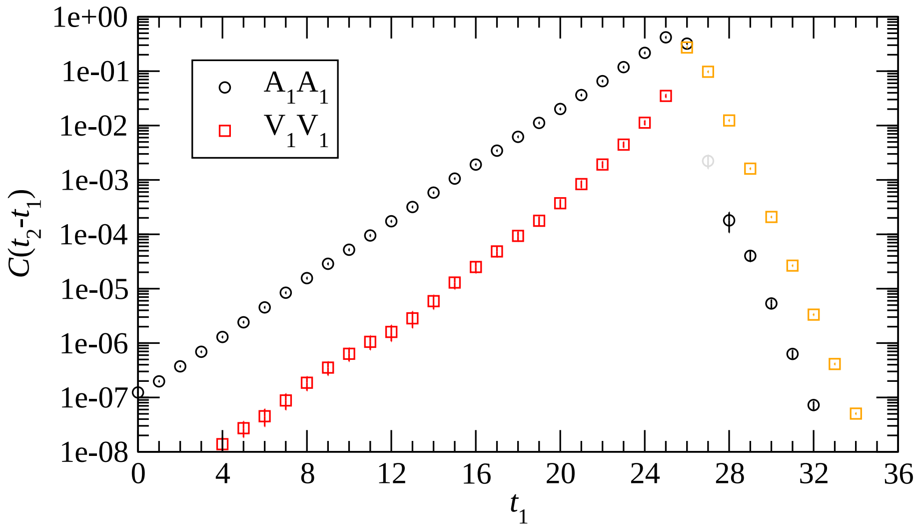


Figure 2: Correlation function $C^{JJ}_{\mu\nu}(t;\mathbf{q})$ plotted as a function of t_1. The other current t_2 is set at $t_2 = 26$. Data for the AA channel (circles) and for the VV channel (squares) both in the spatial direction $\mu\nu = 11$. The points below $t_1 = t_2$ represent the physical decay while those above are unphysical ($b \to bb\bar{c}$). The data at β = 4.35 ($1/a \simeq 3.6$ GeV) and $m_b = (1.25)^4 m_c$ are shown.

$M_D)^2 - q^2 - M_B^2)/2M_B$. The analytic structure of $T(v\cdot q)$ in the complex plane of $v \cdot q$ is shown in Figure 1. The cuts where the imaginary part appears are shown by thick lines. The formula (1.3) only refers to an integral over the physical cut; a contribution from the other (unphysical) process exists and it should be possible to separately identify each contribution in the lattice calculation and in the continuum analytical calculation.

2. Lattice calculation

The method to calculate the matrix element $T(v\cdot q, q^2)$ is described in [1]. It reduces to a calculation of the four-point function corresponding to the matrix element

$$C^{JJ}_{\mu\nu}(t;\mathbf{q}) = \int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \frac{1}{2M_B} \langle B(\mathbf{0})|J^{\dagger}_{\mu}(\mathbf{x},t) J_{\nu}(0)|B(\mathbf{0})\rangle, \tag{2.1}$$

which is different from $T_{\mu\nu}(v\cdot q, q^2)$ defined in (1.2) as it does not include the Fourier transform in the t direction. We use the standard sequential source method to calculate the four-point function.

The correlator $C^{JJ}_{\mu\nu}(t;\mathbf{q})$ for the case of zero spatial momentum insertion ($\mathbf{q} = \mathbf{0}$) is shown as a function of t_1 ($t = t_2 - t_1$ and $t_2 = 26$) in Figure 2. The currents J are spatial axial-vector A_k (circles) or spatial vector V_k (squares). The lattice data are obtained with Möbius domain-wall fermions for both sea and valence quarks. We take an example obtained at an inverse lattice spacing 3.6 GeV and pion mass 300 MeV. The charm quark mass is tuned to the physical value and the bottom quark mass is $m_b = (1.25)^4 m_c$, which is lower than the physical value. The same set of ensembles is also used for the studies of $B \to D^{(*)}\ell\nu$ form factors [2] and $B \to \pi\ell\nu$ form factors [3]. We use the code set Iroiro++ for the numerical calculations [4].

Going to the left from the point $t_1 = t_2$, we observe an exponential fall-off essentially due to the charm quark propagating between the two currents. This corresponds to the physical decay mode of

$B \to X_c$. At large time separations, the correlation function is dominated by the ground state, which is the D or D^* meson depending on the channel. In the zero-recoil limit, the temporal channel $V_0^\dagger V_0$ is saturated by D (0^-), while the spatial channel $A_k^\dagger A_k$ ($k=1$, circles) is well described by D^* (1^-). At short distances, on the other hand, the correlation function contains contributions from excited states including $D\pi$, $D\pi\pi$, ... *etc.*, which can be seen as a departure from the exponential fall-off. It does not seem to be very significant in this particular case.

The plot in Figure 2 also shows the wrong parity channel $V_k^\dagger V_k$ ($k=1$, squares), which is two orders of magnitude smaller than $A_k^\dagger A_k$. This corresponds to the decays to the states of 1^+, which is likely one of the P-wave states D^{**}.

On the other side ($t_1 > t_2$) we can see a steeper exponential decay corresponding to the unphysical process $B \to bb\bar{c}$.

We finally perform the "Fourier transform" (or Laplace transform) in the time direction as

$$T_{\mu\nu}^{JJ}(\omega,\mathbf{q}) = \int_0^\infty dt e^{\omega t} C_{\mu\nu}^{JJ}(t;\mathbf{q}) \tag{2.2}$$

to obtain the matrix element at an unphysical kinematical point $p_X = (\omega, -\mathbf{q})$, $q = (M_B - \omega, \mathbf{q})$. When $C_{\mu\nu}^{JJ}(t;\mathbf{q})$ is saturated by the ground state, $e^{-m_{D^{(*)}}t}$, the integral gives a pole $1/(m_{D^{(*)}} - \omega)$, which represents the propagator of the $D^{(*)}$ meson.

Figure 3 shows the result of the integral as a function of $\omega = M_B - q_0$. (A derivative of $T(v\cdot q)$ in terms of ω is shown to avoid the ultraviolet divergence.) The variable ω plays the role of the energy injected to the final charmed hadron states. If we assume saturation by the ground state, the amplitude may be written in terms of the corresponding form factors. In the zero-recoil limit, they are defined by

$$\langle D(\mathbf{0})|V_0|B(\mathbf{0})\rangle = 2\sqrt{M_B M_D} h_+(1), \quad \langle D^*(\mathbf{0})|A_k|B(\mathbf{0})\rangle = 2\sqrt{M_B M_D} h_{A_1}(1)\varepsilon_k^*, \tag{2.3}$$

and the structure functions are given as

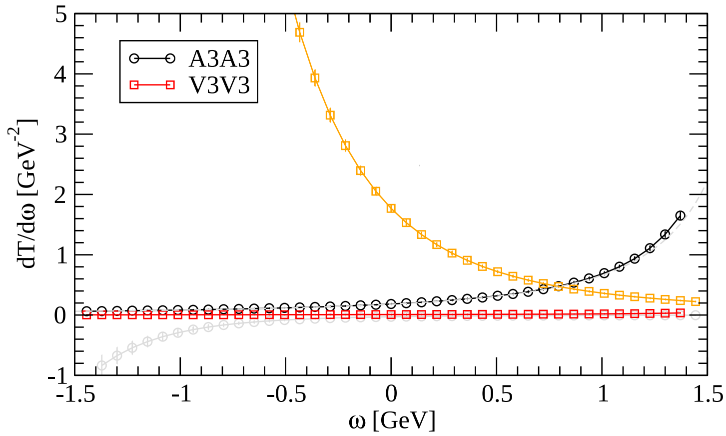
$$T_{00}^{VV}(\omega,\mathbf{0}) = \frac{|h_+(1)|^2}{M_D - \omega}, \quad T_{kk}^{AA}(\omega,\mathbf{0}) = \frac{|h_{A_1}(1)|^2}{M_{D^*} - \omega}. \tag{2.4}$$

The zero-recoil form factors $h_+(1)$ and $h_{A_1}(1)$ are normalized to unity in the heavy quark limit. The correction due to finite m_b and m_c makes them different from 1, and our results are roughly consistent with the known values [5] as well as with the mass dependence studied in [6, 7].

The wrong-parity channel can also be estimated assuming the saturation by the lowest-lying states, which are the P-wave states of spin-parity 1^+: D_1 (2,421 MeV) and D_1^* (2,427 MeV). Using the estimates of [8] for the corresponding form factors $g_{V_1}(1)$ and $f_{V_1}(1)$, we draw the dashed curve in Figure 3 (bottom panel). It is not perfectly consistent with the lattice data, but explains the size of the contribution fairly well.

3. Comparison with heavy quark expansion

In the region of ω away from the hadronic poles, such as the D or D^* meson poles, the heavy quark expansion is applicable. The motion of the b quark inside the initial B meson may be taken into account by the heavy quark expansion [9, 10].



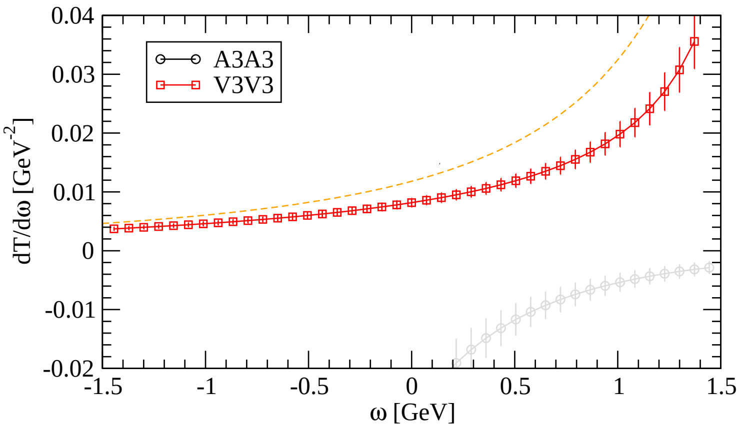


Figure 3: Structure functions $dT_{11}^{JJ}/d\omega$ for both $JJ = A_k^\dagger A_k$ (circles) and $V_k^\dagger V_k$ (squares) channels. The top panel depicts the contributions from the physical (black and red) and unphysical (gray and orange) amplitudes. The bottom panel magnifies the contribution of the wrong parity channel, where only the wrong parity channel (red) is visible other than the unphysical contribution (gray).

A comparison of the lattice data with the estimates within heavy quark expansion is shown in Figure 4. Here, a derivative of $T_{33}(\omega)$ with respect to ω is plotted; the perturbative calculation is valid away from the hadronic resonances, *i.e.* in the small ω region. In this plot, the perturbative results (still at the tree level) are without (dashed curves) and with (solid curves) the $1/m_b^2$ corrections are shown. At the order of $1/m_b^2$, two hadronic parameters appear: $\mu_G^2 = \frac{1}{2M_B}\langle B|\bar{Q}\frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}Q|B\rangle$ and $\mu_\pi^2 = \frac{1}{2M_B}\langle B|\bar{Q}(i\vec{D})^2 Q|B\rangle$. The spin-magnetic term μ_G^2 is well determined by the experimental data for the B-B^* splitting while the kinetic term μ_π^2 is not very accurately known. Here we take two representative values: $\mu_\pi^2 = 0$ (thin curves) and 0.5 GeV^2 (thick curves). The $1/m_b^3$ correction is also known (black curve) but only for the sum of vector and axial vector channels [11].

Agreement between the lattice data and the heavy quark expansion is improved by adding the $1/m_b^2$ corrections especially when we choose $\mu_\pi^2 = 0.5$ GeV^2 but a significant deviation is still visible. Of course, higher order corrections of order α_s and α_s^2 are necessary for more serious comparison. A calculation of the corresponding perturbative coefficients is underway. They may be obtained by performing Cauchy's integral starting from the existing one-loop calculation of the

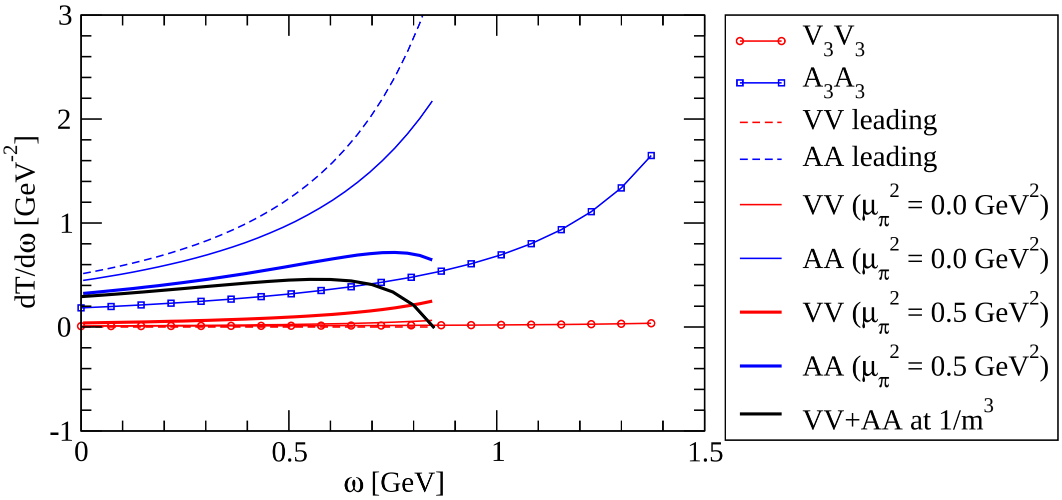


Figure 4: Lattice results for $dT_{33}/d\omega$ for the vector channel (red) and axial-vector channel (blue) are compared with the corresponding estimates from heavy quark expansion. The estimates at the leading order (dashed) and the $1/m_b^2$ order (solid) are shown.

$b \to c\ell\nu$ differential decay rate [12, 13], but one needs to be careful because the contour integral runs toward unphysical kinematical regions where extra cuts show up. Such studies should then be extended to include the $1/m_b^2$ terms, for which the one-loop calculation is available for the physical kinematics [14].

4. Discussions

This work presents our on-going effort to compare the lattice calculation of the structure function relevant to inclusive B meson decays to the corresponding estimate within heavy quark expansion. The goal would be to provide another test of the perturbative expansion applied for B decays and to extract the hadronic parameters, such as μ_π^2, without recourse to fitting to the experimental data.

The idea to calculate the matrix elements relevant to inclusive processes may be applied to other processes. An interesting example is the ℓN scattering (see [15]). This is typically studied in the high energy region (hence the terminology, "deep" inelastic scattering), but the formulation itself is also valid in the low-energy region, which would be a main application of the lattice calculation.

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References

[1] S. Hashimoto, *Inclusive semi-leptonic B meson decay structure functions from lattice QCD*, *PTEP* **2017** (2017), no. 5 053B03 [1703.01881].

[2] **JLQCD** Collaboration, T. Kaneko, Y. Aoki, B. Colquhoun, H. Fukaya and S. Hashimoto, *$B \to D^{(*)} \ell \nu$ form factors from $N_f = 2+1$ QCD with Möbius domain-wall quarks*, in *36th International Symposium on Lattice Field Theory (Lattice 2018) East Lansing, MI, United States, July 22-28, 2018*, 2018. 1811.00794.

[3] B. Colquhoun, S. Hashimoto and T. Kaneko, *Heavy quark scaling of $B \to \pi \ell \nu$ form factors with Möbius domain wall fermions*, in *36th International Symposium on Lattice Field Theory (Lattice 2018) East Lansing, MI, United States, July 22-28, 2018*, 2018. 1811.00227.

[4] G. Cossu, J. Noaki, S. Hashimoto, T. Kaneko, H. Fukaya, P. A. Boyle and J. Doi, *JLQCD IroIro++ lattice code on BG/Q*, 2013. 1311.0084.

[5] **MILC** Collaboration, J. A. Bailey *et. al.*, *$B \to D \ell \nu$ form factors at nonzero recoil and $|V_{cb}|$ from 2+1-flavor lattice QCD*, *Phys. Rev.* **D92** (2015), no. 3 034506 [1503.07237].

[6] S. Hashimoto, A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, *Lattice QCD calculation of anti-B —> D lepton anti-neutrino decay form-factors at zero recoil*, *Phys. Rev.* **D61** (1999) 014502 [hep-ph/9906376].

[7] S. Hashimoto, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, *Lattice calculation of the zero recoil form-factor of anti-B —> D* lepton anti-neutrino: Toward a model independent determination of |V(cb)|*, *Phys. Rev.* **D66** (2002) 014503 [hep-ph/0110253].

[8] F. U. Bernlochner, Z. Ligeti and D. J. Robinson, *Model independent analysis of semileptonic B decays to D^{**} for arbitrary new physics*, *Phys. Rev.* **D97** (2018), no. 7 075011 [1711.03110].

[9] B. Blok, L. Koyrakh, M. A. Shifman and A. I. Vainshtein, *Differential distributions in semileptonic decays of the heavy flavors in QCD*, *Phys. Rev.* **D49** (1994) 3356 [hep-ph/9307247]. [Erratum: Phys. Rev.D50,3572(1994)].

[10] A. V. Manohar and M. B. Wise, *Inclusive semileptonic B and polarized Lambda(b) decays from QCD*, *Phys. Rev.* **D49** (1994) 1310–1329 [hep-ph/9308246].

[11] M. Gremm and A. Kapustin, *Order 1/m(b)**3 corrections to B –> X(c) lepton anti-neutrino decay and their implication for the measurement of Lambda-bar and lambda(1)*, *Phys. Rev.* **D55** (1997) 6924–6932 [hep-ph/9603448].

[12] M. Trott, *Improving extractions of |V(cb)| and m(b) from the hadronic invariant mass moments of semileptonic inclusive B decay*, *Phys. Rev.* **D70** (2004) 073003 [hep-ph/0402120].

[13] V. Aquila, P. Gambino, G. Ridolfi and N. Uraltsev, *Perturbative corrections to semileptonic b decay distributions*, *Nucl. Phys.* **B719** (2005) 77–102 [hep-ph/0503083].

[14] A. Alberti, T. Ewerth, P. Gambino and S. Nandi, *Kinetic operator effects in $\bar{B} \to X_c l \nu$ at $O(\alpha_s)$*, *Nucl. Phys.* **B870** (2013) 16–29 [1212.5082].

[15] A. J. Chambers, R. Horsley, Y. Nakamura, H. Perlt, P. E. L. Rakow, G. Schierholz, A. Schiller, K. Somfleth, R. D. Young and J. M. Zanotti, *Nucleon Structure Functions from Operator Product Expansion on the Lattice*, *Phys. Rev. Lett.* **118** (2017), no. 24 242001 [1703.01153].

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