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Scattering in a Euclidean formulation of relativistic quantum mechanics

W. N. Polyzou[*†]
University of Iowa
E-mail: polyzou@uiowa.edu

Gordon Aiello
University of Utah - Asia campus
E-mail: gordon.aiello@utah.edu

Philip Kopp
University of Iowa
E-mail: philip-kopp@uiowa.edu

Abstract: A Euclidean formulation of relativistic quantum mechanics is discussed. Representations of the Hilbert space inner product and Poincaré generators are all expressed in terms of Euclidean space-time variables. The formulation does not require analytic continuation and can be used to directly calculate scattering observables. A toy model is used to demonstrate the feasibility of performing scattering calculations using the suggested computational methods.

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[*]Speaker.
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1. Euclidean relativistic quantum theory

Relativistic treatments of quantum mechanics are needed to investigate the dynamics and structure of physical systems on scales smaller than the Compton wavelength of the constituent particles of the system. This is particularly relevant for understanding strong-interaction dynamics and structure at sub-nucleon distance scales. While quantum field theory provides an elegant and consistent treatment of relativity and quantum mechanics, it is difficult formulate controlled approximations that preserve relativistic invariance in kinematic regions where the interactions are strong. Finite degree of freedom models that provide a consistent treatment of relativity and quantum mechanics can also be formulated, however models that also satisfy cluster properties normally require generating complicated frame-dependent many-body interactions [1].

The Euclidean axioms [2] of quantum field theory have the feature that the locality axiom is logically independent of the other axioms. This suggests an alternate path to formulate relativistic quantum models with a finite number of degrees of freedom that retain all of the other desirable properties of the field theory. In addition, in the Euclidean case, the Green's functions of the field theory are formally related to the Lagrangian of the theory by a Euclidean path integral. This can be used to include dynamical constraints on models that are motivated by local field theory.

One of the surprising properties of the Euclidean construction is that the quantum mechanical Hilbert space and all of the Poincaré generators can be represented in a purely Euclidean framework without performing an explicit analytic continuation in Euclidean time. In this work I discuss this construction along with how it can be used to perform scattering calculations that are normally not considered natural in a Euclidean framework.

The dynamical input to the Euclidean formulation of relativistic quantum mechanics is a collection of N-point Euclidean Green functions,

$$\{G_{E:mk}(x_m,\cdots,x_1;y_1,\cdots,y_k)\} \qquad m+k=N \tag{1.1}$$

where $\{x_i\}$ are final Euclidean coordinates and $\{y_i\}$ are initial Euclidean coordinates. In a local theory there is only one Green function for any pair $m+k=N$, however if locality is not required there can be several for each N. These Green functions must be Euclidean invariant or covariant, symmetric or antisymmetric with respect to interchange of the initial or final Euclidean coordinates among themselves, satisfy cluster properties, be tempered distributions, and satisfy reflection positivity.

The elements of a relativistic quantum theory are (1) a Hilbert space, (2) a unitary representation of the Poincaré group satisfying (3) cluster properties and (4) a spectral condition. The Euclidean reconstruction discussed below has all of these properties.

A dense set of elements of the Hilbert space are finite sequences of functions of Euclidean space-time variables

$$\psi(x) := (\psi_1(x_{11}), \psi_2(x_{21},x_{22}),\cdots) \tag{1.2}$$

satisfying the positive relative-time support condition

$$\psi_n(x_{n1},x_{n2},\cdots,x_{nn}) = 0 \qquad \text{unless} \qquad 0 < x^0_{n1} < x^0_{n2} < \cdots < x^0_{nn}. \tag{1.3}$$

The symmetry properties of the Green's functions imply that as long as the Euclidean time support of these functions are satisfied for some ordering, the coordinates can be relabeled so (1.3) holds.

The physical Hilbert space inner product is

$$\langle\psi|\phi\rangle_M = (\theta\psi, G_E\phi)_E = \sum_{nk}\int d^{4n}x d^{4k}y \psi_n^*(\theta x_{n1}, \theta x_{n2}, \cdots, \theta x_{nn})\times$$

$$G_{E:nk}(x_{nn}, \cdots, x_{1n}; y_{1k}, \cdots, y_{kk})\phi_k(y_{k1}, y_{k2}, \cdots, y_{kk}) \tag{1.4}$$

where θ is the Euclidean time reflection operator $\theta x := \theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x})$.

Reflection positivity is the condition that $\langle\psi|\psi\rangle_M \geq 0$. This ensures that (1.4) has the properties of a Hilbert space inner product.

Relativistic invariance follows from the condition that the determinant of the following 2×2 matrices,

$$X_m := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad X_e := \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}, \tag{1.5}$$

is preserved under the linear transformation $X \to X' = AXB^t$ for $\det(A) = \det(B) = 1$. These transformations, which preserve the Euclidean and Lorentz line elements

$$\mathbf{det}(X_M) = t^2 - \mathbf{x}^2 \qquad \mathbf{det}(X_E) = -(\tau^2 + \mathbf{x}^2), \tag{1.6}$$

define complex Lorentz and complex orthogonal transformations. It follows that the real Euclidean transformations form a subgroup of the complex Poincaré group. This relation be exploited to relate generators of the 4-dimensional Euclidean group to generators of the Poincaré group.

In the spinless case the relations lead to the following representation of the Poincaré generators on the Euclidean representation of the Hilbert space (on each component of ψ):

$$H\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = \sum_{k=1}^{n} \frac{\partial}{\partial x^0_{nk}} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \tag{1.7}$$

$$\mathbf{P}\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = -i\sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{nk}} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \tag{1.8}$$

$$\mathbf{J}\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = -i\sum_{k=1}^{n} \mathbf{x}_{nk} \times \frac{\partial}{\partial \mathbf{x}_{nk}} \psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) \tag{1.9}$$

$$\mathbf{K}\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}) = \sum_{k=1}^{n} (\mathbf{x}_{nk}\frac{\partial}{\partial x^0_{nk}} - x^0_{nk}\frac{\partial}{\partial \mathbf{x}_{nk}})\psi_n(x_{n1}, x_{n2}, \cdots, x_{nn}), \tag{1.10}$$

which are the generators of time translation, space translation, rotations and rotationless Lorentz boosts respectively. These generators satisfy the Poincaré commutation relations, and are formally Hermitian with respect to the inner product (1.4). In these expressions all of the variables and derivatives are Euclidean. If the Green functions satisfy cluster properties then these generators also satisfy cluster properties. In addition, if the Green functions satisfy reflection positivity, then it follows that the Hamiltonian above satisfies a spectral condition.

2. Scattering theory

Given a Hilbert space and a Hamiltonian satisfying cluster properties, scattering observables can be defined and calculated using the same methods that are used in non-relativistic scattering theory. In a quantum theory the S-matrix is the probability amplitude for scattering from an initial state to a final state, $S_{fi} := \langle \psi_+ | \psi_- \rangle$. The scattering state vectors can be expressed in terms of free state vectors $|\psi_{0\pm}\rangle$ that are seen asymptotically in the detector after the collision or in the beam and target before the collision, $|\psi_\pm\rangle = \Omega_\pm |\psi_{0\pm}\rangle$. Wave operators $\Omega_\pm$ for multichannel scattering have the general structure

$$\Omega_\pm |\psi_{0\pm}\rangle = \lim_{t\to\pm\infty} \sum e^{iHt} \underbrace{\prod_n |\phi_n, \mathbf{p}_n, \mu_n\rangle \underbrace{e^{-ie_n t}}_{e^{-iH_0 t}} \underbrace{f_n(\mathbf{p}_n, \mu_n)}_{|\psi_{0\pm}\rangle} d\mathbf{p}_n}_{J} =: \lim_{t\to\pm\infty} e^{iHt} J e^{-iH_0 t} |\psi_{0\pm}\rangle \tag{2.1}$$

where $|\phi_n, \mathbf{p}_n, \mu_n\rangle$ represents an elementary or bound system with total momentum $\mathbf{p}_n$, magnetic quantum number μ_n, and energy e_n. $f_n(\mathbf{p}_n, \mu_n)$ represents a localized wave packet with the mean momentum of the particle or bound sub-system. The mapping J [1] above is called an injection operator. It is a mapping from a Hilbert space of scattering asymptotes to the physical Hilbert space.

Haag-Ruelle scattering is the field-theory version of the above. In the field theory case the operator J is expressed as a suitably symmetrized product of operators that create single-particle states out of the vacuum. While this requires solving the one-body problem, the benefit is that the limits in (2.3) ([3][4][5]) are strong limits.

In the Euclidean case for a two-particle initial state a candidate for the injection operator J is [6]

$$J : \langle x_1 | \phi_1, \mathbf{p}_1 \rangle \langle x_2 | \phi_2, \mathbf{p}_2 \rangle = h_1(\nabla_1^2)\delta(x_1^0 - \tau_1) h_2(\nabla_2^2)\delta(x_2^0 - \tau_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2}, \qquad \tau_2 > \tau_1 \tag{2.2}$$

where the Euclidean Laplacians, ∇_i^2, are the mass squared operators for each particle or subsystem, and the $h_i(m^2)$ are smooth functions that are 1 when m is the mass of the i^{th} particle (or subsystem) and 0 on the rest of the mass spectrum. The delta function in the Euclidean times ensures the time-support condition.

The reason that this is only a candidate is because $h_2(\nabla^2)$ is not analytic in ∇^2, so it could transform a wave function satisfying the relative-time support conditions to one that does not, leading to a range that is out of Hilbert space. This will not happen if polynomials in ∇^2 are complete in this space. To establish this, note that a sufficient condition for completeness is that the Stieltjes moments

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} \rho(m) m^{2n} dm \tag{2.3}$$

where $\tau = \tau_1 + \tau_2 > 0$ satisfy Carleman's condition [7]

$$\sum_{n=0}^{\infty} |\gamma_n|^{-\frac{1}{2n}} > \infty. \tag{2.4}$$

This will hold as long as the Lehmann weight $\rho(m)$ in (2.6) is polynomially bounded [6]. This ensures that the Haag-Ruelle functions $h(\nabla^2)$ can be approximated by polynomials.

A sufficient condition for the convergence of the limit (2.3) that defines the scattering wave operators is the Cook condition [8], which in the Euclidean representation for 2-2 scattering has the form

$$\int_a^\infty \|(HJ - JH_0)e^{\mp iH_0t}|\psi_0\rangle\|_M dt < \infty \tag{2.5}$$

where a is a constant and

$$\|(HJ - JH_0)\Phi e^{\mp iH_0t}|\psi_0\rangle\|_M^2 := (\psi_0 e^{\pm iH_0t}(J^\dagger H - H_0J^\dagger)\theta G_E(HJ - JH_0)e^{\mp iH_0t}|\psi_0)_E. \tag{2.6}$$

The important observation is that because the injection operator asymptotically projects on one-body states, the definitions imply that the contribution to the integral (2.9) due to the disconnected parts of the Green function vanishes [9][6]. All that remains is the connected part, which, if the spectrum has a mass gap, is expected to fall off like t^{-3} for large t. This suggest that the scattering problem is mathematically well defined in this Euclidean representation.

3. Computational considerations

There are a number of tricks that can facilitate the computation of scattering observables in the Euclidean case [10]. The first is to use the invariance principle [11] which implies

$$\lim_{t\to\pm\infty} e^{iHt}Je^{-iH_0t}|\psi\rangle = \lim_{t\to\pm\infty} e^{if(H)t}Je^{-if(H_0)t}|\psi\rangle \qquad \text{for} \qquad f(x) = -e^{-\beta x}. \tag{3.1}$$

This gives

$$\lim_{t\to\pm\infty} e^{iHt}Je^{-iH_0t}|\psi\rangle = \lim_{n\to\infty} e^{\mp ine^{-\beta H}}Je^{i\pm ne^{-\beta H_0}}|\psi\rangle. \tag{3.2}$$

Since $\sigma(e^{-\beta H}) \in [0,1]$, this means that $e^{\mp ine^{-\beta H}}$ can be uniformly approximated by a polynomial in $e^{-\beta H}$ where $\beta > 0$ is a parameter that can be adjusted for convergence. The inequality

$$|e^{inx} - P(x)| < \varepsilon \qquad x \in [0,1] \tag{3.3}$$

leads to the uniform operator inequality

$$|||e^{ine^{-\beta H}} - P(e^{-\beta H})||| < \varepsilon \tag{3.4}$$

with the same ε in (3.3) and (3.4). This is useful in the Euclidean case because $e^{-n\beta H}$ simply translates the Euclidean time to the right by $n\beta$.

This method can be used to calculate sharp-momentum transition matrix elements. The approximate expression has the form

$$\langle \mathbf{k}_f|T(E+i0)|\mathbf{k}_i\rangle \approx \langle \psi_{f0}|(J^\dagger H - H_0J^\dagger)P(e^{-\beta H})Je^{ine^{-\beta H_0}}|\psi_i\rangle \tag{3.5}$$

where the initial and final wave packets must be sufficiently narrow, β must be chosen based on the energy scale, n must be sufficiently large and the polynomial $P(x)$ must accurately approximate e^{2inx} on $[0,1]$. These approximations must be done in the proper order. (1) First choose a sufficiently narrow wave packet in momentum space. (2) Choose β based on the energy scale. (3) For the choice of β and wave packet choose n large enough for convergence. (4) Given the n from step 3 construct a polynomial approximation to e^{2inx} for $x \in [0,1]$.

This computational method was tested for scattering from a separable potential with the range of a pion exchange interaction and a strength chosen to produce a bound state with the binding energy of a Deuteron:

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = -\frac{\lambda}{(m_\pi^2+\mathbf{k}'^2)(m_\pi^2+\mathbf{k}^2)}. \tag{3.6}$$

While this is not a Euclidean calculation, it is exactly solvable and provides a means to precisely test the use of the invariance principle, narrow wave packets and the polynomial approximation to $e^{ine^{-\beta H}}$ in the computation of sharp-momentum transition matrix elements. In this calculation the sharp-momentum transition matrix elements were extracted directly from S-matrix elements rather than using (3.5).

The result of this exploration was that convergence was achieved for a wide range of momenta, between 50 MeV and 2 GeV [10]. The parameters of the approximations were chosen to get better than a 1% error in the scattering amplitude. The largest source of error (by far) was the wave packet width. For wave packets chosen to give an error better than 1% the n values were a few hundred and the degree of the polynomial was slightly higher. The polynomials were accurately approximated using a Chebyshev expansion. The parameter β was chosen so βE was a number of order unity. The results are shown in the table

Table 1: Sharp-momentum transition matrix elements

$k0$	Real T	Im T	% error
0.05	2.18499e-1	-1.03160e+0	0.0982
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.2	-1.01512e-1	-4.61420e-2	0.0981
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.4	-1.39007e-2	-1.44974e-3	0.0997
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.6	-3.34091e-3	-1.24434e-4	0.0952
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.8	-1.14188e-3	-1.93605e-5	0.0965
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.0	-4.85708e-4	-4.37769e-6	0.0967
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.2	-2.39235e-4	-1.27439e-6	0.0968
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.4	-1.30818e-4	-4.44560e-7	0.0955
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.6	-7.73390e-5	-1.77573e-7	0.0992
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.8	-4.85672e-5	-7.87802e-8	0.0956
1.9	-3.92110e-5	-5.42037e-8	0.0967
2.0	-3.20000e-5	-3.80004e-8	0.0966

These calculations support the possibility of directly performing scattering calculations in a Euclidean representation. The main challenge for using this method is how to include dynamics. The physics is in the Euclidean Green functions; these have to be computed or modeled. Reflection positivity puts strong constraints on these models. A structure theorem for reflection positive distributions is an important goal of this research program. This is needed to identify acceptable models that have Hilbert space inner products. For a theory like QCD both cluster properties and reflection positivity are limited to initial and final states being local color singlets. For this reason a formulation directly in terms of gauge-invariant degrees of freedom is another important goal.

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