

The critical endpoint in the 2d U(1) gauge-Higgs model at topological angle $\theta = \pi$

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We study 2d U(1) gauge Higgs systems with a θ -term. For properly discretizing the topological charge as an integer we introduce a mixed group- and algebra-valued discretization (MGA scheme) for the gauge fields, such that the charge conjugation symmetry at $\theta = \pi$ is implemented exactly. The complex action problem from the θ -term is overcome by exactly mapping the partition sum to a worldline/worldsheet representation. Using Monte Carlo simulation of the worldline/worldsheet representation we study the system at $\theta = \pi$ and show that as a function of the mass parameter the system undergoes a phase transition. Determining the critical exponents from a finite size scaling analysis we show that the transition is in the 2d Ising universality class. We furthermore study the U(1) gauge Higgs systems at $\theta = \pi$ also with charge 2 matter fields, where an additional \mathbb{Z}_2 symmetry is expected to alter the phase structure. Our results indicate that for charge 2 a true phase transition is absent and only a rapid crossover separates the large and small mass regions.

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1. Introduction

One of the many exciting features of quantum field theories is that they allow for the introduction of topological terms. Such terms may alter the symmetry structure of the system and consequently give rise to new physics. Since topological terms capture global properties of a system, non-perturbative techniques are needed to study the corresponding physics.

In principle lattice field theory is a suitable non-perturbative approach, as long as two key challenges can be overcome: The lattice discretization of topological terms is not straightforward due to the absence of the notion of smoothness of the fields. The second challenge is the fact that topological terms typically give rise to a complex action, such that the Boltzmann factor cannot be used as a probability weight in a Monte Carlo simulation ("complex action problem").

Here we sketch an approach that solves the two challenges for 2d U(1) gauge Higgs models, which constitute a class of systems interesting as toy models for high energy physics, as well as in condensed matter theory. The approach is based on a mixed group- and algebra-valued lattice discretization of the gauge fields (MGA discretization) that correctly implements the symmetries related to the U(1) θ -term and combine this with an exact mapping to a worldline/worldsheet representation that solves the complex action problem.

To be more specific, the simplest model we study with the new discretization approach is the 2d U(1) gauge Higgs model which in the continuum is described by the action

$$S = \int_{\mathbb{T}^2} d^2x \left(|D_\mu \phi|^2 + m^2 |\phi|^2 + \lambda |\phi|^4 + \frac{1}{2e^2} F_{12}^2 + i\theta \frac{1}{2\pi} F_{12} \right). \quad (1.1)$$

$\phi(x) \in \mathbb{C}$ denotes the charged scalar field and $A_\mu(x) \in \mathbb{R}$ the U(1) gauge field. $D_\mu = \partial_\mu + iA_\mu$ is the U(1) covariant derivative and $F_{12} = \partial_1 A_2 - \partial_2 A_1$ denotes the field strength tensor. m is the bare mass, λ the coupling for the quartic self-interaction, e the electric charge and θ the topological angle. We study the theory on a 2-torus \mathbb{T}^2 , where (for sufficiently smooth fields) the flux of F_{12} is quantized in integer units of 2π , such that the topological charge $Q_{top} = \frac{1}{2\pi} \int d^2x F_{12}$ is an integer. The partition function of the system is given by $Z = \int D[\phi] \int D[A] e^{-S[\phi, A, \theta]}$.

Charge conjugation transforms the fields as $\phi(x) \rightarrow \phi(x)^*$, $A_\mu(x) \rightarrow -A_\mu(x)$, and the field strength changes its sign $F_{12} \rightarrow -F_{12}$ under this transformation. The gauge field and the matter field parts of the action remain invariant, while the topological charge Q_{top} changes its sign. Since Q_{top} is an integer, charge conjugation is a symmetry not only at the trivial value $\theta = 0$, but also at $\theta = \pi$. This \mathbb{Z}_2 symmetry is expected to be intact for small mass m , but broken at large m . It has been conjectured [1] that at intermediate mass values there is a second order phase transition in the universality class of the 2d Ising model. Establishing this conjecture from an ab-initio lattice calculation was one of the goals of this project, where we use the new MGA scheme to discretize the topological charge as an integer (for more details and motivation see [2]). Thus charge conjugation symmetry at $\theta = \pi$ is implemented exactly and a mapping to a worldline/worldsheet representation with only real and positive weights solves the complex action problem.

Also more general systems of the type (1.1) are of interest and need to be studied in a suitable lattice formulation – in particular matter fields with a higher charge and also generalizations to more than two flavors. New symmetries appear and the phase structure changes. As a preview to future work for this type of systems we present first results for the U(1) gauge Higgs model at $\theta = \pi$ but with scalar fields of charge 2.

2. Mixed group- and algebra-valued lattice discretization (MGA discretization)

We begin the discussion of the MGA discretization with the lattice action $S_H[\phi, U]$ for the matter field (the lattice spacing a is set to $a = 1$),

$$S_M[\phi, U] = \sum_{x \in \Lambda} \left[M |\phi_x|^2 + \lambda |\phi_x|^4 - \sum_{\mu=1}^2 \left(\phi_x^* U_{x,\mu} \phi_{x+\hat{\mu}} + c.c. \right) \right], \quad (2.1)$$

where the mass parameter M is related to the bare mass m via $M = 4 + m^2$. The gauge fields couple via the group-valued link variables $U_{x,\mu} \in U(1)$. We parameterize the link variables in the form $U_{x,\mu} = e^{iA_{x,\mu}}$, with the algebra-valued lattice gauge fields $A_{x,\mu} \in \mathbb{R}$.

The gauge field action and the θ -term will now be discretized with the algebra-valued fields $A_{x,\mu}$. For this step we note that due to the use of the group-valued link variables $U_{x,\mu}$, the matter field action (2.1) is invariant under the shifts

$$A_{x,\mu} \rightarrow A_{x,\mu} + 2\pi k_{x,\mu}, \quad k_{x,\mu} \in \mathbb{Z}, \quad (2.2)$$

and the discretization of the gauge field action and topological term will have to take into account this invariance. Using the group-valued fields $A_{x,\mu}$, a natural definition of the field strength is $F_{x,12} = A_{x+\hat{1},2} - A_{x,2} - A_{x+\hat{2},1} + A_{x,1} \equiv F_x$, that is assigned to the plaquettes of the lattice, which in 2d can be labeled by the coordinate x of the lower left corner of the plaquette. However, this definition of F_x is not invariant under the shifts (2.2), where it transforms as $F_x \rightarrow F_x + 2\pi(k_{x+\hat{1},2} - k_{x,2} - k_{x+\hat{2},1} + k_{x,1})$, i.e., F_x is shifted by multiples of 2π . In order to recover invariance we implement the following discretization strategy: The continuum field strength $F_{12}(x)$ is replaced by $A_{x+\hat{1},2} - A_{x,2} - A_{x+\hat{2},1} + A_{x,1} + 2\pi n_x$, and the plaquette-based auxiliary variables n_x are summed over all integers. Obviously this construction recovers the invariance under the shifts (2.2).

Using this prescription we can write the Boltzmann factor $B_G[A]$ for the algebra-valued gauge field, which takes into account the gauge field action and the θ -term as ($\beta \equiv 1/e^2$)

$$B_G[A] = \prod_{x \in \Lambda} \sum_{n_x \in \mathbb{Z}} e^{-\frac{\beta}{2}(F_x + 2\pi n_x)^2 - i\frac{\theta}{2\pi}(F_x + 2\pi n_x)} = \sum_{\{n\}} e^{-\frac{\beta}{2} \sum_x (F_x + 2\pi n_x)^2 - i\theta \sum_x n_x}. \quad (2.3)$$

In the second step we have introduced $\sum_{\{n\}} \equiv \prod_x \sum_{n_x \in \mathbb{Z}}$ for the sum over all configurations of the auxiliary variables n_x . Furthermore, in that step we have written the product \prod_x as a sum over x in the exponent and used the fact that $\sum_x F_x = 0$ for a lattice with periodic boundary conditions. Consequently this step identifies the topological charge as $Q_{top} = \sum_x n_x$, which obviously is an integer, such that the first criterion for our lattice discretization is obeyed. We remark that at $\theta = 0$ our Boltzmann factor (2.3) reduces to the well known Villain form [3].

Our MGA lattice discretization is completed by combining the building blocks into the lattice path integral for the partition sum,

$$Z = \int D[A] B_G[A] Z_M[U], \quad Z_M[U] = \int D[\phi] e^{-S_M[\phi, U]}, \quad (2.4)$$

with the corresponding measures defined as $\int D[\phi] = \prod_x \int_{\mathbb{C}} \frac{d\phi_x}{2\pi}$ and $\int D[A] = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi}$.

We conclude our discussion by showing that at $\theta = \pi$ the partition sum (2.4) with matter field action (2.1) and Boltzmann factor (2.3) is indeed invariant under charge conjugation. Charge

conjugation is implemented as in the continuum via $\phi_x \rightarrow \phi_x^*$ and $A_{x,\mu} \rightarrow -A_{x,\mu}$. The latter implies $U_{x,\mu} \rightarrow U_{x,\mu}^*$, which together with $\phi_x \rightarrow \phi_x^*$ ensures the invariance of $Z_M[U]$. For the field strength we find again $F_x \rightarrow -F_x$, which in the quadratic term $(F_x + 2\pi n_x)^2$ of the gauge field Boltzmann factor (2.3) can be compensated by transforming also the auxiliary variables via $n_x \rightarrow -n_x$. This implies for the transformation of the Boltzmann factor with the topological charge $e^{-i\theta \sum_x n_x} \rightarrow e^{+i\theta \sum_x n_x}$ which is invariant for $\theta = \pi$ (and $\theta = 0$).

3. Representation with worldlines and worldsheets

Having found an approach that discretizes the topological charge as an integer and thus exactly implements the charge conjugation symmetry at $\theta = \pi$, we now come to solving the complex action problem by transforming the partition sum to a worldline/worldsheet representation. This transformation has been discussed for U(1) gauge Higgs systems in, e.g., [4, 5] and for the MGA discretization is derived in detail in [2]. Thus we here only provide a short sketch of the derivation and mainly discuss the final form we use for the numerical simulation.

The partition function $Z_M[U]$ for the matter field in a background U of the compact link variables is clearly a gauge invariant functional. The only gauge invariant quantities one can form with the link variables correspond to products of link variables $U_{x,\mu}$ placed along closed loops. Such closed loops can be described by integer valued flux variables $j_{x,\mu} \in \mathbb{Z}$ assigned to the links of the lattice. The value $j_{x,\mu}$ for the flux indicates how often a link is run through by loops, where negative values correspond to fluxes in negative direction. The requirement that the loops are closed is implemented by enforcing zero divergence $\vec{\nabla} \cdot \vec{j}_x = \sum_\mu [j_{x,\mu} - j_{x-\hat{\mu},\mu}] = 0$ at every site of the lattice. The contribution of a link variable $U_{x,\mu}$ is then simply given by $(U_{x,\mu})^{j_{x,\mu}}$. Thus the matter field partition sum can be written as

$$Z_M[U] = \sum_{\{j\}} W_M[j] \prod_x \delta(\vec{\nabla} \cdot \vec{j}_x) \prod_{x,\mu} (U_{x,\mu})^{j_{x,\mu}} = \sum_{\{j\}} W_H[j] \prod_x \delta(\vec{\nabla} \cdot \vec{j}_x) \prod_{x,\mu} e^{iA_{x,\mu} j_{x,\mu}}, \quad (3.1)$$

where we have defined $\sum_{\{j\}} = \prod_{x,\mu} \sum_{j_{x,\mu} \in \mathbb{Z}}$ to denote the sum over all configurations of the flux variables $j_{x,\mu}$. The zero divergence condition is implemented by a product of Kronecker deltas (here denoted by $\delta(n) \equiv \delta_{n,0}$) at all sites. The gauge field dependence is the product of $(U_{x,\mu})^{j_{x,\mu}}$ over all links, where in the second step in (3.1) we have already inserted $U_{x,\mu} = e^{iA_{x,\mu}}$.

The configurations of the flux variables $j_{x,\mu}$ come with real and positive weight factors $W_H[j]$ that can be determined by an expansion of the nearest neighbor Boltzmann factors of the matter fields and a subsequent integration over the matter fields ϕ_x (see [2, 4, 5] for their derivation).

The next step for finding the worldline/worldsheet representation is to represent the gauge field Boltzmann factor $B_G[A]$ by its Fourier transform. Since the Boltzmann factor $B_G[A]$ is 2π -periodic in the F_x , the Fourier representation will depend on the gauge fields in the form $\prod_x e^{iF_x p_x}$ where the Fourier modes $p_x \in \mathbb{Z}$ are assigned to the plaquettes and referred to as "plaquette occupation numbers". Since the Boltzmann factor (2.3) is Gaussian, the Fourier transforms can be computed in closed form, such that one finds (see, e.g., [2] for details)

$$B_G[A] = \sum_{\{p\}} e^{-\frac{1}{2\beta} \sum_x \left(p_x + \frac{\theta}{2\pi}\right)^2} \prod_x e^{iF_x p_x} = \sum_{\{p\}} e^{-\frac{1}{2\beta} \sum_x \left(p_x + \frac{\theta}{2\pi}\right)^2} \prod_x e^{iA_{x,1}[p_x - p_{x-2}] e^{-iA_{x,2}[p_x - p_{x-1}]}. \quad (3.2)$$

Here $\sum_{\{p\}} = \prod_x \sum_{p_x \in \mathbb{Z}}$ denotes the sum over all configurations of the plaquette occupation numbers. In the second step we have reorganized the gauge field dependence already in terms of the non-compact gauge fields $A_{x,\mu}$. The configurations of the plaquette occupation numbers come with a Gaussian weight, and the topological angle determines the position of the center of the Gaussian.

In a final step we integrate over the gauge fields $A_{x,\mu}$ at all links. The factors $e^{iA_{x,\mu} j_{x,\mu}}$ from the matter field partition sum $Z_M[U]$ and the factors $e^{iA_{x,1}[p_x - p_{x-2}]} e^{-iA_{x,2}[p_x - p_{x-1}]}$ from the Boltzmann factor $B_G[A]$ are linked together by this integration to a new set of constraints that now live on the links of the lattice. We thus obtain the final form of the worldline/worldsheet representation,

$$Z = \sum_{\{p,j\}} e^{-\frac{1}{2\beta} \sum_x \left(p_x + \frac{\theta}{2\pi}\right)^2} W_H[j] \prod_x \delta(\vec{\nabla} \cdot \vec{j}_x) \delta(j_{x,1} + p_x - p_{x-2}) \delta(j_{x,2} - p_x + p_{x-1}). \quad (3.3)$$

The partition function is a sum over all configurations of the plaquette occupation numbers p_x and the flux variables $j_{x,\nu}$. They come with real and positive weight factors and two types of constraints: The zero divergence constraint that ensures conservation of the matter flux, as well as link-based constraints that enforce the vanishing combined flux from the matter flux $j_{x,\mu}$ on a link and the neighboring plaquettes that contain the link. One finds that admissible configurations are closed loops of matter flux which are filled with occupied plaquettes, i.e., plaquettes with $p_x \neq 0$ form patches (2d surfaces) that are bounded by matter flux. Since all weights in (3.3) are real and positive, numerical simulations can be done in terms of the flux variables $j_{x,\mu}$ and the plaquette occupation numbers p_x and the complex action problem is solved. Suitable update schemes that properly take into account the constraints are discussed in [4].

The observables we consider here are the expectation value of the topological charge density $q = Q_{top}/V$, where V is the volume of the lattice and the corresponding susceptibility χ_t . They correspond to the first and second derivative of $\ln Z$ with respect to the topological angle θ . We can evaluate these derivatives using the dual form (3.3) of the partition sum and obtain the two observables in terms of the first and second moments of the plaquette occupation numbers. For details of this derivation and the technical aspects of the numerical simulation we refer to [2]. For a lattice study of the same system based on the Wilson discretization see [6].

4. Numerical results

As we have outlined in the introduction, we aim at studying the system at $\theta = \pi$ as a function of the mass parameter. At small $M = 4 + m^2$ one expects that charge conjugation symmetry is intact, while at large M it should be broken [1]. For some critical value M_c a second order phase transition in the 2d Ising universality class is expected. A suitable infinite volume order parameter is the expectation value $\langle q \rangle$ of the topological charge (q is odd under charge conjugation), which on a finite lattice needs to be replaced by $\langle |q| \rangle$.

In the left hand side plot of Fig. 1 we show $\langle |q| \rangle$ at $\theta = \pi$ as a function of M and indeed we observe the transition from a symmetric phase with $\langle |q| \rangle = 0$ at small M into a broken phase with $\langle |q| \rangle \neq 0$ at sufficiently large M . The corresponding susceptibility in the rhs. plot of Fig. 1 shows maxima that grow with the volume, which is an indication of a phase transition.

In [2] we implemented a detailed finite volume scaling analysis in order to determine the critical exponents for the transition. The results are $\nu = 1.003(11)$, $\beta = 0.126(7)$ and $\gamma = 1.73(7)$,

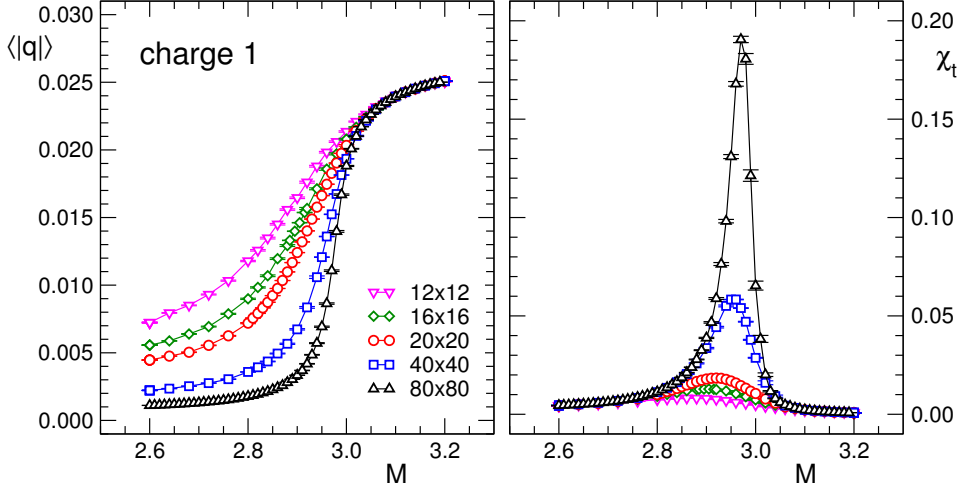


Figure 1: The topological charge density $\langle |q| \rangle$ and the susceptibility χ_t at $\theta = \pi$ for different volumes. We show the charge 1 results for $\beta = 3.0$, $\lambda = 0.5$ and plot the observables as a function of $M = 4 + m^2$.

which are in good agreement with the 2d Ising exponents $\nu = 1$, $\beta = 0.125$ and $\gamma = 1.75$. Thus by combining our MGA discretization with the worldline/worldsheet representation we were able to establish the conjectured critical point in the 2d Ising universality class.

We already remarked that in the future we will consider generalizations of the simple $U(1)$ gauge Higgs system with a topological term we have studied so far. Models with more than one flavor or different charges have different symmetries and may have altered anomaly matching conditions [7, 8, 9] such that the phase structure will be changed. As a first step towards this direction we here show results for the $U(1)$ gauge Higgs model with scalar fields of charge 2. The MGA discretization proceeds as for charge 1, but now in the action (2.1) for the matter fields the link variables $U_{x,\mu} = e^{iA_{x,\mu}}$ are replaced by $(U_{x,\mu})^2 = e^{i2A_{x,\mu}}$. In this case one has an additional \mathbb{Z}_2

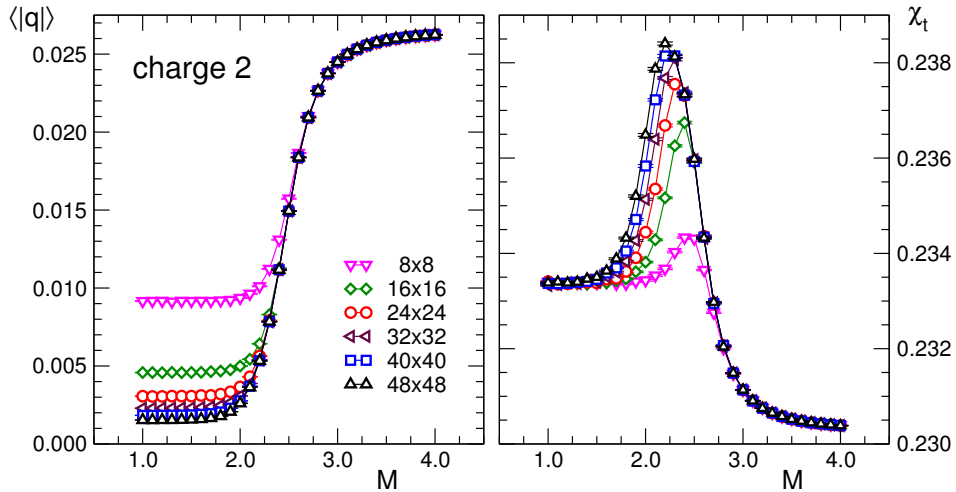


Figure 2: The topological charge density $\langle |q| \rangle$ and the susceptibility χ_t at $\theta = \pi$ for different volumes. We show the charge 2 results for $\beta = 3.0$, $\lambda = 0.5$ and plot the observables as a function of $M = 4 + m^2$.

symmetry under $U_{x,\mu} \rightarrow -U_{x,\mu}$, which in terms of the $A_{x,\mu}$ is given by $A_{x,\mu} \rightarrow A_{x,\mu} + \pi$.

While it looks like the charge 2 model is a simple rescaling of the charge 1 theory, the two are in fact different. Namely the charge 2 model has a \mathbb{Z}_2 center symmetry with an order parameter being the charge 1 Wilson loop. There is a mixed anomaly between the \mathbb{Z}_2 center symmetry and the charge conjugation symmetry. This is evident from the fact that gauging the \mathbb{Z}_2 center turns the theory into the usual charge 1 theory at topological angle $\theta = \pi/2$, which does not have charge conjugation symmetry (see [7, 8, 9] and references therein for related discussions).

As a consequence the anomaly between the \mathbb{Z}_2 center symmetry and charge conjugation symmetry must be saturated by breaking one or the other. Since the \mathbb{Z}_2 center symmetry (a 1-form symmetry) typically does not break in 1+1 dimensions due to instanton effects (in analogy to how instantons restore a discrete ordinary 0-form symmetry in quantum mechanics), we cannot have a phase without charge conjugation symmetry breaking, and there should be no phase transition.

In Fig. 2 we show our observables $\langle |q| \rangle$ and χ_t as a function of M for the charge 2 case at $\theta = \pi$. Again we observe transitory behavior at some critical value of M where the expectation value $\langle |q| \rangle$ rises quickly. However, the volume dependence is different and the rhs. plot clearly shows the absence of volume scaling for the peaks of the topological susceptibility. This indicates that no critical behavior emerges in the charge 2 model and the small and large mass regions are only separated by a crossover. This observation provides a first example for how the changed symmetry content may alter the pattern of a transition originating from the presence of a topological term.

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