PROCEEDINGS OF SCIENCE

Topology and geometry for physicists

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These lecture notes are based on a series of lectures given in 2017 at the XIII Modave Summer School in Mathematical Physics. These notes are therefore aimed at beginning PhD students in theoretical physics and cover topics in homotopy theory, homology and cohomology, as well as fibre bundles, with applications in condensed matter systems, electromagnetism and Yang-Mills gauge theories.

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Note to the reader

These lecture notes are based on a five hour lecture course given at the XIII Modave Summer School in Mathematical Physics. The course was aimed at beginning PhD students in theoretical physics and aim to introduce some of the important concepts in topology and geometry, in particular homotopy theory, homology and cohomology and fibre bundles, which the students are likely to encounter during their careers. Applications of these concepts are discussed at the end of the respective sections. Due to the shortness of the course and its target audience, the emphasis in these notes is not on mathematical rigour but instead on conveying the important concepts in a way that is hopefully intuitive to physicists. Therefore, proofs of theorems are only given when they are instructive.

On the other hand, the notes include exercises interspersed throughout the text. Any reader who wants to gain more than a superficial understanding of the material, should attempt at least a large proportion of the exercises. Particularly difficult exercises are marked with an asterik.

These notes assume a basic knowledge of topology and differential manifolds, to the standard introduced in a typical Master's course on general relativity. There are several excellent books where more details can be found that were not covered in these notes. In particular, I can recommend the books "Geometry, topology and physics" by Nakahara, as well as "Geometry and topology for physicists" by Nash & Sen. For the more mathematically minded readers, the book "Algebraic Topology" by Hatcher is a fantastic resource, available for download for *free* at `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`. There are also many very good online resources, for example the lecture notes of the Edinburgh Mathematical Physics Group postgraduate course on "Gauge theories" which are available at `https://empg.maths.ed.ac.uk/Activities/GT/`.

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1. Topology

Topology is the study of continuous deformations. We wish to identify and spaces which can be continuously deformed into another. For example, a circle, a triangle and a box have the same *topology*. Recall that continuity can be defined in terms of open sets. It does not require either the notions of "smoothness" or "distance".

1.1 Basic notion of topology.

Definition: Two topological spaces, X, Y, are **homeomorphic** if there exists a continuous map $f : X \longrightarrow Y$, *with continuous inverse* $f^{-1} : Y \longrightarrow X$. We then write $X \sim Y$.

This is the key concept of topology! From the perspective of topology, we view two homeomorphic spaces as equivalent. More formally, if X and Y are topological spaces with $X \sim Y$ and $f : X \longrightarrow Y$ the homeomorphism, then the topology of X induces the topology of Y under f.

Theorem 1.1: Homeomorphism is an equivalence relation.

Exercise 1.1: Prove this, i.e. prove the following properties for topological spaces X, Y, Z.

(i) Reflexivity

$$X \sim X\,. \tag{1.1}$$

(ii) Symmetry

$$X \sim Y \iff Y \sim X\,. \tag{1.2}$$

(iii) Transitivity

$$\text{If } X \sim Y \text{ and } Y \sim Z\text{, then } X \sim Z\,. \tag{1.3}$$

There are some properties of topological spaces which are invariant under homeomorphisms, i.e. if $X \sim Y$ then they have that same property. Such properties, which are the same on any equivalence class of homeomorphic spaces, are called topological invariants. They play a crucial in topology and, as we will see, physics. An open problem in topology is to find a set of topological invariants (i.e. invariant under homeomorphism, defined below) such that if they agree for two spaces, those two spaces are homeomorphic.

Exercise 1.2: Show that the dimension of $\mathbb{R}^n$ is a topological invariant.

Exercise 1.3: Show that compactness is a topological invariant.

Exercise 1.4: Show that connectedness is a topological invariant.

We now continue this idea of identifying objects related by continuous deformations. For example, we wish to view continuous maps which can be continuously deformed into one another

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as equivalent. This leads to the idea of homotopy. To give the definition, let us first introduce our shorthand for an interval

$$I = [0,1] = \{t|0 \leq t \leq 1\} . \tag{1.4}$$

Its boundary is the two-point set $\partial I = \{0,1\}$.

Definition: Two continuous maps between topological spaces $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are **homotopic** if there exists a continuous function

$$H : X \times I \longrightarrow Y , \tag{1.5}$$

such that

$$H(x,0) = f(x) , \qquad H(x,1) = g(x) . \tag{1.6}$$

Here continuity also refers to continuity in $t \in I$. We write $f \sim g$ and call H a homotopy between f and g. You should think of H as a continuous family of maps $H_t : X \longrightarrow Y$ parameterised by the value t.

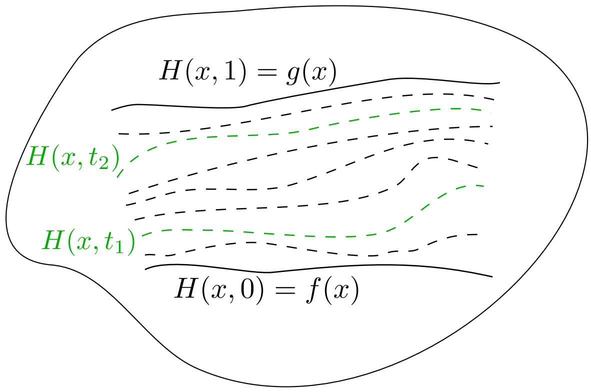


Figure 1: The two functions $f(x)$ and $g(x)$ are homotopic $f \sim g$ with $H(x,t)$ a homotopy between them. The different dashed curves correspond to $H(x,t)$ for fixed, but different, values of t. The green curves are $H(x,t_1)$ and $H(x,t_2)$.

Theorem 1.2: Homotopy is an equivalence relation.

Exercise 1.5: Prove this.

1.2 What about geometry?

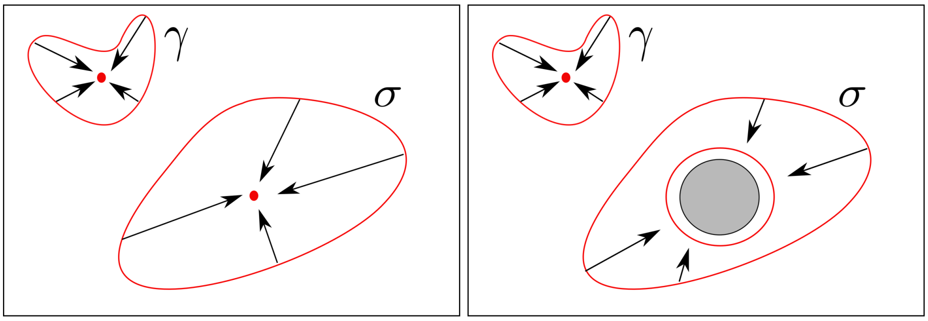
We have just defined some basic notions of topology. Given the title of the course, you may wonder what the basic notions of geometry are that we will be using. This inevitably raises the question of what exactly we mean by geometry versus topology. For the purposes of this course, we will view topology as the study of any global structures of spaces, while geometry will be the study of local structures.

For example, remember that a manifold of dimension n looks locally like $\mathbb{R}^n$ and therefore, locally, all manifolds are the same. Hence, we will consider the study of manifolds as falling under topology. On the other hand, a manifold with a Riemannian metric locally looks like $\mathbb{R}^n$ with some metric, which is not necessarily the flat Riemannian metric. Therefore, manifolds with Riemannian metrics have local structure and are studied in geometry, in the subfield known as Riemannian geometry.

According to the above definition, in this course we will mostly be concerned with topology. The exception are chapter 3.4, dealing with Hodge theory which relies on a Riemannian metric, and chapter 5 in which we will study connections on fibre bundles, which provide the fibre bundle with a local structure.

2. Homotopy groups

Remarkably, the set of homotopy equivalence classes of maps contain topological information about that space. For example, consider different loops in a space with a hole and one without a hole, as in figure 2.



(a) Two loops in the plane. (b) Two loops in the plane with a hole.

Figure 2: The two loops γ and σ in the plane can be contracted to a point. In the plane with a hole, γ is still contractible because it does not enclose the hole. However, σ can no longer be contracted to a point because it encircles the hole.

It is clear that in the space without a hole, all loops can be contracted to a point. For example, as shown in figure 2a, the loops γ and σ can both be contracted to a point. However, in the space with a hole, any loop surrounding the hole cannot be contracted to a point, as shown in figure 2b. Furthermore, it is intuitively clear that if we consider loops which contain a fixed point x_0 in the plane, they can all be continuously deformed into one another. This is not true of loops containing a fixed point x_0 in the plane with a hole. By contrast, these loops can instead be labelled by an integer which counts how often they wind around the hole.

As we just saw, the set of loops which can be continuously deformed into one another contain topological information. We will now make these intuitive notions precise by defining "homotopy groups".

2.1 First homotopy group

To extract the topological information from spaces as we intuitively discussed above, we need to consider *loops*. Let us first introduce the concept of a path.

Definition: A **path** γ in X from x_0 to x_1 is a continuous map

$$\gamma : I \longrightarrow X \,, \tag{2.1}$$

with

$$\gamma(0) = x_0 \,, \qquad \gamma(1) = x_1 \,. \tag{2.2}$$

This also allows us to define a stronger notion of connectedness, called path-connectedness.

Definition: A topological space X is called **path-connected** if there exists a path between any pair of points $x_0, x_1 \in X$.

We can now define loops which will play a crucial role in the following.

Definition: A **loop** at $x_0 \in X$ is a path γ in X with

$$\gamma(0) = \gamma(1) = x_0 \,. \tag{2.3}$$

The space of all loops at $x_0 \in X$ or **loop space** at $x_0 \in X$ is denoted $\mathscr{C}_{x_0}(X)$.

We can similarly describe a loop by $\gamma : S^1 \longrightarrow X$ but it will often be useful to highlight the base point x_0. Loops are great because one can "multiply" them together by following them in succession. In fact, we will soon show that homotopy equivalence classes of loops at a point $x_0 \in X$ form a group!

Let us formalise our intuitive notion of multiplying loops.

Definition: The product of loops

$$\star : \mathscr{C}_{x_0}(X) \otimes \mathscr{C}_{x_0}(X) \longrightarrow \mathscr{C}_{x_0}(X) \,, \tag{2.4}$$

is defined as follows. For any two loops at $x_0 \in X$, $\gamma, \sigma \in \mathscr{C}_{x_0}(X)$, the **product loop** $\rho = \sigma \star \gamma \in \mathscr{C}_{x_0}(X)$ is given by

$$\rho(t) = \begin{cases} \gamma(2t) \,, & 0 \leq t \leq 1/2 \,, \\ \sigma(2t-1) \,, & 1/2 \leq t \leq 1 \,. \end{cases} \tag{2.5}$$

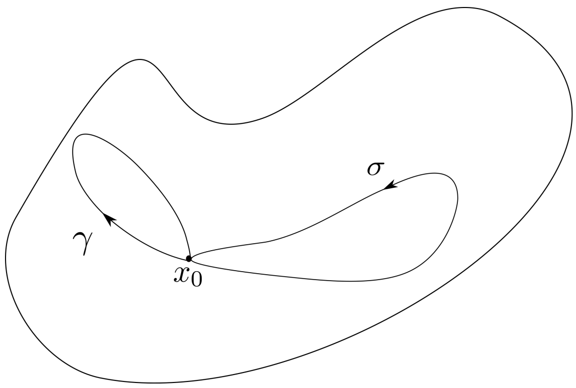


Figure 3: The product of two loops γ and σ at the same point x_0 is formed by following first γ and then σ at twice their normal "speeds". The product is by construction another loop at x_0.

Definition: Consider the loop γ at $x_0 \in X$. Its **inverse loop** γ^{-1} is defined as

$$\gamma^{-1}(t) = \gamma(1-t) \,, \qquad 0 \leq t \leq 1 \,. \tag{2.6}$$

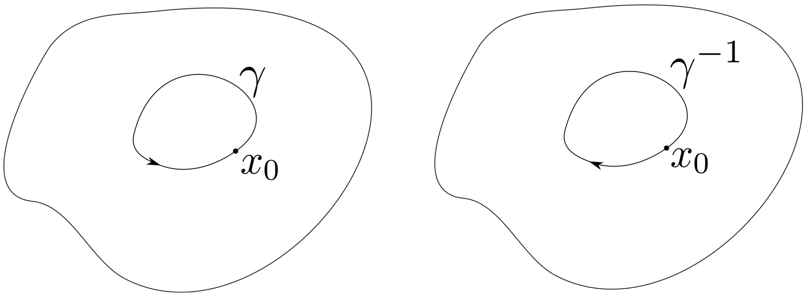


Figure 4: A loop γ at x_0 and its inverse loop γ^{-1} which is obtained by following γ in reverse.

Definition: The **constant loop** at $x_0 \in X$, e, is defined as

$$e(t) = x_0 \,, \qquad 0 \leq t \leq 1 \,. \tag{2.7}$$

It is instructive to think about whether these operations make the space of loops at x_0 into a group. There are several reasons why this does not work. For a start, the clearest candidate of an identity element is the constant loop. Yet this does not work. Think of t as representing time. Then $\gamma \star e \neq \gamma$ because $\gamma \star e$ includes a "pause" for half the time, as compared to γ. We somehow want to ignore these pauses in the group operation. Similarly, $\gamma \star \gamma^{-1} \neq e$. Clearly, the space of loops with the product operation defined above does not yield a group. (Even if it did, this would clearly have to be an infinite-dimensional group, and thus very hard to work with.)

However, all is not lost. We can use the earlier definition of homotopy to define homotopic loops. However, we need to change the definition slightly to ensure that the base point is changed by the homotopy.

Definition: Two loops γ, σ at $x_0 \in X$ are **homotopic loops** if there exists a continuous map

$$H : I \times I \longrightarrow X \,, \tag{2.8}$$

such that

$$\begin{aligned} H(t,0) &= \gamma(t) \,, & 0 \leq t \leq 1 \,, \\ H(t,1) &= \sigma(t) \,, & 0 \leq t \leq 1 \,, \\ H(0,s) &= H(1,s) = x_0 \,, & 0 \leq s \leq 1 \,. \end{aligned} \tag{2.9}$$

Then we write $\gamma \sim \sigma$ and we call H a **loop homotopy** between γ and σ.

Theorem 2.1: Homotopy of loops is an equivalence relation.

Proof. The proof follows from Exercise 1.5. □

Definition: We will call the homotopy equivalence class of a loop γ at $x_0 \in X$ the **homotopy class** of γ and denote it by $[\gamma]$.

One can easily show that the above definitions for the product and inverse loops are well-defined on homotopy classes of loops.

Exercise 2.1: Consider loops γ_0, σ_0, γ_1, σ_1 at $x_0 \in X$. Show that:

1. $\sigma_0 \sim \sigma_1 \Rightarrow \sigma_0^{-1} \sim \sigma_1^{-1}$.
2. $\sigma_0 \sim \sigma_1$, $\gamma_0 \sim \gamma_1 \Rightarrow \gamma_0 \star \sigma_0 \sim \gamma_1 \star \sigma_1$.

This shows that it makes sense to define the product $\star$ on homotopy classes of loops.

Working with homotopy classes immediately alleviates the two problems we mentioned above. The equivalence class of constant loops $[e]$ really is the identity of the product $\star$ defined on homotopy classes and $[f] \star [f^{-1}] = [e]$. This leads to the following theorem

Theorem 2.2: The set of homotopy classes of loops at x_0 forms a group under the loop product, with the constant loop at x_0 as the identity and the inverse given by the homotopy class of the inverse loop.

Proof. We have already seen that the product of loops gives another loop and is well-defined on homotopy classes of loops. The above remarks also show that there is an identity and inverse. It remains to show that the product of homotopy classes of loops is associative, as you are asked to show in the following exercise. $\square$

Exercise 2.2: Show that the product of homotopy classes of loops is associative.
Hint: Consider three loops, α, β and $\gamma : I \longrightarrow X$.

(i) Show that $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$ are given by

$$(\alpha \star \beta) \star \gamma(t) = \begin{cases} \alpha(4t) & 0 \leq t \leq \frac{1}{4}, \\ \beta(4t-1) & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ \gamma(2t-1) & \frac{1}{2} \leq t \leq, \end{cases} \tag{2.10}$$

and

$$\alpha \star (\beta \star \gamma)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2}, \\ \beta(4t-2) & \frac{1}{2} \leq t \leq \frac{3}{4}, \\ \gamma(4t-3) & \frac{3}{4} \leq t \leq 1. \end{cases} \tag{2.11}$$

(ii) Using the above result, find a homotopy between $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$.

Definition: This group of homotopy classes of loops at $x_0 \in X$ with the loop product is called the **fundamental group** or **first homotopy group** at x_0 and is denoted $\pi_1(X, x_0)$.

You may worry that the fundamental group depends on the base point, x_0, chosen. However, we will now show that for a path-connected space, the fundamental groups at any two points are isomorphic. Therefore, for a path-connected space one can just speak of *the* fundamental group.

Let us begin by defining a multiplication law for paths, not just loops.

Definition: The **product path** ρ from x_0 to $x_2 \in X$ of a path γ from x_0 to $x_1 \in X$ with a path σ from x_1 to $x_2 \in X$ is given by

$$\rho(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2, \\ \sigma(2t-1), & \frac{1}{2} \le t \le 1. \end{cases} \tag{2.12}$$

We denote it by $\rho = \sigma \star \gamma$.

Exercise 2.3: Show that the product path is a path, i.e. that it is continuous.

Just like for loops we can define an inverse path.

Definition: The **inverse path** γ^{-1} from x_1 to $x_0 \in X$ of a path γ from x_0 to $x_1 \in X$ is given by

$$\gamma^{-1}(t) = \gamma(1-t). \tag{2.13}$$

Using these concepts we can now prove the following theorem.

Theorem 2.3: If there is a path γ from x_0 to $x_1 \in X$, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ are isomorphic.

Proof. The idea of the proof is to use γ and its inverse γ^{-1} to turn a loop at x_0 into a loop at x_1 and vice versa as illustrated in figure 5.

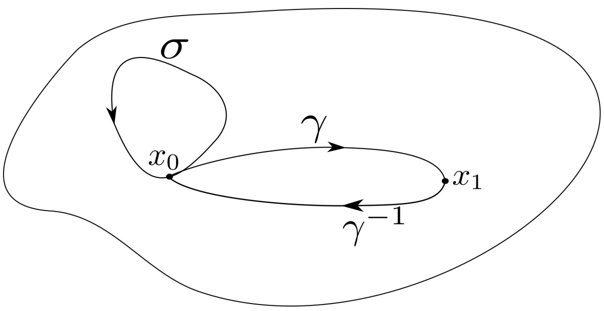


Figure 5: Using a path γ from x_0 to x_1 and its inverse path γ^{-1} we can turn the loop σ at x_0 into a loop at x_1.

Following this idea, we first define the maps between fundamental groups

$$\begin{aligned} \gamma^* &: \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)\,, \\ \gamma^{-1*} &: \pi_1(X,x_1) \longrightarrow \pi_1(X,x_0)\,, \end{aligned} \tag{2.14}$$

by

$$\begin{aligned} \gamma^* &: ([\sigma],x_0) \longrightarrow \left([\gamma\star\sigma\star\gamma^{-1}],x_1\right)\,, \\ \gamma^{-1*} &: ([\sigma],x_1) \longrightarrow \left([\gamma^{-1}\star\sigma\star\gamma],x_0\right)\,. \end{aligned} \tag{2.15}$$

One can easily show that these maps define an isomorphism between the of fundamental groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$, as you are asked to do in the following exercises. □

Exercise 2.4: Show that γ^* is well-defined on homotopy classes, i.e. if $\sigma \sim \sigma'$ are homotopic loops at x_0 then $[\gamma^*\sigma] = [\gamma^*\sigma']$.

Exercise 2.5: Show that γ^* defines a homomorphism, i.e. for any two loops σ, σ' at $x_0 \in X$,

$$[\gamma^*(\sigma)\star\gamma^*(\sigma')] = [\gamma^*(\sigma\star\sigma')]\,. \tag{2.16}$$

Exercise 2.6: Show that $\gamma^{-1*}\star\gamma^* = e_0$ where e_0 represents the equivalence class of the constant loop at x_0.

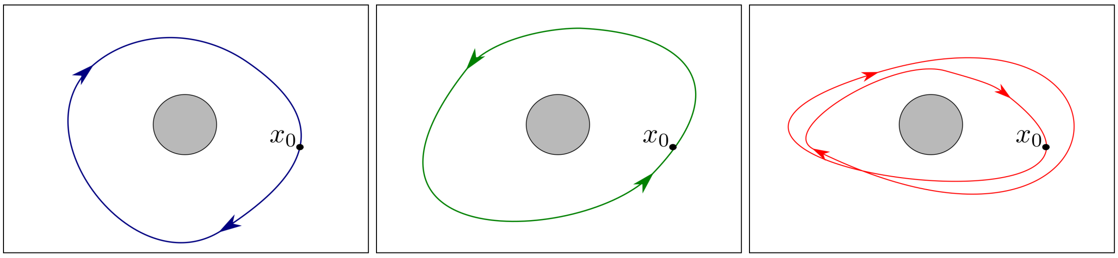
Corollary: If X is a path-connected topological space then for any $x_0,x_1 \in X$, the fundamental groups $\pi_1(X,x_0) \simeq \pi_1(X,x_1)$.

2.1.1 Examples of homotopy groups

Example 2.1: Figure 6 shows the punctured plane, i.e. $\mathbb{R}^2$ with a hole. We can assign to each loop at x_0 a winding number $n \in \mathbb{Z}$ counting how many times the loop encircles the hole clockwise. If $n < 0$, the loop encircles the hole counter-clockwise. The product loop of two loops with winding numbers n and m has winding number $n+m$. Thus, we have that $\pi_1(X) \simeq \mathbb{Z}$, where X is the punctured plane.

Example 2.2: Note that the fundamental group need not necessarily be abelian. To see an example of a non-Abelian fundamental group consider a plane with two holes cut out as in figure 7. Consider the loop α at x_0 which winds around the left hole while staying below the right hole and the loop β at x_0 which winds around the left hile while staying above the right hole. Clearly, the two loops are not homotopic $\alpha \not\simeq \beta$ as loops at x_0, i.e with x_0 fixed. However, using γ which winds around the right hole counter-clockwise we can turn α into a loop that is homotopic to β:

$$\gamma^{-1}\star\alpha\star\gamma \simeq \beta\,, \tag{2.17}$$



(a) A loop with winding no. 1. (b) A loop with winding no. -1. (c) A loop with winding no. 2.

Figure 6: The punctured plane has a non-trivial first homotopy group $\pi_1 \simeq \mathbb{Z}$ and hence homotopy classes of loops can be labelled by a winding number, counting how many times they wind the hole clockwise. The figures show three loops at x_0 belonging to different homotopy classes.

and thus

$$[\gamma]^{-1} \star [\alpha] \star [\gamma] = [\beta] \neq [\alpha] \, . \tag{2.18}$$

However, this implies that the fundamental group of the double punctured plane cannot be Abelian since

$$[\gamma] \star [\alpha] \neq [\alpha] \star [\gamma] \, . \tag{2.19}$$

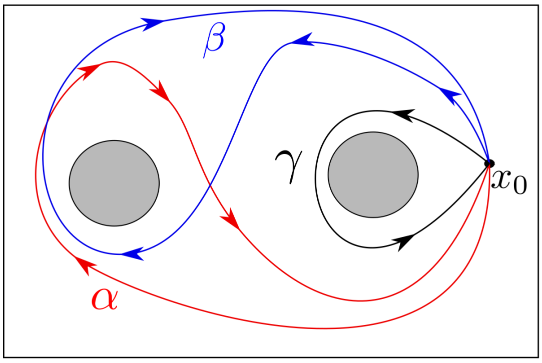


Figure 7: The plane with two punctures has a non-Abelian first homotopy group. This can be seen by considering the loops α and β at x_0 which wrap the left hole while below / above the right hole and the loop γ at x_0 which wraps the right hole but not the left. Clearly, $\alpha \neq \beta$ but $\gamma^{-1} \star \alpha \star \gamma \simeq \beta$ from which it follows that the first homotopy group is non-Abelian.

2.2 Homotopy type and deformation retraction

We will now introduce two operations on topological spaces, other than homeomorphisms, which leave the fundamental group invariant, and are thus useful in calculating homotopy groups. Rather incredibly, these operations can even change the dimension of the topological spaces, yet the fundamental group stays invariant. This fact already shows that homotopy groups are not enough to fully characterise the topology of the spaces.

Definition: Two spaces are of the same **homotopy type** if we have continuous maps f and g,

$$f : X \longrightarrow Y\,, \qquad g : Y \longrightarrow X\,, \tag{2.20}$$

such that

$$f \circ g \sim 1_Y\,, \qquad g \circ f \sim 1_X\,. \tag{2.21}$$

Theorem 2.4: If two topological spaces X and Y are path-connected and of the same homotopy type, then

$$\pi_1\left(X, x_0\right) \simeq \pi_1\left(Y, y_0\right)\,,\ \forall x_0 \in X,\, y_0 \in Y\,. \tag{2.22}$$

Proof. The proof can be found in Nash & Sen, chapter 3.. □

Corollary: If two topological spaces are homeomorphic $X \sim Y$ and are path-connected then

$$\pi_1(X, x_0) \simeq \pi_1(Y, y_0)\,. \tag{2.23}$$

This corollary shows that, as promised, the fundamental group is a topological invariant.

Example 2.3: A circle and a circle with a line attached, as shown in figure 8, are of the same homotopy type but are not homeomorphic.

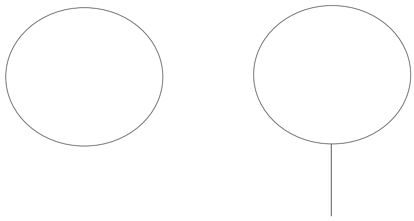


Figure 8: The circle and the circle with a line attached are of the same homotopy type.

Example 2.4: Two circles attached by a line, the figure eight and two circles divided by a line, as shown in figure 9, are all of the same homotopy type. However, they are not homeomorphic.

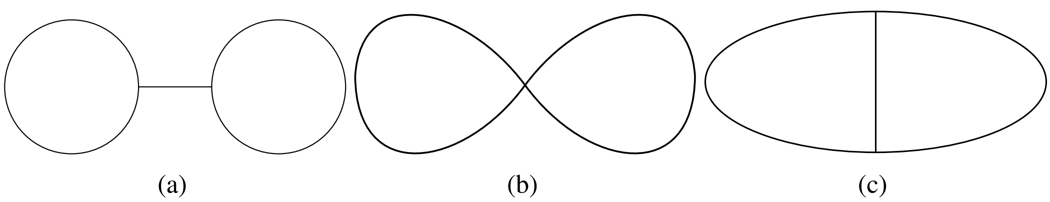


Figure 9: Three homotopic spaces that are not homeomorphic.

We can also define an operation which captures the intuitive notion of continuously shrinking a topological space onto some subset. This is called a deformation retract.

Definition: A subset $A \subset X$ of a topological space is called a **deformation retract** if there exists a continuous map

$$r : X \longrightarrow A \,, \tag{2.24}$$

with $r|_A = 1_A$, i.e. $r(a) = a \,\forall a \in A$, and there exists another continuous function

$$H : X \times I \longrightarrow X \,, \tag{2.25}$$

with

$$\begin{aligned} H(x,0) &= x \,, \\ H(x,1) &= r(x) \,, \\ H(a,t) &= a \;\forall a \in A \, t \in I \,. \end{aligned} \tag{2.26}$$

The continuous map r defined here is called a **retract**.

Note that the condition for a deformation retract is stronger than requiring just a retract that is homotopic to the identity because we also require this homotopy to act like the identity on the deformation retract for all $t \in I$. Deformation retracts are important for the following reason.

Theorem 2.5: If X is a path-connected topological subspace and A a deformation retract of X, then $\pi_1(X,a) \simeq \pi_1(A,a)$ with $a \in A$.

Let us emphasise once more that the fundamental group is not enough to distinguish topological spaces. For example, we will show in the following examples, deformation retracts of a topological space can be of a different dimension. Nonetheless the fundamental group between two such spaces is the same.

Example 1: The point $\{0\}$ is a deformation retract of $\mathbb{R}^n$. The retract just maps $r : \mathbb{R}^n \longrightarrow \{0\}$ and the homotopy is given by

$$H(x,t) = tx \,, \quad t \in I \; x \in \mathbb{R}^n \,. \tag{2.27}$$

Thus, we find $\pi_1(\mathbb{R}^n, 0) = 0$.

Definition: A topological space X that can be deformation retracted to a point is called a **contractible space**.

Proposition: All contractible spaces have trivial fundamental group.

Proof. This follows immediately from the fact that the fundamental group of a point is trivial. □

Example 2: S^{n-1} is a deformation retract of $D^n - \{0\}$. Here D^n refers to the n-dimensional disc defined as

$$D^n = \left\{ (x_1, \ldots, x_n) \,|\, |x_1|^2 + \ldots + |x_n|^2 \leq 1 \right\} \,. \tag{2.28}$$

The deformation retract is obtained via

$$H(x,t) = (1-t)\,x + t\frac{x}{|x|}\,. \tag{2.29}$$

Thus,

$$\pi_1\left(D^2 - \{0\}, x_0\right) \simeq \pi_1\left(S^1, x_0\right) \simeq \mathbb{Z}\,. \tag{2.30}$$

One also often encounters product spaces, for which the following theorem tells us how to find the fundamental group.

Theorem 2.6: For two path-connected topological spaces X, Y, we have

$$\pi_1\left(X \times Y, x_0 \times y_0\right) \simeq \pi_1\left(X, x_0\right) \oplus \pi_1\left(Y, y_0\right),\ x_0 \in X\,,\ y_0 \in Y\,. \tag{2.31}$$

Exercise 2.7: Prove this.

Example: Using $T^n = \underbrace{S^1 \otimes \ldots \otimes S^1}_{n \text{ copies}}$, we find $\pi_1(T^n) = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n \text{ copies}}$.

Exercise 2.8: What is π_1 of a cylinder?

These tools should allow you to calculate the homotopy groups of most topological spaces you will encounter. We end our discussion of the fundamental group by noting that one can often calculate it using a triangularisation of the space, called a simplicial complex. However, we will not go into any detail of this construction, but refer the interested reader to the relevant chapters in the books by Nakahara (chapter 4), Nash & Sen (chapter 3), or for the more mathematically minded to Hatcher (chapter 3).

2.3 Higher homotopy groups

We can generalise the concepts that we just met to higher dimensions. We start by introducing n-loops. Firstly, we denote by I_n the n-cube: $[0,1]^n$ and by ∂I_n its boundary.

$$\begin{aligned} I^n &= \left\{(t_1,\ldots,t_n)\,|0 \leq t_i \leq 1\right\}\,, \\ \partial I^n &= \left\{(t_1,\ldots,t_n)\,|0 \leq t_i \leq 1\,,\ \text{some } t_i = 0 \text{ or } 1\right\}\,. \end{aligned} \tag{2.32}$$

Definition: A ***n*-loop** based at $x_0 \in X$ is a continuous map

$$\gamma : I_n \longrightarrow X\,, \tag{2.33}$$

such that

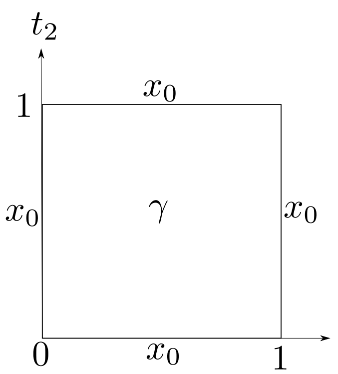
$$\gamma|_{\partial I_n} = x_0\,. \tag{2.34}$$

We will typically denote a point in I_n by t with coordinates $(t_1,\ldots,t_n)$ and $0 < t_i < 1$. The boundary ∂I_n is then the set where $t_i = 0$ or $t_i = 1$ for any $i = 1,\ldots,n$. This means that we require $\gamma(t_1,\ldots,t_n) = x_0$ when $t_i = 0$ or $t_i = 1$ for any i.

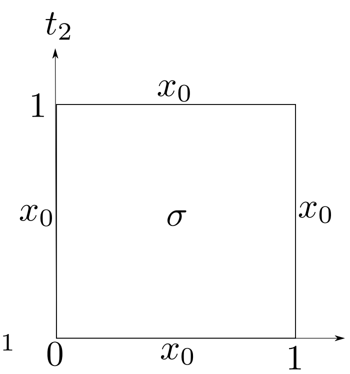
We can continue and define the product of n-loops as follows.

Definition: Let γ, σ be two n-loops at $x_0 \in X$. Then their **product loop** $\rho = \sigma \star \gamma$ is an n-loop at $x_0 \in X$ defined as

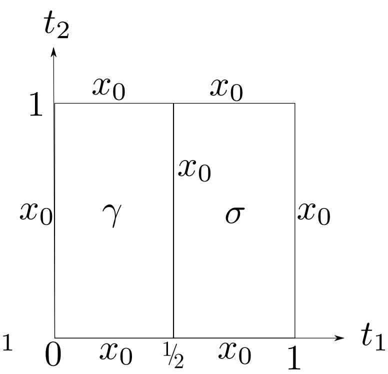
$$\rho(t_1,\ldots,t_n) = \begin{cases} \gamma(2t_1,t_2,\ldots,t_n)\,, & 0 \leq t_1 \leq 1/2\,, \\ \sigma(2t-1,t_2,\ldots,t_n)\,, & 1/2 \leq t_1 \leq 1\,. \end{cases} \tag{2.35}$$



(a) The 2-loop γ.



(b) The 2-loop σ.



(c) The product 2-loop $\rho = \sigma \star \gamma$.

Figure 10: The product of 2-loops.

We can also define homotopic n-loops.

Definition: Two n-loops γ, σ at $x_0 \in X$ are **homotopic** if there exists a continuous map

$$H : I \times I_n \longrightarrow X\,, \tag{2.36}$$

such that

$$\begin{aligned} &H(0;t) = \gamma(t)\,, \\ &H(1;t) = \sigma(t)\,, \\ &H(s,t) = x_0\,, \quad \text{if } t \in \partial I_n\,. \end{aligned} \tag{2.37}$$

Exercise 2.9: Show that homotopy of n-loops is an equivalence relation.

We will again refer to these equivalence classes as homotopy classes and denote them by $[\,]$.

Definition: The **constant n-loop** e at $x_0 \in X$ is defined as

$$e : I_n \longrightarrow x_0\,. \tag{2.38}$$

Definition: The **inverse n-loop** γ^{-1} of an n-loop at x_0 is the map

$$\gamma^{-1} : I_n \longrightarrow X\,, \tag{2.39}$$

defined as

$$\gamma^{-1}(t_1,\ldots,t_n) = \gamma(1-t_1,\ldots,t_n)\,. \tag{2.40}$$

Lemma The product and inverse of n-loops at $x_0 \in X$ are well-defined on homotopy classes.

Exercise 2.10: Show this.

Theorem 2.7: The homotopy classes of n-loops at $x_0 \in X$ with product, inverse as defined above and identity the constant n-loop at x_0 define a group.

Exercise 2.11: Prove this!

It should not be surprising that the higher homotopy groups satisfy the same theorems regarding path-connected, contractible and product spaces.

Theorem 2.8: If there is a path f in X, a topological space, from x_0 to x_1 then

$$\pi_n(X,x_0) \simeq \pi_n(X,x_1)\,. \tag{2.41}$$

Corollary: If X is a path-connected topological space then $\pi_n(X,x_0) \simeq \pi_n(X,x_1)$ for any pair of points x_0, $x_1 \in X$.

Theorem 2.9: If two path-connected topological spaces X and Y are of the same homotopy type, then

$$\pi_n(X,x_0) \simeq \pi_n(Y,y_0)\,,\; x_0 \in X\,,\; y_0 \in Y\,. \tag{2.42}$$

Theorem 2.10: If $A \subset X$ is a deformation retract of X then

$$\pi_n(X,a) \simeq \pi_n(A,a)\,,\; a \in A\,. \tag{2.43}$$

Corollary: If a topological space X is contractible, then $\pi_n(X,x_0) = 0$.

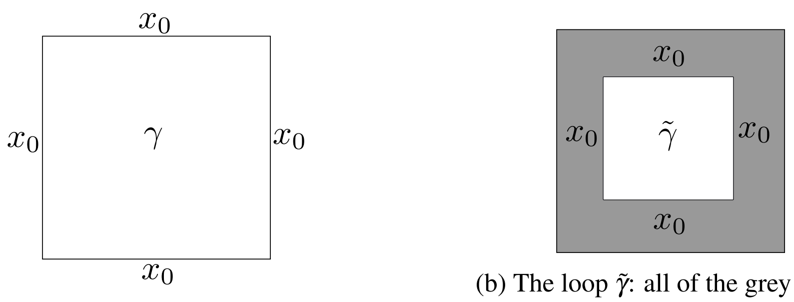
Theorem 2.11: If X and Y are path-connected topological spaces, then

$$\pi_n(X\times Y, x_0 \times y_0) \simeq \pi_n(X,x_0) \oplus \pi_n(Y,y_0)\,,\; x_0 \in X\,,\; y_0 \in Y\,. \tag{2.44}$$

Despite these similarities, the higher homotopy groups differ in a crucial way from the fundamental group: they are all abelian!

Theorem 2.12: The n-dimensional homotopy groups $\pi_n(X,x_0)$ are abelian for $n>1$.

Proof. First note that the n-loop γ at $x_0 \in X$ is homotopic to the n-loop $\tilde{\gamma}$ obtained by "thickening" the boundary as shown in figure 11. Now one can easily show that the homotopy groups are



(a) The loop γ

(b) The loop $\tilde{\gamma}$: all of the grey area is mapped into x_0.

Figure 11: The loop γ and the homotopic "thickened" loop $\tilde{\gamma} \sim \gamma$.

abelian by considering the product of any two n-loops at x_0, $\sigma \star \gamma$, and following the sequence of steps shown in figure 12.

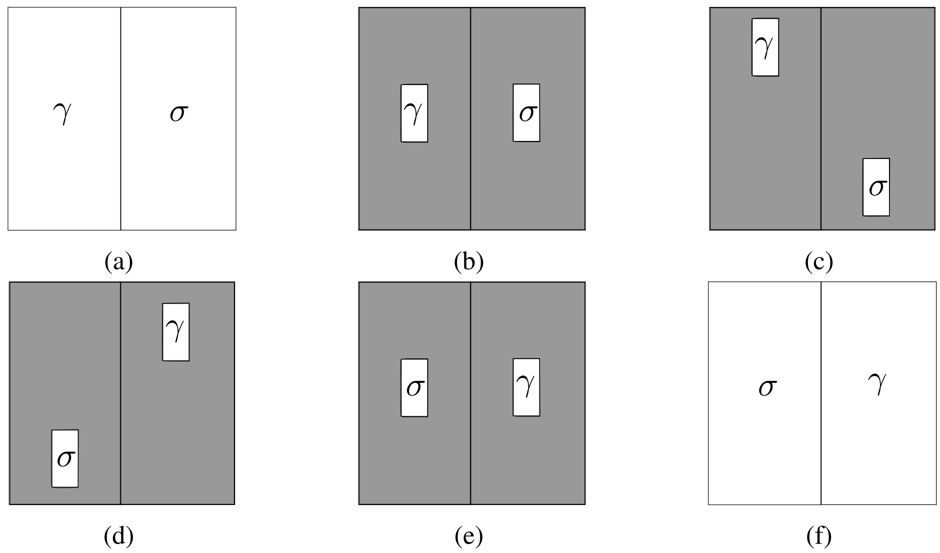


Figure 12: After "thickening", γ and σ can be slided past each other. From this it follows that $\gamma \star \sigma \sim \sigma \star \gamma$ for n-loops with $n \geq 2$.

□

Exercise 2.12*: Why does the above argument fail for 1-loops? In other words, what is the crucial difference between n-loops for $n > 1$ and 1-loops that makes the higher-dimensional homotopy groups abelian?

2.4 Applications in physics

Consider a phase transition in a condensed matter system, for example in a ferromagnet which develops a spontaneous magnetisation, characterised by the average magnetisation vector $\mathbf{m}$, below a certain "critical temperature", T_c, i.e.

$$\begin{aligned} T > T_c\,, \quad \mathbf{m} = 0\,, \\ T < T_c\,, \quad \mathbf{m} \neq 0\,. \end{aligned} \tag{2.45}$$

More generally, associated with a phase transition, there will be some **order parameter** that becomes non-zero after the transition and which breaks a global symmetry of the system. For example, in the example of a ferromagnet the magnitude of the average magnetisation is fixed by the temperature, $|\mathbf{m}| = m(T)$. Therefore at fixed T we have

$$\mathbf{m} \in S^2\,, \tag{2.46}$$

and any $\mathbf{m}$ breaks the symmetry group of the condensed matter system $\mathrm{SO}(3) \longrightarrow \mathrm{SO}(2)$. The space in which the order parameter, here $\mathbf{m}$, can take values is called the **order parameter space**, $\mathscr{M}$. Thus, for the ferromagnet we have $\mathscr{M} = \mathrm{SO}(3)$.

In general, there may be regions where the order parameter is ill-defined, called **defects**. Consider for example a system undergoing a phase transition where the order parameter in different regions becomes mutually incompatible. At the boundary of the two regions, the order parmeter will be ill-defined.

Defects are usually given different names depending on their co-dimension, e.g. in three dimensions

- a point defect is called a **monopole**
- a line defect is called a **vortex**
- a surface defect is called a **domain wall**.

Most importantly, these defects are stable because of topology! As we will see, the defects are topologically distinct from the vacuum configuration and therefore no smooth process that destabilise the defects.

Furthermore, we can use homotopy to see what kind of defects are possible in a particular condensed matter system. To see this, consider enclosing a defect in a condensed matter system X by a S^n, where $n = d - m - 1$ with d the dimension of X and m the dimension of the defect. We can take the sphere to be sufficiently big so that the system is in thermal equilibrium everywhere along the S^n, i.e. the defect is far away that it is "no longer felt". Then along the S^n we would have $\mathbf{m} \in M$ and hence we can associate with the defect an element of the homotopy group $\pi_n(X)$. In particular, only if $\pi_n(X) \neq 0$ is it possible to have a defect! Moreover, defects can also combine according to the group product of $\pi_n(X)$.

Example 2.5: Defects in 3-dimensional ferromagnets: As an example, consider the ferromagnet again with $M = S^2$. A defect is only possible if $\pi_n(S^2) \neq 0$. However, we have

$$\begin{aligned} \pi_1(S^2) &= 0\,, \\ \pi_2(S^2) &= \mathbb{Z}\,, \\ \pi_{n\geq 3}(S^2) &= 0\,. \end{aligned} \tag{2.47}$$

Using $n = d - m - 1$ we see that in a 3-dimensional ferromagnet, only monopoles can exist.

Example 2.6: So-called liquid crystals have a *nematic* phase (nematic is Greek for thread-like) in which the rod-shaped molecules making up the liquid crystal align themselves. However, the molecules are not directed, meaning that opposite orientations are identical. Therefore, the order parameter space in the nematic phase is

$$\mathscr{M} = S^2/\mathbb{Z}_2 = \mathbb{RP}^2\,. \tag{2.48}$$

As a result, nematic liquid crystal defects are classified by $\pi_n(\mathbb{RP}^2)$. We have

$$\begin{aligned} \pi_1(\mathbb{RP}^2) &= \mathbb{Z}_2\,, \\ \pi_2(\mathbb{RP}^2) &= \mathbb{Z}\,. \end{aligned} \tag{2.49}$$

We see that a 3-dimensional nematic liquid crystal system only admits two kinds of vortices, with a composition law like $\mathbb{Z}_2$, while it admits infinitely many monopoles, which can be composed together like the integers under addition.

3. Homology and cohomology

3.1 Homology

A careful exposition of homology is too lengthy for this course. Here we will take a short-cut by not being as careful and defining homology in terms of submanifolds of a differential manifold and their boundaries.

Definition: Given a manifold M, a p**-chain** a_p is a formal sum of smooth oriented p-dimensional submanifolds of M, call them N_i, so that

$$a_p = \sum_i r_i N_i \,, \tag{3.1}$$

where $r_i \in \mathbb{F}$ are just some coefficients taken in a field $\mathbb{F}$. When $r \in \mathbb{R}$ we have a **real** p**-chain**, whereas when $r \in \mathbb{C}$ we have a **complex** p**-chain**.

Note: Unless otherwise specified, we will only consider real p-chains.

Because we are using oriented p-dimensional submanifolds we can integrate p-forms over these. Thus, we can think of p-chains as something we can integrate a p-form over and the coefficients r_i then just define the weight of the various integrals over N_i:

$$\int_{\sum_i r_i N_i} = \sum_i r_i \int_{N_i} \,. \tag{3.2}$$

Definition: The space of p-chains of a manifold M over the field $\mathbb{F}$ is a vector space (and hence also Abelian group), called the p**-th chain group**. It is labelled by $C_p(M,\mathbb{F})$.

Definition: The **boundary operator** ∂ associates to each manifold M its boundary ∂M. It maps a manifold of dimensions p to a manifold of dimension $p-1$.

Theorem 3.1:

$$\partial^2 = 0 \,. \tag{3.3}$$

Proof. The boundary of a boundary vanishes and so $\partial^2 M = 0$ for all M, i.e. $\partial^2 = 0$. This can be proven rigorously by "triangulating" your manifolds using singular homology, but is beyond the scope of these lectures. □

Definition: We define the **boundary operator** to act on p-chains by linearity:

$$\partial \sum_i r_i N_i = \sum_i r_i \partial N_i \,. \tag{3.4}$$

Thus,

$$\partial : C_p(M,\mathbb{F}) \longrightarrow C_{p-1}(M,\mathbb{F}) \,. \tag{3.5}$$

Sometimes we will use a subscript p on ∂ to emphasise it acts on p-chains, i.e. $\partial_p : C_p(M,\mathbb{F}) \longrightarrow C_{p-1}(M,\mathbb{F})$.

Definition: A **chain complex** is a sequence of Abelian groups, $\ldots, C_2, C_1, C_0, C_{-1}, \ldots$ and homomorphisms $\partial_i : C_i \longrightarrow C_{i-1}$ such that $\partial_i \cdot \partial_{i+1} = 0$. This is usually denoted as

$$\ldots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow \ldots \,. \tag{3.6}$$

The chain complexes of a manifold define a chain complex with $C_{n>\dim M} = 0$ and $C_{n<0} = 0$. The chain complex can thus be represented as

$$0 \xrightarrow{i} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} \xrightarrow{C}{}_0 \xrightarrow{\partial_0} 0 \,, \tag{3.7}$$

where i is just the inclusion map.

For any chain complex, the relation between the image, $\mathrm{Im}\,\partial$, and the kernel, $\mathrm{Ker}\,\partial$, of the boundary map is interesting. Hence the image and Kernel are usually given names.

Definition: A ***p*-cycle** is a p-chain $z_p \in C_p$ without boundary, i.e.

$$\partial z_p = 0 \,. \tag{3.8}$$

Theorem 3.2: The set of p-cycles $Z_p(M)$ is a subgroup of $C_p(M)$, called the ***p*-cycle group**. Thus,

$$Z_p(M) = \{ z_p \in C_p(M) \mid \partial_p z_p = 0 \} = \mathrm{Ker}\,\partial_p \,. \tag{3.9}$$

Definition: If $c \in C_p(M)$ and $\exists$ some $d \in C_{p+1}(M)$ such that

$$c = \partial_{p+1} d \,, \tag{3.10}$$

then c is called a ***p*-boundary**.

Theorem 3.3: The set of p-boundaries $B_p(M)$ is a subgroup of $C_p(M)$, called the ***p*-boundary group**. Thus,

$$B_p(M) = \{ a_p \in C_p \mid a_p = \partial_{p+1} a_{p+1} \text{ for some } a_{p+1} \in C_{p+1} \} = \mathrm{Im}\,\partial_{p+1} \,. \tag{3.11}$$

> **Theorem 3.4:** $B_p(M) \subset Z_p(M)$.

Proof. This follows from $\partial^2 = 0$. □

While neither $Z_p(M)$ nor $B_p(M)$ are topological invariants, their quotient is.

> **Definition:** The p**-th homology group** of a manifold M is the quotient group
>
> $$H_p(M) = \frac{Z_p(M)}{B_p(M)} \,. \tag{3.12}$$
>
> Thus, H_p is the set of p-cycles with two cycles deemed equivalent if they differ by a boundary
>
> $$a_p \sim a_p + \partial c_{p+1} \,. \tag{3.13}$$

Example 3.1: All points are 0-cycles since they have no boundary and any two points are the boundary of a curve. On each connected component we identify all points by the equivalence relation and so $H_0 = \mathbb{R}^c$ where c is the number of connected components.

Example 3.2: Consider T^2. As shown in figure 13 below, there are only two homologously different 1-cycles, labelled a and b in 13a. Any other 1-cycles are either a boundary, such as b' in 13b, or together with a or b form a boundary.

(a) The a and b cycles are the only homologously distinct 1-cycles.

(b) $a_1 \sim a_2$ since $a_1 - a_2$ is the boundary of the shaded region enclosed. b' is a trivial 1-cycle as it is the boundary of the shaded space enclosed.

Figure 13: T^2 has only two homologously distinct 1-cycles a and b, as shown in 13b. Any other 1-cycles are either a boundary or differ from a or b by a boundary, as shown in 13a.

Example 3.3: The homology groups of T^2: $H_0 = \mathbb{R}$ since the torus is connected. H^1 is generated by two different 1-cycles as we discussed above and so $H_1 = \mathbb{R}^2$. Finally, the only 2-chain without boundary is the T^2 itself and so we have $H_2 = \mathbb{R}$. You may wonder why the T^2 is not the boundary of the space it encloses (just as a sphere is the boundary of the ball it encloses). The reason is that the enclosed space is itself not part of the T^2 manifold.

Example 3.4: The homology groups of S^n: $H_0 = \mathbb{R}$ since the n-sphere is connected. $H^k = 0$ $\forall\ 0 < k < n$ because each (hyper-)circle is the boundary of some half-sphere. $H^n = \mathbb{R}$ since the n-sphere has no boundary and is itself not a boundary.

Theorem 3.5: $H_n(M,\mathbb{Z})$ always takes the form

$$\underbrace{\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}}_{m\text{ times}}\oplus\underbrace{\mathbb{Z}_{k_1}\oplus\ldots\oplus\mathbb{Z}_{k_1}}_{\text{“torsion”}}\,, \tag{3.14}$$

where the finite Abelian groups are known as the **torsion** of the n-th homology group.

Finally, let us state some results that are useful when calculating homology groups.

Theorem 3.6: If two topological spaces X and Y are of the same homotopy type, then

$$H_n(X) \simeq H_n(Y)\,. \tag{3.15}$$

Proof. See Hatcher chapter 2. □

Theorem 3.7: If two topological spaces A is a deformation retract of X then

$$H_n(X) \simeq H_n(A)\,. \tag{3.16}$$

Proof. See Hatcher chapter 2. □

Corollary: If X is contractible, then $H_n(X) = 0$.

It is also often useful to know the homology of a product space.

(Künneth theorem) Theorem 3.8:

$$H_k(X\times Y) \simeq \bigoplus_{i+j=k} H_i(X)\otimes H_j(Y)\,. \tag{3.17}$$

3.2 Cohomology

I will assume some familiarity with differential forms, in particular integrating forms over manifolds but will introduce many of the concepts that we will need.

First, let us label by $\Omega^p(M)$ the space of smooth p-forms. Let us define objects which should be familiar from differential geometry.

Definition: Let M be a differentiable manifold. Then we define the **exterior derivative** acting on p-forms as

$$d_p : \Omega^p(M) \longrightarrow \Omega^{p+1}(M) \,. \tag{3.18}$$

Often we will drop the subscript p. For a p-form ω, with components in local coordinates, x^μ,

$$\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \,, \tag{3.19}$$

we define the action of d by

$$d\omega = \frac{1}{p!} \partial_{[\mu_1} \omega_{\mu_2 \ldots \mu_{p+1}]} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} \,. \tag{3.20}$$

Exercise 3.1: Let M be a manifold. Show, using local coordinates x^μ on M, that the exterior derivative of a 1-form $\omega \in \Omega^1(M)$ satisfies

$$d\omega(X,Y) = X\left[\omega(Y)\right] - Y\left[\omega(X)\right] - \omega([X,Y]) \ \forall X, Y \in TM \,. \tag{3.21}$$

Generalise the above result for p-forms, i.e. for $\omega \in \Omega^p(M)$ and all $X_i \in TM$,

$$\begin{aligned} d\omega(X_1, \ldots, X_{p+1}) = & \sum_{i=1}^{p} (-1)^{i+1} X_i \omega\left(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}\right) \\ & + \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots \hat{X}_j, \ldots, X_{p+1}\right) \,, \end{aligned} \tag{3.22}$$

where $\hat{X}$ means that the vector X is omitted in the evaluation of ω.

Exercise 3.2: Show using both the coordinate-dependent definition and the coordinate-free definition of exercise 3.1 that

$$d^2 = 0 \,. \tag{3.23}$$

The set of all spaces of differential p-forms form a co-chain complex

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{d_0} \ldots \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M) \xrightarrow{d_{p+1}} \ldots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0 \,. \tag{3.24}$$

Definition: A p-form $\omega \in \Omega^p(M)$ is called **closed** if it is in the kernel of d, i.e.

$$d\omega = 0\,, \tag{3.25}$$

and **exact** if it is the image of d, i.e. $\exists\, \chi \in \Omega^{p-1}(M)$ such that

$$\omega = d\chi\,. \tag{3.26}$$

We denote by

$$Z^p = \{\omega_p \mid \omega_p \in \Omega^p(M)\,,\, d\omega_p = 0\}\,, \tag{3.27}$$

the space of closed p-forms and by

$$B^p = \left\{\omega_p \mid \omega_p \in \Omega^p(M)\,,\, \exists \beta_{p-1} \in \Omega^{p-1} \text{ such that } \omega_p = d\beta_{p-1}\right\}\,, \tag{3.28}$$

the space of exact p-forms.

Definition: The p-th **deRham cohomology group** is defined as

$$H^p = \frac{Z^p}{B^p}\,. \tag{3.29}$$

This means that H^p is the space of closed p-forms where we identify any two closed p-forms which differ by an exact p-form:

$$\omega_p \sim \omega_p + d\beta_{p-1}\ \forall\, \beta_{p-1} \in \Omega^{p-1}\,. \tag{3.30}$$

Given some closed p-form $\omega \in \Omega^p$ we define its equivalence class, the **cohomology class**

$$[\omega] \in H^p\,, \tag{3.31}$$

as the space of closed p-forms which differ from ω by an exact p-form. ω is called a **representative** of the cohomology class.

Note: Just as for homology, the group operation for the cohomology group is addition.

Example 3.5: H^0 is the space of constant functions on the manifold. This is because 0-forms are functions and there are no (-1)-forms hence no "exact functions". Thus, on a connected manifold we have $H^0 = \mathbb{R}$. If we have more than one connected component of the manifold then we can define a constant function on each as a generating element and so we have $H^0 = \mathbb{R}^c$ where c is the number of connected components.

Example 3.6: On an n-dimensional manifold, an n-form is always closed and so H^n is generated by the volume form, if it exists. On an orientable manifold, there is a globally well-defined volume form which generates $H^n = \mathbb{R}$ whereas for a non-orientable manifold $H^n = 0$.

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Example 3.7: $H^2\left(T^2\right) = \mathbb{R}^2$. Let us label the coordinates on the T^2 by coordinates x and y, subject to the identifications $x \sim x+1$, $y \sim y+1$. There are only two closed 1-forms which are not exact: dx and dy. Despite their suggestive form, dx and dy are not exact because the "functions" x and y do not respect the torus identifications and thus are not globally well-defined.

From the examples above we see that the dimensions of the cohomology groups are important. For example, they count the number of connected components of the manifold, or indicate whether the manifold is orientable.

Definition: We define the **Betti numbers**

$$b_p = \dim H^p\,, \tag{3.32}$$

to be the dimension of the cohomology groups.

Theorem 3.9: For an oriented, compact, closed Riemannian manifold, $b_p = b_{n-p}$.

Proof. This can easily be shown using Hodge theory, which we will introduce in section 3.4. □

Definition: The **Euler characteristic** of a manifold is defined as the alternating sum of the Betti numbers:

$$\chi = \sum_{p=0}^{n} (-1)^p b^p\,. \tag{3.33}$$

Corollary: The Euler characteristic vanishes for odd-dimensional manifolds.

Proof. This follows immediately from the identity $b_p = b_{n-p}$. □

It should be clear that the existence of closed but not exact forms (and hence harmonic forms) is closely related to the topology of the manifold. Let us consider a simple example to see how this happens. Consider a closed 1-form ω which can thus *locally* be written as

$$\omega = d\chi\,, \tag{3.34}$$

for some function χ. In $\mathbb{R}^n$ we could then construct χ by taking

$$\chi(x) = \int_y^x \omega(\xi)_\mu d\xi^\mu\,. \tag{3.35}$$

In $\mathbb{R}^n$ this is a valid construction since χ is independent of the path chosen between y and x. (Changing y just corresponds to shifting χ by a constant which is of course going to leave ω invariant). This is because for two different paths γ_1 and γ_2 from y to x the difference in the definition of χ is just

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\partial U} \omega = \int_U d\omega = 0\,, \tag{3.36}$$

where ∂U is the boundary of the region U enclosed by the two curves γ_1, γ_2. Thus, the χ obtained by this procedure is well-defined. This is not true for a general manifold, as we can see by looking at the torus. For any two points, different paths connecting them are not in general the boundary of a region.

By now, you should have recognised the discussion of curves that are or are not boundaries of a region as homology. Thus, we see that the cohomology of a manifold is closely related to its homology. We will next make this more precise.

3.3 De Rham's Theorems

Given a closed p-form and a p-cycle, it is natural to integrate one over the other. This defines a natural inner product between Z^p and Z_p.

Definition: Given a closed p-form $\omega_p \in Z^p$ and a p-cycle $a_p \in Z_p$, we define the **period** of ω_p over a_p as

$$\pi\left(a_p, \omega_p\right) = \int_{a_p} \omega_p \,. \tag{3.37}$$

Theorem 3.10: The period function π defined above is a function on homology and cohomology classes, i.e.

$$\pi : H_p \otimes H^p \longrightarrow \mathbb{R} \,. \tag{3.38}$$

Proof. The period function as defined in (3.37) above looks like a map from $Z_p \otimes Z^p \longrightarrow \mathbb{R}$. We wish to show that the period of any element of a cohomology class over any cycles separated by a boundary is the same. Consider thus the closed p-form $\omega'_p = \omega_p + d\beta_{p-1}$ and p-cycle $a'_p = a_p + \delta c_{p+1}$. Then using Stoke's theorem we find that the period is

$$\begin{aligned}
\pi(a'_p, \omega'_p) &= \int_{a_p+\delta c_{p+1}} \left(\omega_p + d\beta_{p-1}\right) \\
&= \int_{a_p} \omega_p + \int_{a_p} d\beta_{p-1} + \int_{\delta c_{p+1}} \omega_p + \int_{\delta c_{p+1}} d\beta_{p-1} \\
&= \int_{a_p} \omega_p + \int_{\delta a_p} \beta_{p-1} + \int_{c_{p+1}} d\omega_p + \int_{c_{p+1}} d^2\beta_{p-1} \\
&= \int_{a_p} \omega_p \,.
\end{aligned} \tag{3.39}$$

Thus the period evaluated on different representatives of the same (co-)homology class is the same. □

There are two important theorems involving the period, known as deRham's theorems, which together show that the p-th cohomology and p-th homology groups are isomorphic to each other. We will only state these theorems here.

(deRham's 1st) Theorem 3.11: Let $\{z_i\}$ be a basis for H_p. Then given any set of numbers α_i, $i = 1, \ldots \dim(H_p)$, there exists a closed p-form $\omega \in Z^p$ such that $\pi(z_i, \omega) = \alpha_i$.

(deRham's 2nd) Theorem 3.12: If all the periods of a closed p-form $\omega \in Z^p$ vanish then ω is exact.

If $\{z_i\}$ is a basis for H_p and $\{\omega^i\}$ is a basis for H^p then the period matrix

$$\pi_i{}^j = \pi\left(z_i, \omega^j\right) \tag{3.40}$$

is invertible and thus H_p and H^p are isomorphic. The isomorphism can be made more concrete using Poincaré duality (and recalling that $H^p \simeq H^{n-p}$).

(Poincaré Duality) Theorem 3.13: Given any p-cycle $a \in Z_p$ there exists an $(n-p)$-form α, the so-called **Poincaré dual** of a, such that for any closed p-form $\omega \in Z^p$

$$\int_a \omega = \int_M \alpha \wedge \omega\,. \tag{3.41}$$

Roughly speaking, you could think of α as the localised volume form on a plus exact forms so that α is smooth (and hence a differential form). The necessity of including exact terms is why α can only be defined when integrating closed p-forms $\omega \in Z^p$ over a.

Finally, this implies that the theorems about homotopy type, deformation retracts and product spaces also holds for cohomology, e.g. two manifolds of the same homotopy type have isomorphic de Rham cohomology. In particular, this implies that any contractible space has trivial cohomology, which is equivalent to the Poincaré Lemma: any closed p-form is locally exact.

3.4 Hodge theory

We will now consider Riemannian manifolds (M, g), i.e. manifolds equipped with a Riemannian metric, to gain a new perspective on cohomology using "Hodge theory". In particular, the Riemannian metric will allow us to build a differential operator which leaves the rank of differential forms invariant, $\Delta : \Omega^p(M) \longrightarrow \Omega^p(M)$, in contrast to the exterior derivative which changes the rank of the differential forms.

Definition: Let (M,g) be an n-dimensional oriented Riemannian manifold. Then we define the **Hodge dual** acting on p-forms as

$$\star : \Omega^p(M) \longrightarrow \Omega^{n-p}(M) . \tag{3.42}$$

For a p-form ω, with components in local coordinates as in (3.19), we define $\star$ as

$$\star\omega = \frac{1}{p!\,(n-p)!}\varepsilon_{\mu_1\ldots\mu_p\mu_{p+1}\ldots\mu_n} g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\omega_{\nu_1\ldots\nu_p} dx^{\mu_{p+1}}\wedge\ldots\wedge dx^{\mu_n} , \tag{3.43}$$

where $\varepsilon_{\mu_1\ldots\mu_n}$ is the n-dimensional alternating tensor which takes values $\pm\sqrt{g}$ depending on whether $(\mu_1\ldots\mu_n)$ is an even or odd permutation of $(1\ldots n)$.

Note: The Hodge dual $\star$ depends on the metric whereas the exterior derivative d does not!

Exercise 3.3: Show that for a p-form ω

$$\star\star\,\omega = (-1)^{p(n-p)}\,\omega . \tag{3.44}$$

We can use the Hodge dual to define an inner product on p-forms.

Definition: Let (M,g) be an oriented Riemannian manifold. Then the **inner product** on p-forms,

$$(\,,\,) : \Omega^p(M)\otimes\Omega^p(M) \longrightarrow \mathbb{R} , \tag{3.45}$$

is defined as

$$(\alpha,\beta) = \int_M \alpha\wedge\star\beta \;\; \forall\,\alpha,\beta\in\Omega^p(M) . \tag{3.46}$$

Theorem 3.14: Let (M,g) be an oriented Riemannian manifold. The inner product for p-forms as defined above is symmetric and positive-definite.

Proof. It is easy to show that for p-forms $\alpha,\beta\in\Omega^p(M)$ the inner product is

$$(\alpha,\beta) = \int_M \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}\beta^{\mu_1\ldots\mu_p}\sqrt{g}d^nx , \tag{3.47}$$

where $\beta^{\mu_1\cdots\mu_p}$ are the components of β raised with the inverse metric. It is now clear that the inner product is symmetric. We can also see that

$$(\alpha,\alpha) = \int_M \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}\alpha^{\mu_1\cdots\mu_p}\sqrt{g}\,d^nx > 0 \;\forall\,\alpha\neq 0 \tag{3.48}$$

is positive-definite because the metric g is positive-definite. □

Exercise 3.4: Show that

$$\alpha_p \wedge \star\beta_p = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}\beta^{\mu_1\ldots\mu_p}\sqrt{g}dx^1 \wedge \ldots \wedge dx^n\,. \tag{3.49}$$

Since we now have a symmetric inner product on the vector space of p-forms, we can define adjoints of any operator. In particular, we define the adjoint of the exterior derivative $d^\dagger$, called the codifferential:

Definition: Let (M,g) be an oriented Riemannian manifold. The **codifferential**

$$d^\dagger : \Omega^p(M) \longrightarrow \Omega^{p-1}(M) \tag{3.50}$$

is defined for any $\alpha_p \in \Omega^p(M)$ and $\beta_{p-1} \in \Omega^{p-1}(M)$ as

$$(\alpha_p, d\beta_{p-1}) = \left(d^\dagger\alpha_p, \beta_{p-1}\right)\,. \tag{3.51}$$

Let us from now onwards assume that we have a compact manifold without boundary, $\partial M = 0$. We call manifolds without boundary closed. This implies that the codifferential does not contain boundary terms.

Theorem 3.15: Let (M,g) be an n-dimensional oriented, compact, closed Riemannian manifold. Then the codifferential is given by

$$d^\dagger = (-1)^{pn-n+1}\star d\star\,. \tag{3.52}$$

Proof. Let us begin by using the fact that the inner product is symmetric so that

$$(\alpha_p, d\beta_{p-1}) = \int_M d\beta_{p-1} \wedge \star\alpha_p\,, \tag{3.53}$$

and integrate by parts

$$(\alpha_p, d\beta_{p-1}) = -(-1)^{p-1}\int_M \beta_{p-1} \wedge d\star\alpha_p\,. \tag{3.54}$$

Now use the fact that $d\star\alpha_p$ is a $n-p+1$ form and so

$$\star\star d\star\alpha = (-1)^{(n-p+1)(p-1)}\,d\star\alpha\,. \tag{3.55}$$

Thus, we can write (3.54) as

$$(\alpha_p, d\beta_{p-1}) = (-1)^{(n-p+1)(p-1)+p}\int_M \beta_{p-1} \wedge \star(\star d\star\alpha_p)\,. \tag{3.56}$$

Now, let us simplify the exponent of -1 by noticing that $p(p-1)$ is always even. Finally comparing to the definition of the codifferential (3.51) we find

$$d^\dagger = (-1)^{pn-n+1}\star d\star\,. \tag{3.57}$$

This completes the proof. □

Corollary: The codifferential $d^\dagger$ is nilpotent, i.e. $d^\dagger d^\dagger = 0$.

Proof. This follows from the definition. For a p-form:

$$d^\dagger d^\dagger = (-1)^n \star d \star \star d \star = (-1)^{p(n-p)+n} \star d^2 \star = 0\,, \tag{3.58}$$

because $d^2 = 0$. □

Exercise 3.5: Using local coordinates, write down the explicit expression for $d^\dagger$ acting on p-forms in 3 dimensions.

Exercise 3.6: Show that for a p-form $\omega \in \Omega^p$ with components in local coordinates

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,, \tag{3.59}$$

the codifferential acts as

$$d^\dagger \omega = -\frac{1}{(p-1)!}\nabla^\sigma \omega_{\sigma\mu_2\ldots\mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}\,. \tag{3.60}$$

Hint: Recall that $\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}V^\mu\right) = \nabla_\mu V^\mu$.

Definition: A p-form $\omega \in \Omega^p(M)$ is called **co-closed** if it is in the kernel of $d^\dagger$, i.e.

$$d^\dagger \omega = 0\,, \tag{3.61}$$

and **co-exact** if it is the image of $d^\dagger$, i.e. $\exists\, \chi \in \Omega^{p+1}(M)$ such that

$$\omega = d^\dagger \chi\,. \tag{3.62}$$

We will denote by

$$\bar{Z}^p = \left\{\omega_p \mid \omega_p \in \Omega^p(M)\,,\, d^\dagger \omega_p = 0\right\}\,, \tag{3.63}$$

the space of co-closed p-forms and by

$$\bar{B}^p = \left\{\omega_p \mid \omega_p \in \Omega^p(M)\,,\, \exists \beta_{p+1} \in \Omega^{p+1} \text{ such that } \omega_p = d^\dagger \beta_{p+1}\right\}\,, \tag{3.64}$$

the space of exact p-forms.

We now have an operator that raises the rank of a p-form and one that lowers it. Thus, we can define an operator that takes p-forms to p-forms. This generalises our notion of the Laplacian operators acting on functions.

Definition: Let (M, g) be a compact, closed Riemannian manifold. Then we define the **Hodge-deRham operator**

$$\Delta : \Omega^p(M) \longrightarrow \Omega^p(M) \tag{3.65}$$

by

$$\Delta = dd^\dagger + d^\dagger d\,. \tag{3.66}$$

Definition: We call a p-form $\omega \in \Omega^p(M)$ a **harmonic** p-form if it lies in the kernel of Δ, i.e.

$$\Delta\omega = 0\,. \tag{3.67}$$

We label the vector space of harmonic p-forms as $\mathscr{H}^p(M)$.

Exercise 3.7: Using local coordinates, write down the explicit expression of the Hodge-deRham operator acting on p-forms in 3 dimensions.

Exercise 3.8:

(i) Show that for a p-form $\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ the Hodge operator acts as

$$\Delta\omega = -\left(\frac{1}{p!}\nabla^\sigma\nabla_\sigma\omega_{\mu_1\ldots\mu_p} + \frac{1}{(p-1)!}\left[\nabla_{\mu_1}, \nabla_\nu\right]\omega^\nu{}_{\mu_2\ldots\mu_p}\right) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,, \tag{3.68}$$

where $\omega^\nu{}_{\mu_2\ldots\mu_p} = g^{\sigma\nu}\omega_{\sigma\mu_2\ldots\mu_p}$. Therefore, it is a generalisation of the scalar Laplacian $\Delta f = -\nabla^\mu\nabla_\mu f$ for a function $f \in C^\infty(M)$.

(ii) Show that

$$\left[\nabla_\mu, \nabla_\nu\right]\omega^\nu{}_{\mu_2\ldots\mu_p} = -R_{\sigma\mu}\,\omega^\sigma{}_{\mu_2\ldots\mu_p} - \omega^\nu{}_{\sigma[\mu_3\ldots\mu_p}R^\sigma{}_{\mu_2]\mu\nu}\,, \tag{3.69}$$

where $R^\sigma{}_{\mu\nu\rho}$ is the Riemann curvature tensor and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ is the Ricci tensor. *Hint:* Use normal coordinates so that the connection vanishes but the derivatives of the connection are non-vanishing.

(iii) Use the result of the previous two parts to show that

$$\begin{aligned}\Delta\omega = -\Bigg(&\frac{1}{p!}\nabla^\sigma\nabla_\sigma\omega_{\mu_1\ldots\mu_p} - \frac{1}{(p-1)!}R_{\sigma[\mu_1}\omega^\sigma{}_{\mu_2\ldots\mu_p]} \\ &+ \frac{1}{(p-2)!}\omega^\nu{}_{\sigma[\mu_3\ldots\mu_p}R^\sigma{}_{\mu_1\mu_2]\nu}\Bigg) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\,.\end{aligned} \tag{3.70}$$

Theorem 3.16: A harmonic form is both closed and co-closed.

Proof. A harmonic form α satisfies

$$\Delta\alpha = dd^\dagger\alpha + d^\dagger d\alpha = 0\,. \tag{3.71}$$

Take the inner product of the above with α:

$$\left(\alpha, dd^\dagger\alpha\right) + \left(\alpha, d^\dagger d\alpha\right) = \left(d\alpha, d\alpha\right) + \left(d^\dagger\alpha, d^\dagger\alpha\right) = 0\,. \tag{3.72}$$

But since the inner product is positive-definite, the two terms must vanish independently. Thus we have

$$\left(d\alpha, d\alpha\right) = 0\,, \qquad \left(d^\dagger\alpha, d^\dagger\alpha\right) = 0\,. \tag{3.73}$$

But again because the inner product is positive-definite, this means that

$$d\alpha = 0\,, \qquad d^\dagger\alpha = 0\,, \tag{3.74}$$

and so α is both closed and co-closed. This completes the proof. □

(Hodge) Theorem 3.17: Let (M,g) be a compact, closed Riemannian manifold. Then any p-form $\omega \in \Omega^p(M)$ has a *unique* decomposition into a harmonic, exact and co-exact part, i.e.

$$\omega = \alpha + d\beta + d^\dagger\gamma\,, \tag{3.75}$$

for some $\beta \in \Omega^{p-1}(M)$, $\gamma \in \Omega^{p+1}(M)$ and $\alpha \in \mathscr{H}^p(M)$. This is called the **Hodge decomposition**.

Proof. We will not prove the full theorem but we will show that the decomposition is unique. Assume that there are two different decompositions

$$\omega = \alpha + d\beta + d^\dagger\gamma = \alpha' + d\beta' + d^\dagger\gamma'\,, \tag{3.76}$$

with $\alpha, \alpha' \in \mathscr{H}^p(M)$. Let us denote by

$$\tilde{\alpha} = \alpha' - \alpha\,, \qquad \tilde{\beta} = \beta' - \beta\,, \qquad \tilde{\gamma} = \gamma' - \gamma\,. \tag{3.77}$$

Then we have that

$$\tilde{\alpha} + d\tilde{\beta} + d^\dagger\tilde{\gamma} = 0\,. \tag{3.78}$$

It remains to show that each term vanishes separately. To do so, act with d on the above equation to obtain

$$dd^\dagger\tilde{\gamma} = 0\,. \tag{3.79}$$

Taking the inner product of this with $\tilde{\gamma}$ we find

$$\left(\tilde{\gamma}, dd^\dagger\tilde{\gamma}\right) = \left(d^\dagger\tilde{\gamma}, d^\dagger\tilde{\gamma}\right) = 0\,. \tag{3.80}$$

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Because the inner product is positive-definite this implies

$$d^\dagger \tilde{\gamma} = 0 \,. \tag{3.81}$$

We can repeat this argument to find

$$d\tilde{\beta} = 0 \,, \tag{3.82}$$

and hence $\tilde{\alpha} = 0$. Thus, the two decompositions are the same. □

Note: The Hodge decomposition implies that we can decompose the vector space of p-forms as

$$\Omega^p(M) = \mathscr{H}^p(M) \oplus B^p(M) \oplus \bar{B}^p(M) \,. \tag{3.83}$$

We will from now onwards often drop the argument (M).
Corollary: A closed form $\omega_p \in B^p$ can always be written as

$$\omega_p = \alpha + d\beta \,, \tag{3.84}$$

where $\alpha \in \mathscr{H}^p$ is harmonic, and similarly for a co-closed form.

Exercise 3.9: Prove the above, i.e. that a closed p-form can always be written as the sum of a harmonic and an exact p-form.

The cohomology group is manifestly independent of the metric since it is only defined using the exterior derivative which does not require a metric. Thus, it is a topological property of the manifold. The space of harmonic forms, on the other hand, clearly depends on the metric since $d^\dagger$ is defined with respect to a metric. Thus, it may seem like these two vector spaces measure very different things. However, this is not the case as the following theorem shows.

Theorem 3.18: The space of harmonic p-forms $\mathscr{H}^p$ and the p-th cohomology group H^p are isomorphic.

Proof. We saw previously that a closed form can always be written as the unique sum of a harmonic and exact form. This defines the isomorphism. □

Corollary: Every cohomology class contains a unique harmonic representative.

(Poincaré duality) Theorem 3.19: A p-form ω is harmonic if and only if $\star\omega$ is harmonic.

Proof. A p-form ω harmonic if and only if $d\omega = d^\dagger\omega = 0$. Consider now

$$d^\dagger \star \omega = (-1)^c \star d\omega \,, \tag{3.85}$$

and

$$d \star \omega = (-1)^{c'} \star d^\dagger \omega \,, \tag{3.86}$$

where c and c' are some integers (the factors of (-1) are unimportant here). Recall that $\star\star = (-1)^{p(n-p)}$ and hence $\star$ is invertible. Thus, we see that

$$d^\dagger \omega = 0 \Longleftrightarrow d \star \omega = 0 \,, \qquad d\omega = 0 \Longleftrightarrow d^\dagger \star \omega = 0 \,. \tag{3.87}$$

This completes the proof. □

Corollary: H^p and H^{n-p} are isomorphic.

Proof. This follows from the above together with the fact that the p-th deRham cohomology group and space of harmonic p-forms is isomorphic: $H^p \simeq \mathscr{H}^p$. □

3.5 Difference between homology and cohomology

We have seen that homology and cohomology are both defined in terms of the image and kernel of a nilpotent operator ∂ and d, respectively. So you might be wondering what the difference between homology and cohomology is in a more general setting. The answer is that homologies and cohomologies behave differently under maps of their underlying manifolds, in the same way that vector fields and differential forms behave differently: Homology groups are pushed-forward while cohomology groups are pulled-back along maps between their underlying manifolds.

Push-forwards and pull-backs

Consider a smooth map between two manifolds M and N, $f : M \longrightarrow N$. We can use this to pull back functions on N to functions on M. For example, from any function $\sigma : N \longrightarrow \mathbb{R}$ we can define a new function $f^*\sigma = f \circ \sigma : M \longrightarrow \mathbb{R}$ by composing with f as shown in figure 14.
We can also use $f : M \longrightarrow N$ to push-forward vector fields from M to N. For a vector $X_p \in T_pM$, we can define $X_{*\,f(p)}$ by

$$X_p(f^*\sigma) \equiv X_{*\,f(p)}(\sigma) \,. \tag{3.88}$$

Thus, using a map $f : M \longrightarrow N$, we can generate a map between from TN to TM, $f_* : TM \longrightarrow TN$. This can be used to push-forward any vector fields on M to a vector field on N.
On the other hand, differential forms can be pulled back from N to M using f because they transform contravariantly to vector fields. For a differential form $\omega \in T^*(N)$, we can define the pull-back $\omega^* \in T^*(M)$ by

$$\omega^*(X) = \omega(X_*) \,, \forall\, X \in TM \,. \tag{3.89}$$

The fact that cohomology is pulled-back along maps between the underlying manifolds allows us to give more structure to cohomology groups. In particular, the groups can be made into a ring by defining a **cup product**: $\cup : H^n(M) \otimes H^m(M) \longrightarrow H^{n+m}(M)$. The first step in defining such a product is by using the tensor product

$$H^n(M) \otimes H^m(M) \longrightarrow H^{n+m}(M \times M) \,, \tag{3.90}$$

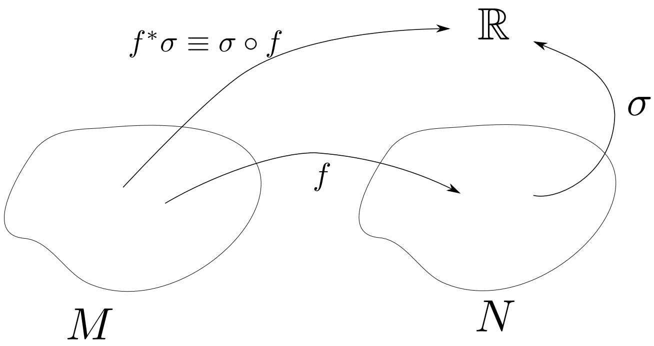


Figure 14: Using a function $f : M \longrightarrow N$ between manifolds, we can turn any function $\sigma : N \longrightarrow \mathbb{R}$ into a function $f^*\sigma : M \longrightarrow \mathbb{R}$ by composing with f.

which can also be defined in the same way for homology groups. However, we now need a way to relate $H^{n+m}(M \times M)$ to $H^{n+m}(M)$. For homology groups this is not possible in general. However, it is possible for generao cohomologies thanks to the pull-back property. In particular, one can use the diagonal map of a manifold diag : $M \longrightarrow M \times M$ to induce the pull-back

$$\text{diag}^* : H^n(M \times M) \longrightarrow H^n(M) . \tag{3.91}$$

This allows us to define a cup product as required for cohomologies. For de Rham cohomology the cup product is formed from the wedge product which induces a well-defined a product on cohomologies.

Exercise 3.10: Show that the wedge product of differential forms induces a well-defined product on cohomologies, i.e consider $\alpha = \alpha' + d\omega$ and $\beta = \beta' + d\gamma$ with α' and β' closed. Show that $\alpha \wedge \beta \sim \alpha' \wedge \beta'$, i.e. they differ only by closed forms.

3.6 Applications in physics

You will hopefully recall from an advanced course on electromagnetism that we can describe the electromagnetic field by a 2-form $F \in \Omega^2(M)$ where M is the four-dimensional spacetime we are considering. Take x^μ to be the standard flat coordinates on $\mathbb{R}^4$ with $\mu = 0, \dots, 3$. We can write

$$x^\mu = \left(x^0, x^i\right) , \tag{3.92}$$

where $i = 1, \dots, 3$ and x^i are coordinates parameterising the spacelike hypersurfaces $x_0 =$ const. In this coordinate system we identify the electric and magnetic fields as

$$\begin{aligned} F_{0i} &= E_i , \\ F_{ij} &= \varepsilon_{ijk} B^k , \end{aligned} \tag{3.93}$$

where $\varepsilon_{ijk} = \pm 1$ is the alternating symbol on $\mathbb{R}^3$. You also probably remember that Hodge duality realises electromagnetic duality:

$$F \longrightarrow \star F , \tag{3.94}$$

implies

$$E_i \longrightarrow B_i\,, \qquad B_i \longrightarrow -E_i\,. \tag{3.95}$$

Hopefully, you will also remember how to write the source-less Maxwell equations in terms of differential forms. They are given by

$$dF = 0\,, \qquad d\star F = 0\,. \tag{3.96}$$

The fact that $dF = 0$ means that we can locally write $F = dA$ with A a local 1-form. This is the usual description of th electromagnetic field in terms of a gauge potential A. In fact, on $\mathbb{R}^4$, we can do this globally, so that $A \in \Omega^1(M)$.

Now let us consider sources. The equations now become

$$dF = \rho_m\,, \qquad \star d\star F = \rho_e\,, \tag{3.97}$$

where $\rho_{m/e}$ are the electric and magnetic charges, respectively. We will now focus on the magnetic charge for two reasons. Firstly, *we do not require a metric* and thus our analysis will reduce to topology. Secondly, we notice that in the presence of magnetic charges F cannot be exact where $\rho_m \neq 0$. Thus, we cannot express $F = dA$ in terms of a vector potential everywhere. However, we might be forgiven to think that we can introduce a well-defined vector potential A away from the magnetic charges such that $F = dA$ there. However, this is not so.

To see why, let us consider the integral version of Maxwell's equations by integrating (3.98) over a three-manifold Σ with boundary $\partial\Sigma$ which encloses the magnetic charge and using Stoke's Theorem to obtain

$$\int_\Sigma dF = \int_{\partial\Sigma} F = \int_\Sigma \rho_m = Q_m\,, \tag{3.98}$$

where Q_m is the magnetic charge enclosed in the volume Σ.

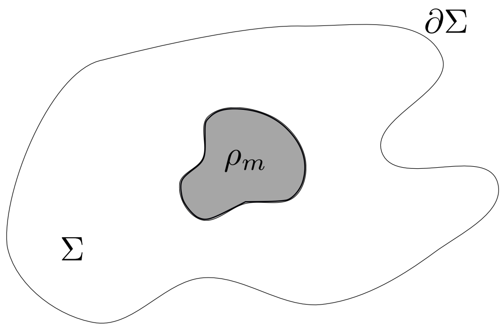


Figure 15: The 3-manifold Σ is bounded by $\partial\Sigma$ and includes a region with non-zero magnetic charge ρ_m, shaded above.

From our earlier discussion on homology we know that boundaries have no boundaries themselves and thus $\partial\Sigma$ is closed. Thus, if F were exact on $\partial\Sigma$, $F|_{\partial\Sigma} = dA|_{\partial\Sigma}$, then using Stokes' Theorem again we would necessarily have

$$Q_m = \int_{\partial\Sigma} F = \int_{\partial\Sigma} dA = \int_{\partial^2\Sigma} A = 0\,. \tag{3.99}$$

We see that we cannot even introduce a well-defined vector potential on any two-surface enclosing the magnetic charge! However, $dF = 0$ on $\partial\Sigma$ and so we see that $F \in H^2(\partial\Sigma)$. Therefore the possible field strengths F must be elements of the second cohomology group of $\partial\Sigma$.

Let us try and understand this in a little more detail by looking at a point charge, $\rho_m = g\,\delta(\mathbf{r})$. Recall that the Maxwell equations are linear and so we can learn everything by considering point charges. This will also nicely lead us into our next topic, fibre bundles. As we saw before, we can smoothly deform the surface Σ without changing the value of the integral, as long as we do not cross the loci of magnetic charge. This allows us to smoothly deform Σ into the ball of radius 1 centred at $\mathbf{r} = 0$. Its boundary is now the 2-sphere $\partial\Sigma = S^2$.

To avoid the above issues, we need to look at open subsets of S^2. A natural set of open subsets to use is given by a choice of charts. Let us use the northern and southern hemispheres ($U_{N/S}$) intersecting on the equator times a strip:

$$\begin{aligned} U_N &= \{(\theta,\phi)\,|0 \leq \theta \leq \pi/2+\varepsilon\,, 0 \leq \phi \leq 2\pi\}\,, \\ U_S &= \{(\theta,\phi)\,|\pi/2-\varepsilon \leq \theta \leq \pi\,, 0 \leq \phi \leq 2\pi\}\,. \end{aligned} \tag{3.100}$$

The intersection $U_I = U_N \cup U_S$ is given by the equator times the strip I_ε of width ε:

$$U_I = \{(\theta,\phi)\,|\pi/2-\varepsilon \leq \theta \leq \pi/2+\varepsilon\,, 0 \leq \phi \leq 2\pi\}\,. \tag{3.101}$$

It is clear that U_I is homotopic to an interval times a circle: $U_I \sim I \times S^1$, see figure 16.

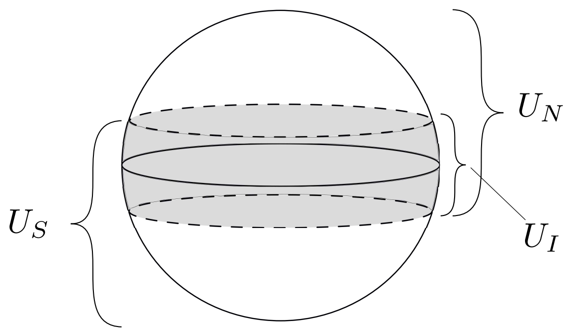


Figure 16: Open subsets U_N and U_S covering the northern and southern hemisphere on the sphere and their intersection U_I, the shaded region.

On each of the open subsets U_N, U_S we have

$$dF_{N/S} = 0\,. \tag{3.102}$$

But since the northern / southern hemispheres are contractible, $H^2(U_{N/S}) = 0$, and thus $dF_{N/S}$ implies $F_{N/S} = dA_{N/S}$. Furthermore, on the intersection U_I these two field strengths must coincide. Otherwise the field strength would not be well-defined! Physically, we see this because F is uniquely determined by Maxwell's equation (3.97) without reference to coordinates. Therefore, we must have

$$F_N = F_S\,, \text{ on } U_I\,, \tag{3.103}$$

which implies

$$A_N = A_S + \Lambda_{NS}\,, \tag{3.104}$$

with $d\Lambda_{NS} = 0$, i.e. $\Lambda_{NS} \in \Omega^1(U_I)$ is closed but not necessarily exact. In physics, this is exactly what we call a gauge transformation. We know even more than this because of (3.99). We can decompose the S^2 integral into an integral over U_N and an integral over U_S

$$\begin{aligned}\int_{S^2} F &= \int_{U_N} F_N + \int_{U_S} F_S = \int_{U_N} dA_N + \int_{U_S} dA_S \\ &= \int_{\partial U_N} A_N + \int_{\partial U_S} A_S \\ &= \int_{U_I} (A_N - A_S) \\ &= \int_{U_I} \Lambda_{NS}\,.\end{aligned} \tag{3.105}$$

We see that if $Q_m \neq 0$, Λ_{NS} cannot be exact. Instead $\Lambda_{NS} \in H^1(S^1) \in \mathbb{Z}$. We see that magnetic monopoles are classified by integers. Without more physics or mathematics, this is as far as we can go.

Physically we know that if we couple the gauge field to some charged field, e.g. a scalar ψ, then accompanying the $U(1)$ gauge transformation (3.104), the charged field transforms as $\psi_N = \psi_S \exp(-ie\phi)$ with $\Lambda_{NS} = d\phi$. Since the wavefunction must be single-valued we have

$$\frac{e\,[\phi(2\pi) - \phi(0)]}{\hbar c} \in 2\pi\mathbb{Z}\,. \tag{3.106}$$

However, from (3.105) we know that $\phi(2\pi) - \phi(0) = \int_{U_I} \Lambda_{NS} = Q_m$ and hence

$$\frac{eQ_m}{\hbar c} \in 2\pi\mathbb{Z}\,,. \tag{3.107}$$

This is the Dirac charge quantisation condition.

Exercise 3.11: Show that for the magnetic point charge, $dF = g\,\delta(\mathbf{r})$, the gauge potential can be chosen to be

$$\begin{aligned}A_N &= -g\,(1+\cos\theta)\,d\phi\,, \\ A_S &= g\,(1-\cos\theta)\,d\phi\,.\end{aligned} \tag{3.108}$$

Even though A is not a 1-form on M, it appears naturally in this context. We will formalise what A is via fibre bundles in the next chapter.

4. Fibre bundles

In physics, we often encounter differentiable manifolds (or more generally topological spaces) which locally appear like a product of two manifolds but not globally. Perhaps the most famous example of this is in gauge theory, which in each spacetime patch seems to be making use of the gauge group G, viewed as a differential manifold, times the spacetime patch. However, globally this product breaks down because gauge transformations in spacetime shift the position in G. We will return to a concrete example.

Mathematically, such spaces are known as fibre bundles. Perhaps the easiest example to visualise are the cylinder and the Möbius strip, which both locally look like a rectangle $I \times I$. However, while globally the cylinder is a product space $I \times S^1$, the Möbius strip is not, see figure 17.

We formalise these intuitive notions as follows.

Definition: (E, π, B, F, G) is called a **fibre bundle** if

(i) E is a topological space called the **total space**.

(ii) B is a topological space called the **base space**.

(iii) F is a topological space called the **fibre**.

(iv) A surjection $\pi : E \longrightarrow B$ called the **projection**. The inverse image of a point $p \in B$, $\pi^{-1}(p) = F_p \sim F$ is called the fibre at p.

(v) A group G of homeomorphisms of F called the **structure group**.

(vi) A set of open coverings $\{U_\alpha\}$ of B with homeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$, called the **local trivialisations** such that

$$\pi \circ \phi_\alpha^{-1}(p, f) = p \,\forall\, p \in U_\alpha,\, f \in F\,. \tag{4.1}$$

(vii) For any fixed $p \in U_\alpha \cap U_\beta \neq \emptyset$, the maps

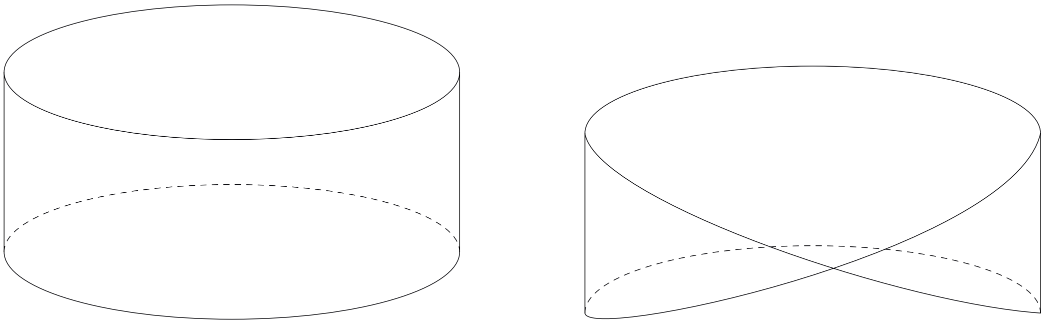
$$t_{\alpha\beta} \equiv \phi_{\alpha,p} \circ \phi_{\beta,p}^{-1} : F \longrightarrow F\,, \tag{4.2}$$

are elements of the structure group G, i.e.

$$t_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G\,. \tag{4.3}$$

The maps $t_{\alpha\beta}$ are known as **transition functions**.

If we require smoothness (physics applications usually do), we should replace “topological space” everywhere in the definition with “differentiable manifold”, and require $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ to be a diffeomorphism rather than a homeomorphism.



(a) (b)

Figure 17: The cylinder shown in (a) is globally a product while the Möbius strip shown in (b) is not.

Exercise 4.1: Show that the transition functions of fibre bundles satisfy the following **cocycle conditions** by construction.

(i) For each fixed $p \in U_\alpha$,

$$t_{\alpha\alpha}(p) = 1\,. \tag{4.4}$$

(ii) For each fixed $p \in U_\alpha \cap U_\beta \neq \emptyset$,

$$t_{\alpha\beta}(p) t_{\beta\alpha}(p) = 1\,. \tag{4.5}$$

(iii) For each fixed $p \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$,

$$t_{\alpha\beta}(p) t_{\beta\gamma}(p) = t_{\alpha\gamma}(p)\,. \tag{4.6}$$

Definition: A **trivial fibre bundle** is a fibre bundle whose total space $E \sim B \times F$ is homeomorphic (diffeomorphic) to the direct product of the base space B and the fibre F.

Fibre bundles are often denoted as

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

with the structure group G left implicit. Sometimes, the projection π is also not shown explicitly.

Another commonly used notation is to simply state the projection map

$$\pi : E \longrightarrow B\,, \tag{4.7}$$

to denote the fibre bundle E with base B and projection π. In this notation one does not explicitly refer to the fibre F or the structure group G.

Note that the fibre bundle is defined asymmetrically, with only a projection from E onto the base B but not onto the fibre F. If we could also define a projection $\pi' : E \longrightarrow F$ then the fibre bundle would be trivial, i.e. $E \sim B \times F$.

Example 4.1: As can be seen from figure 18, the Möbius strip M can be described as the fibre bundle

$$\begin{array}{ccc} I & \longrightarrow & M \\ & & \downarrow \pi \\ & & S^1 \end{array}$$

with structure group $G = \{e, g\} = \mathbb{Z}_2$, where e denotes the identity and g flips the interval $I = [0,1]$ by $g : I \longrightarrow I$ by $g : t \to 1 - t$.

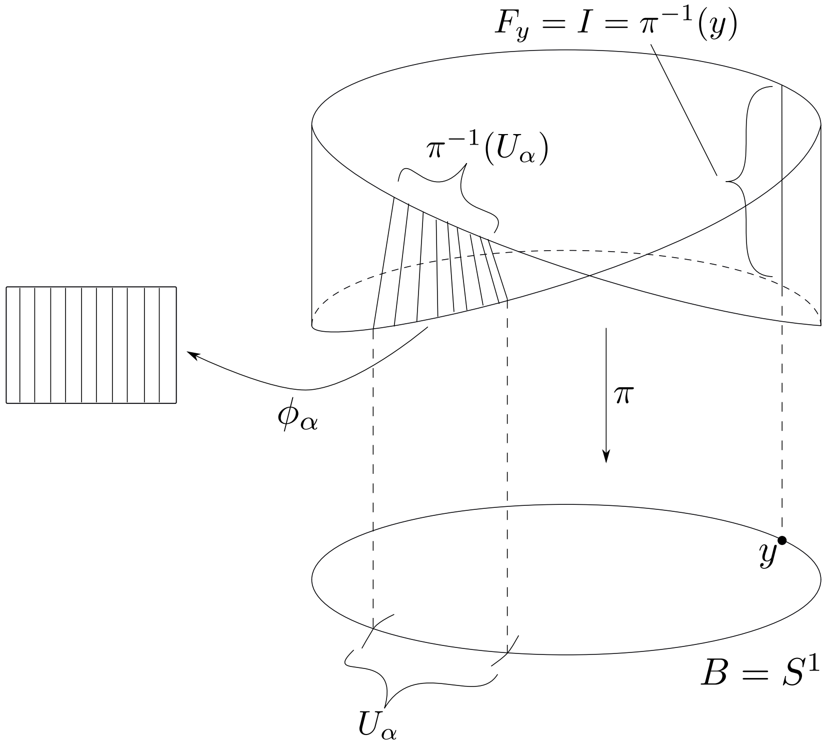


Figure 18: The Möbius strip is a fibre bundle with fibre I, base S^1 and structure group $\mathbb{Z}_2$. The figure shows an open subset $U_\alpha \in S^1$, $\pi^{-1}(U_\alpha)$ and the local trivialisation $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times I$. The fibre above a point $y \in S^1$ is $F = I = \pi^{-1}(y)$.

For example, we can use the open covering $\{U_1, U_2\}$ with intersections $U_1 \cap U_2 = U_A \cup U_B$ as shown in figure 19. The transition functions are then

$$t_{12}(p) = \begin{cases} e\,, & \text{if } p \in A\,, \\ g\,, & \text{if } p \in B\,. \end{cases} \tag{4.8}$$

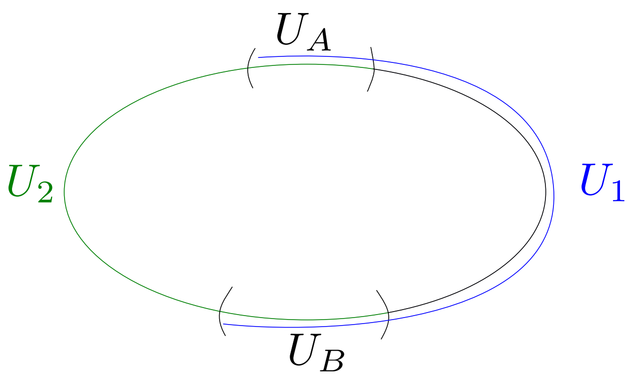


Figure 19: An open covering $\{U_1, U_2\}$ of S^1 with intersections $U_1 \cap U_2 = U_A \cup U_B$.

The fact that the structure group is $\mathbb{Z}_2$ can also be seen by inspection. If we traverse the base S^1 once we find that the fibre I has been reflected. However, traversing the base S^1 twice, we end up with the fibre in its original orientation.

Note that the definition of a fibre bundle is reminiscent of the definition of a manifold with coverings $\{U_i\}$ and trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ being the analogue of coordinate charts, $\phi'_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$. You probably also know that a manifold can be defined via the coordinate charts themselves $\{U_\alpha, \phi_\alpha\}$ (this is how we often define manifolds as physicists by introducing suitable coordinates). The same holds for fibre bundles: knowing $(B, F, G, \{U_\alpha\}, \phi_\alpha)$ we can build the fibre bundle E.

Furthermore, just like for manifolds, we do not view the choice of local trivialisations $\{U_\alpha, \phi_\alpha\}$ as important. Therefore, we define a notion of equivalence of fibre bundles as defined above.

Definition: Two fibre bundles (E, π, F, G, B) and (E', π', F, G, B) are **equivalent** if the local trivialisations

$$\begin{aligned} \phi_\alpha &: \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F\,, \\ \phi'_\alpha &: \pi'^{-1}(U_\alpha) \longrightarrow U_\alpha \times F\,, \end{aligned} \tag{4.9}$$

are such that

$$\lambda_\alpha \equiv \phi'_\alpha \circ \phi_\alpha^{-1} : U_\alpha \times F \longrightarrow U_\alpha \times F\,, \tag{4.10}$$

is a homeomorphism of F with $\lambda_\alpha \in G$ for each $p \in U_\alpha$.

Note that in the above definition we have implicitly assumed that E and E' are decribed by the same open coverings of B, $\{U_\alpha\}$. In fact, the definition of equivalence can be generalised to take into account different coverings $\{U_\alpha\}$ and $\{U'_\alpha\}$.

We will always identify any two equivalent fibre bundles.

Exercise 4.2: Show that

$$t'_{\alpha\beta}(p) = \lambda_\alpha^{-1}(p) t_{\alpha\beta}(p) \lambda_\beta(p)\,,\ \forall\, p \in U_\alpha \cap U_\beta \neq \emptyset\,. \tag{4.11}$$

Soon, we will see how this result is related to Yang-Mills gauge transformations.

Exercise 4.3: Show that if a fibre bundle is trivial, then the transition functions take the form

$$g_{\alpha\beta}(p) = \lambda_\alpha^{-1}(p)\lambda_\beta(p)\,, \tag{4.12}$$

for some homeomorphism $\lambda_\alpha : F \longrightarrow F$.

4.1 Vector Bundles

Example 4.2: In your first general relativity or differential geometry course, you were probably taught to think of the tangent bundle of a differentiable manifold M as the union of tangent spaces

$$TM = \bigcup_{p\in M} T_pM\,. \tag{4.13}$$

In fact, the tangent bundle is a fibre bundle

$$\pi : TM \longrightarrow M\,, \tag{4.14}$$

as can be shown as follows.

Since we define the tangent bundle as the union of all tangent spaces, an element $v \in TM$ consists of a vector $v \in T_pM$ for some $p \in M$. This allows us to define a projection map onto M by

$$\pi(v) = p\,, \tag{4.15}$$

for $v \in T_pM$. The fibre at p is then the tangent space T_pM.

$$\pi^{-1}(p) = T_pM\,, \tag{4.16}$$

which is a vector space of dimension $\dim(M) = n$.

We can use local coordinates on M to construct a local trivialisation as follows. At $p \in U_\alpha \subset M$ we use the local coordinates x^i on U_α so that

$$v = v^i(p)\frac{\partial}{\partial x^i}|_p\,. \tag{4.17}$$

This allows us to define the local trivialisations

$$\phi_\alpha(v) = \left(p, v^i\right)\,. \tag{4.18}$$

Using this, one can easily show that the transition functions are given by

$$t_{\alpha\beta}(p) = \left(\frac{\partial x^i_{(\alpha)}}{\partial x^j_{(\beta)}}\right)\,. \tag{4.19}$$

Because coordinate transformations are invertible, the structure group of TM is $G = \mathrm{GL}(n,\mathbb{R})$.

Exercise 4.4: Using the local trivialisations (4.18), show that

$$t_{\alpha\beta}(p) = \left(\frac{\partial x^i_{(\alpha)}}{\partial x^j_{(\beta)}} \right) . \tag{4.20}$$

Exercise 4.5: Using local coordinates on M, show that T^*M is a fibre bundle $\pi : T^*M \longrightarrow M$. What is the fibre $\pi^{-1}(p)$ for $p \in M$? What are the transition functions and what is the structure group?

In example 4.2 saw that the fibres of TM, $\pi^{-1}(p) = T_pM$, $\forall\, p \in M$ are the tangent spaces at each point $p \in M$. Similarly, in the exercise above, you would have shown that the fibre T^*M is also a vector space. Vector spaces are important because they allow a notion of linear maps acting on them. This motivates the following definition.

Definition: A **vector bundle** is a fibre bundle whose fibre is a vector space and G acts as a *linear* map.

Clearly, TM and T^*M are vector bundles. The fact that G acts as a linear map gives vector bundles interesting properties that we will investigate in the next lecture.

4.2 Principal bundles

Example 4.3: Another important example of a fibre bundle is given by the frame bundle, defined as follows.

Definition: The **frame bundle of a manifold** M, FM, is defined as

$$FM = \bigcup_{p \in M} F_pM , \tag{4.21}$$

where F_pM is the set of ordered bases, or **frames**, of T_pM at $p \in M$.

Note that we can obtain any frame by acting with $\mathrm{GL}(n,\mathbb{R})$ on a given one. For example, consider the open subset $U_\alpha \subset M$ with local coordinates x^i. Then any frame at $p \in U_\alpha$

$$u(p) = \{e_1, \ldots, e_n\}(p) , \tag{4.22}$$

can be written as

$$u_a(p) = \underbrace{u^j{}_a(p)}_{\in \mathrm{GL}(n,\mathbb{R})} \frac{\partial}{\partial x^j}|_p , \quad (1 \leq a \leq n) , \tag{4.23}$$

where $u^j{}_a(p)$ are just the n components (labelled by $j = 1, \ldots, n$) of the vector fields (labelled by $a = 1, \ldots, n$) making up the frame $u(p)$. Note that linear independence of the $u_a(p)$ guarantees that $u^j{}_a(p) \in \mathrm{GL}(n,\mathbb{R})$.

Exercise 4.6: Show that $\pi : FM \longrightarrow M$ is a fibre bundle with the same transition function as TM given in equation (4.19). Show that the fibre is $\mathrm{GL}(n,\mathbb{R})$.
Hint: Use local coordinates on M and equation (4.23) to write any frame as a $\mathrm{GL}(n,\mathbb{R})$ element.

Since any vector space admits a basis, we can in fact generalise the concept of a frame bundle to arbitrary vector bundles $\pi : E \longrightarrow B$ as follows.

Definition: The **frame bundle**, $F(E)$, associated to a vector bundle $\pi : E \longrightarrow B$ is defined as

$$F(E) = \bigcup_{p\in B} F_pB\,, \tag{4.24}$$

where F_pB is the set of ordered bases, or frames, of the vector space $\pi^{-1}(p)$ at $p \in B$.

Exercise 4.7*: Use the local trivialisations of the vector bundle $\pi : E \longrightarrow B$ to construct the frame bundle $\pi' : F(E) \longrightarrow B$ with the same transition functions as the vector bundle. Show that the fibre is $\mathrm{GL}(n,\mathbb{R})$.

Even though $F(E)$ has the same transition functions as E it is a very different kind of fibre bundle because its fibre is not a vector space. Instead, the fibre is the structure group $\mathrm{GL}(n,\mathbb{R})$ itself. Such fibre bundles, whose fibre are the same as the structure group play a crucial role in the study of fibre bundles and theoretical physics.

Definition: A **principal bundle** is a fibre bundle whose fibre is the structure group G. For structure group G, one also often speaks of a **principal G-bundle**.

When introducing vector bundles, we noticed that the vector space structure allows a notion of linear maps, which we indicated gives vector bundles certain properties. What, by contrast, do we gain by having the structure group G as fibre? The answer is that we can use the right action of the group on itself $G\times G \longrightarrow G$ to define a right-action on the G-bundle, P, itself:

$$P\times G \longrightarrow P\,. \tag{4.25}$$

We can define this action using the local trivialisations $\{U_\alpha, \phi_\alpha\}$,

$$\begin{aligned} \phi_\alpha : \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times G\,, \\ \phi_\alpha(u) &= (p, g_\alpha)\,, \end{aligned} \tag{4.26}$$

where $u \in \pi^{-1}(U_\alpha)$ and $p = \pi(u)$. Using this, we define the right action as

$$\phi_\alpha(ua) = (p, g_\alpha a)\ \forall a \in G\,. \tag{4.27}$$

That is, the right action just acts on the fibre, leaving the base invariant. We will often use the notation $R_a : P \longrightarrow P$ to denote right action of $a \in G$ on the principal G-bundle $\pi : P \longrightarrow B$. Since the right-action acts only on the fibre, we have

$$\pi \circ R_g = \pi \ \forall g \in G \,. \tag{4.28}$$

In other words, the following diagram commutes.

$$\begin{tikzcd} P \arrow[rr, "R_g"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & B & \end{tikzcd}$$

Because we defined the right G-action using local trivialisations, we need to make sure that our definition is in fact trivialisation-independent. To do this, we use the form of the transition functions (4.26) and the definition (4.2) to deduce that

$$\left(p, g_\beta\right) = \left(p, t_{\beta\alpha}(p) g_\alpha\right) \,, \tag{4.29}$$

and hence

$$g_\beta = t_{\beta\alpha}(p) g_\alpha \,. \tag{4.30}$$

Next we note that for $u \in P$ and $p = \pi(u) \in U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{aligned} u\,a &= \phi_\beta^{-1}\left(p, g_\beta a\right) \\ &= \phi_\beta^{-1}\left(p, t_{\beta\alpha}(p) g_\alpha a\right) \\ &= \phi_\alpha^{-1}\left(p, g_\alpha a\right) \,. \end{aligned} \tag{4.31}$$

Here in going from the penultimate to the final line, we again used the definition of the transition functions (4.2) to change the trivialisation map ϕ_β^{-1} to ϕ_α^{-1}.

Example 4.4: Perhaps the most famous example of a principal bundle is given by the Hopf fibre bundle.

$$\begin{tikzcd} \mathrm{U}(1) \arrow[r] & S^3 \arrow[d, "\pi"] \\ & S^2 \end{tikzcd}$$

The projection map $\pi : S^3 \longrightarrow S^2$ is commonly known as the **Hopf map**.

To describe this fibre bundle, consider $S^3 \subset \mathbb{C}^2$ defined by

$$|z_1|^2 + |z_2^2| = 1 \,, \tag{4.32}$$

and $S^2 \subset \mathbb{C} \times \mathbb{R}$ defined by

$$|z_0|^2 + x^2 = 1 \,, \tag{4.33}$$

where z_0, z_1 and $z_2 \in \mathbb{C}$ while $x \in \mathbb{R}$. Using these functions on S^3 and S^2, we can define $\pi : S^3 \longrightarrow S^2$ by

$$\pi(z_1, z_2) = \left(\underbrace{2z_1 z_2^*}_{\mathbb{C}}, \underbrace{|z_1|^2 - |z_2|^2}_{\mathbb{R}} \right) . \tag{4.34}$$

Exercise 4.8: Using (4.34), show that $\pi(z_1,z_2) \in S^2$ for all $(z_1,z_2) \in S^3$.

This shows that the image of the projection π is S^2, i.e. $\pi : S^3 \longrightarrow S^2$.

Exercise 4.9: Using (4.34), show that if $(z_1,z_2) \in S^3$ and $(w_1,w_2) \in S^3$ then

$$\pi(z_1,z_2) = \pi(w_1,w_2) \iff (z_1,z_2) = \lambda\,(w_1,w_2) \text{ where } |\lambda| = 1\,. \tag{4.35}$$

Hence $\pi^{-1}(p) = \mathrm{U}(1)$ for $p \in S^2$, and the fibre is $\mathrm{U}(1)$ as required.

Now that we have a projection map, we have to find local trivialisations. For this, we will use "stereographic coordinates" on S^2. These are defined on the open subsets $U_{N/S}$ of S^2 given by

$$U_{N/S} = \left\{ (z_0, x) \mid |z_0|^2 + x^2 = 1\,, \text{ and } x \neq 1 \right\} , \tag{4.36}$$

where $z_0 \in \mathbb{C}$ and $x \in \mathbb{R}$. Thus $U_{N/S}$ cover all of the S^2 except for the north / south pole, respectively. Note that (z_0, x) are three well-defined functions S^2 and *not* coordinates on $U_{N/S}$.

However, stereographic coordinates are defined as follows.

$$\begin{aligned} \text{On } U_S: \quad \chi_S &= \frac{z_0}{1-x} = \frac{2z_1 z_2^*}{1 - |z_1|^2 + |z_2|^2} = \frac{2z_1 z_2^*}{2|z_2|^2} = \frac{z_1}{z_2} \in \mathbb{C}\,, \\ \text{On } U_N: \quad \chi_N &= \frac{z_0^*}{1+x} = \frac{2z_1^* z_2}{1 + |z_1|^2 - |z_2|^2} = \frac{2z_1^* z_2}{2|z_1|^2} = \frac{z_2}{z_1} \in \mathbb{C}\,. \end{aligned} \tag{4.37}$$

Clearly, $\chi_{S/N}$ is only well-defined on $U_{S/N}$.

We now use these local coordinates to define

$$\phi_{S/N} : \pi^{-1}\left(U_{S/N}\right) \longrightarrow U_{S/N} \times \mathrm{U}(1)\,, \tag{4.38}$$

by

$$\begin{aligned} \phi_S(z_1,z_2) &= \left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|} \right) , \\ \phi_N(z_1,z_2) &= \left(\frac{z_2}{z_1}, \frac{z_1}{|z_1|} \right) . \end{aligned} \tag{4.39}$$

Note that we could not have have used $\frac{z_1}{|z_1|}$ on ϕ_S in the $\mathrm{U}(1)$ element because z_1 can vanish on U_S and therefore this combination is not well-defined. Similarly for U_N and $\frac{z_2}{|z_2|}$.

On the overlap $U_N \cap U_S$, i.e. where $z_1 \neq 0$ and $z_2 \neq 0$, we have

$$t_{NS} = \frac{z_1/|z_1|}{z_2/|z_2|} = \frac{z_1}{z_2}\frac{|z_2|}{|z_1|} = \frac{z_0}{|z_0|} \in \mathrm{U}(1)\,. \tag{4.40}$$

Thus, we see that the transition functions are $\mathrm{U}(1)$ valued and therefore the structure group $G = \mathrm{U}(1)$ is the same as the fibre. Hence, S^3 is a principal $\mathrm{U}(1)$-bundle over S^2.

At this stage, you might wonder what other principal $\mathrm{U}(1)$-bundles one can form over S^2. In each case, we would need to use local trivialisations, for example using the covering $\{U_N, U_S\}$ with transition functions $t_{NS} \in \mathrm{U}(1)$. We could smoothly shrink the open subsets $U_{N/S}$ so that their overlap is eventually just given by the equator. Therefore, the transition function would be a map

$$t_{NS} : S^1 \longrightarrow \mathrm{U}(1)\,. \tag{4.41}$$

By now, you know that such maps belong to $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. This implies that we could consider constructing different principal $\mathrm{U}(1)$-bundles over S^2 characterised by an integer. The total spaces of such fibre bundles are known as Lens spaces.

Exercise 4.10*: Find the projection map for

$$\begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \big\downarrow \pi \\ & & S^4 \end{array}$$

Hint: You may find it useful to describe S^7 and S^4 using quaternions. A quaternion $q \in \mathbb{H}$, can be written as the formal sum

$$q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z\,, \tag{4.42}$$

with w, x, y, $z \in \mathbb{R}$ and

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\,, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\,, \\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}\,, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}\,. \end{aligned} \tag{4.43}$$

You may want to use quaternion conjugation

$$* : q \longrightarrow q^*\,, \tag{4.44}$$

defined as

$$q^* = w - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z\,, \tag{4.45}$$

to define a quaternion norm and use this to describe $S^7 \subset \mathbb{H}^2$. You may also want to show that $S^4 \simeq \mathbb{HP}^1$ where $\mathbb{HP}^1 = \mathbb{H}^2/\sim$ with the equivalence relation

$$(q_1, q_2) \sim (\eta\, q_1, \eta\, q_2) \;\; \forall \eta \in \mathbb{H}\,. \tag{4.46}$$

We end this chapter with the important remark, that given any fibre bundle $\pi : E \longrightarrow B$ with structure group G we can always construct the associated principal G-bundle $\pi' : P(E) \longrightarrow B$ which has the same transition functions as $\pi : E \longrightarrow B$. We saw an example of this when we constructed the frame bundle of a manifold M in exercise 4.6, using local trivialisation based on local coordinates on M. This meant that the frame bundle had the same transition functions as TM, which we also constructed using local coordinates on M.

5. Fibre bundles II

The aim of this chapter is two-fold. First, we will start investigating when fibre bundles are trivial. The second half will then introduce connections on fibre bundles and describe their relation to gauge theories.

5.1 Sections of fibre bundles

In many applications, we use maps from the base B into the fibre bundle $\pi : E \longrightarrow B$, e.g. $\sigma : B \longrightarrow E$. For example, vector fields are maps $\sigma : M \longrightarrow TM$ and as you know are pervasive in differential geometry and general relativity. We formalise this concept as follows.

Definition: A section of a fibre bundle $\pi : E \longrightarrow B$ is a continuous map

$$\sigma : B \longrightarrow E \,, \tag{5.1}$$

such that $\pi \circ \sigma = \mathbf{1}|_B$.

Therefore, sections can be viewed as "inverses" of the projection map $\pi : E \longrightarrow B$.

The space of section of a fibre bundle E is often denoted by $\Gamma(E)$. For example, you will have probably come across the notation $\Gamma(TM)$ for the space of vector fields. However, our intuition from the tangent bundle can be misleading, because it is a vector bundle.

Exercise 5.1: Show that every vector bundle has at least one section, the **null section** $\sigma(p) = \mathbf{0}$, or in terms of a local trivialisation $\sigma(p) = (0, p)$.
Hint: Recall that the structure group acts linearly on the fibre of a vector bundle.

However, most fibre bundles *do not admit a section*. Instead, we are lead to define local sections on patches.

Definition: Given a fibre bundle $\pi : E \longrightarrow B$ with open covers $\{U_\alpha\} \subset B$, a local section is a continuous maps

$$\sigma_\alpha : U_\alpha \longrightarrow E \,, \tag{5.2}$$

such that $\pi \circ \sigma_\alpha = \mathbf{1}_{U_\alpha}$.

Exercise 5.2: Use local trivialisations to show that local sections always exist for any fibre bundle $\pi : E \longrightarrow B$.

Transitive and free group actions

An action of a group G on the space X is called:

- Transitive, if $\forall x, y \in X$ $\exists$ some $g \in G$ such that
$$y = g x\,. \tag{5.3}$$
- Free, if for $x \in X$ and $g \in G$
$$x = g x \iff g = e\,, \tag{5.4}$$
where e denotes the identity of G.

The group axioms, in particular the existence of inverses, ensure that the action of G on itself is transitive and free.

Recall that for a principal G-bundle $\pi : P \longrightarrow B$, there is a right G-action on P. This leads to a much stronger relation between local sections and local trivialisations of principal bundles. To do this, we first note that the G-action on G is transitive and free, and is compatible with the projection, i.e.
$$\pi(u) = \pi(u g)\ \forall g \in G\,. \tag{5.5}$$
Therefore, for any two points $u_1, u_2 \in \pi^{-1}(p)$ and any $p \in B$, we can write
$$u_1 = u_2\, g \ \text{ for some unique } g \in G\,. \tag{5.6}$$
This way we can define a map
$$\tau_p : \pi^{-1}(p) \otimes \pi^{-1}(p) \longrightarrow G\,, \tag{5.7}$$
as
$$\tau_p(u_1, u_2) = g\,, \tag{5.8}$$
satisfying $u_1 = u_2\, g$. Note that if we fix a point $u \in P$ the map
$$\tau_u : \pi^{-1}(\pi(u)) \longrightarrow G\,, \tag{5.9}$$
defined as
$$\tau_u(v) \equiv \tau_{\pi(u)}\left(v, u\right)\,, \tag{5.10}$$
is an isomorphism $\pi^{-1}(\pi(u)) \simeq G$. In this isomorphism u is mapped to the identity of G. In other words, by fixing a point $u \in P$ we can construct the isomorphism $\pi(u) \sim G$ by identifying u with the identity in G.

Since a local section, provides us with a continuous map $\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}\left(U_\alpha\right)$, it gives us a continuous assignment of identity elements in $\pi^{-1}\left(U_\alpha\right)$ and thus of homeomorphisms $\pi^{-1}\left(U_\alpha\right) \sim U_\alpha \times G$. In other words, to any local section we can associate a canonical local trivialisation.

Definition: Let $\pi : P \longrightarrow B$ be a principal G-bundle and $\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$ a local section defined on the open subset $U_\alpha \subset B$. The **canonical local trivialisation** $\phi_{\sigma_\alpha} : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ is defined as

$$\phi_{\sigma_\alpha}(u) = \left(\pi(u),\, \tau_{\pi(u)}(\sigma_\alpha(\pi(u)),u)\right) \;\forall\, u \in \pi^{-1}(U_\alpha)\,, \tag{5.11}$$

with inverse given by

$$\phi^{-1}_{\sigma_\alpha}(p, g_\alpha) = \sigma_\alpha(p) g_\alpha \;\forall\, p \in U_\alpha,\, g_\alpha \in G\,. \tag{5.12}$$

Note that $\pi(\sigma_\alpha(p)\, g) = \pi(\sigma_\alpha(p)) = p$ for any $g \in G$ since the G-action was compatible with the projection, i.e.

$$\pi(u) = \pi(u\, g)\; \forall g \in G\,. \tag{5.13}$$

Since we can associate a local trivialisation to each local section, we also immediately obtain the following theorem.

Theorem 5.1: A principal bundle admits a section $\Longleftrightarrow$ it is trivial.

Proof. Consider a principal G-bundle $\pi : P \longrightarrow B$ with a section σ. The canonical trivialisation associated to σ is now a homeomorphism

$$\phi_\sigma : P \longrightarrow B \times G\,, \tag{5.14}$$

with

$$\phi_\sigma(u) = (\pi(u), \tau(\sigma(\pi(u)),u)) \;\forall\, u \in P\,, \tag{5.15}$$

and inverse

$$\phi^{-1}_\sigma(p, g) = \sigma(p) g \;\forall\, p \in B,\, g \in G\,. \tag{5.16}$$

Therefore, P is trivial. □

This is a powerful result because it is often important to know when a principal bundle is trivial. Recall, that we mentioned at the end of the previous chapter that given a fibre bundle $\pi : E \longrightarrow B$ with structure group G, we can construct an associated principal G-bundle $\pi' : P(E) \longrightarrow B$ with the same transition functions as $\pi : E \longrightarrow B$. This allows us to prove the following theorem.

Theorem 5.2: If $P(E)$ is trivial, then E is trivial.

Proof. $P(E)$ and E have the same transition functions. □

Definition: A **parallelisable manifold** of dimension n has n linearly independent vector fields at each point $p \in M$.

Theorem 5.3: A vector bundle $\pi : E \longrightarrow B$ with n linearly independent sections at each point $p \in B$ is a trivial bundle.

Exercise 5.3: Prove theorem 5.1.
Hint: Either use the n linearly independent sections of E to build a global trivialisation or build a section of the frame bundle $P(E)$.

Corollary: A parallelisable manifold has trivial tangent bundle and therefore a trivial frame bundle.

Example 5.1: Lie groups are parallelisable manifolds.

Exercise 5.4: Show that Lie groups are parallelisable manifolds by constructing n linearly independent vector fields at each point $p \in M$.
Hint: Use the right-invariant vector fields on the Lie group.

5.2 Connections on fibre bundles

We will begin by discussing connections on general fibre bundles, before specialising to connections on principal G-bundles. There, we will see that the right G-action allows us to derive several powerful results. Connections on principal G-bundles are also the mathematical setting for gauge theories, as we will see.

5.2.1 Ehresmann connection and parallel transport

We will in the following always require our fibre bundles to be smooth. Given a fibre bundle $\pi : E \longrightarrow B$, a connection on E is a way of mapping a point $u \in E$ to another $u' \in E$ by moving "parallel" to the base B. Recall that the total space of the fibre bundle E is a differential manifold and therefore has a tangent bundle TE. Therefore to move inside E "parallel" to B we would need to construct vector fields on E that are tangent to B in an appropriate sense. However, without any extra structure on the fibre bundle, there is no way canonical way of doing this.

By contrast, there *is* a canonical way of defining vector fields "parallel to the fibre". This asymmetry between defining vector fields parallel to the fibres and those to the base is due to the projection map of the fibre bundle. In particular, we can pull-back the projection map π to the tangent bundle to obain

$$\pi_* : TE \longrightarrow TB \,. \tag{5.17}$$

Since $\pi : E \longrightarrow B$ is surjective, so is π_*.

Definition: Given a fibre bundle $\pi : E \longrightarrow B$, the **vertical subbundle**, VE, of TE is defined as

$$VE = \mathrm{Ker}\, \pi_* \subset TE \,. \tag{5.18}$$

At each point $u \in E$, the vertical subbundle consists of the vectors tangent to the fibre at u, i.e. $V_u E = T_u \left(E_{\pi(u)}\right)$. The subbundle is called vertical the fibres are often visualised as extending vertically from the horizontal base B.

Exercise 5.5: Show that VE is integrable, i.e. for any $X,Y \in \Gamma(VE)$

$$[X,Y] \in \Gamma(VE)\,. \tag{5.19}$$

Bundle maps

The pull-back of the projection map π_* is an example of a "bundle map". A **bundle map** is a pair of continuous maps $\sigma : E_1 \longrightarrow E_2$ and $\sigma' : B_1 \longrightarrow B_2$ between two fibre bundles $\pi_1 : E_1 \longrightarrow B_1$ and $\pi_2 : E_2 \longrightarrow B_2$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\sigma"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\sigma'"] & B_2 \end{tikzcd}$$

commutes. That is, $\sigma' \circ \pi_1 = \pi_2 \circ \sigma$.
The pull-back of the projection map π_* is a bundle map between the tangent bundles to E and B, i.e. between the fibre bundles $\pi_1 : TE \longrightarrow E$ and $\pi_2 : TB \longrightarrow B$.

While the vertical subbundle VE is canonically defined, the fact that there is no projection map onto the fibre of a fibre bundle means that the complement to VE inside TE, which would consist of vectors tangent to the base, is not canonically defined. Instead we have to choose a complement to VE.

Definition: An **(Ehresmann) connection** on a fibre bundle $\pi : E \longrightarrow B$ is a choice of subspaces $H_uE \subset T_uE$ for each $u \in E$ such that

(i) $T_uE = V_uE \oplus H_uE \;\forall u \in E$ (complementarity),

(ii) any smooth vector field $X \in \Gamma(TE)$ is separated into smooth vector fields X_H and X_V with
$$X = X_H + X_V\,, \tag{5.20}$$
with $X_V|_u \in V_uE$ and $X_H|_u \in H_uE$ (smoothness).

By defining HE, an Ehresmann connection on E now gives us a way to move in E in a way that is "parallel" to B. We require the tangent vector along the path we take in E to be horizontal at each point. This can be used to define a horizontal lift of a path $\gamma : I \longrightarrow B$ to E.

Definition: Let $\pi : E \longrightarrow B$ be a fibre bundle with an Ehresmann connection HE, $\gamma : I \longrightarrow B$ a path, and $u_0 \in E$ a point satisfying $\pi(u_0) = \gamma(\gamma(0))$. Then a **horizontal lift** $\tilde{\gamma}_{u_0} : I \longrightarrow E$ of γ through u_0 is a path in E satisfying

(i) $\pi \circ \tilde{\gamma}_{u_0} = \gamma$ with $\tilde{\gamma}_{u_0}(0) = u_0$,

(ii) the tangent vectors to the path $\tilde{\gamma}_{u_0}$ belong to $H_{\tilde{\gamma}_{u_0}(t)}E$ at each point $\tilde{\gamma}_{u_0}(t)$, i.e.

$$\frac{d\tilde{\gamma}_{u_0}}{dt}(t) \in H_{\tilde{\gamma}_{u_0}(t)}E \;\; \forall t \in I\,. \tag{5.21}$$

If only condition (i) is satisfied, $\tilde{\gamma}_{u_0}$ is called a **lift** of γ through u_0.

Note that we fix the startpoint $u_0 = \tilde{\gamma}_{u_0}(0)$ but have left the endpoint $u_1 = \tilde{\gamma}(1)$ of the horizontal unspecified. At this stage, we should address if and when a horizontal lift exists and understand to what extent it is unique. The following theorem (based on the Picard-Lindelöf theorem) shows local existence, i.e. for small $t \in I_\varepsilon = [0,\varepsilon)$, and uniqueness.

Theorem 5.4: Given a path $\gamma : I \longrightarrow B$ and a fibre bundle $\pi : E \longrightarrow B$ with Ehresmann connection HE, then the horizontal lift $\tilde{\gamma}_{u_0}$ of γ through $u_0 \in \pi^{-1}(\gamma(0))$ exists for small $t \in I_\varepsilon = [0,\varepsilon)$ and is unique.

The reason why the horizontal lift may only exist locally is the same as why general integral curves only exist locally: the horizontal lift is given by the solution of an ordinary differential equation which may develop singularities.

We can use the horizontal lift of a path γ to map any point $u_0 \in \pi^{-1}(\gamma(0))$ to another point $u_\varepsilon \in \pi^{-1}(\gamma(\varepsilon))$ for small ε by "parallel transporting" along γ. However, the fact that the horizontal lift γ_{u_0} in general only exists for small t with the range of existence itself depending on u_0 makes a proper definition of parallel transport in the general case awkward. We will soon see that on principal connection, we can define parallel transport without any of these problems.

5.2.2 Curvature of Ehresmann connection

In exercise 5.5 you showed that the vertical subbundle of a fibre bundle $\pi : E \longrightarrow B$ is integrable, i.e. $[X,Y] \in \Gamma(VE)$ for all $X,Y \in \Gamma(VE)$. On the other hand, the vertical subbundle is in general not integrable. Instead, it measures the curvature of the Ehresmann connection HE.

Definition: The **curvature** of an Ehresmann connection HE of a fibre bundle $\pi : E \longrightarrow B$ is a map $R : \Gamma(TE) \otimes \Gamma(TE) \longrightarrow \Gamma(VE)$ given by

$$R(X,Y) = [X_H, Y_H]_V \;\; \forall\, X, Y \in \Gamma(TE)\,, \tag{5.22}$$

where X_H, Y_H are the horizontal parts of X and Y and $_V$ denotes the projection onto the vertical subbundle.

We mentioned in the introduction that connections on fibre bundles will be one of two examples in these lectures which go beyond topology (the other given by Hodge theory which requires a metric). Here we see a manifestation of this in the curvature, which allows us to *locally* distinguish a fibre bundle with Ehresmann connection from a product. Therefore, connections on fibre bundles are not topological objects. Another way to see this is to note that we can use local trivialisations $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ to push forward HE to a subbundle $\phi_{\alpha*}HE \subset T(U_\alpha \times F)$. However, on $U_\alpha \times F$, we can define a "canonical Ehresmann connection" by $\operatorname{Ker}\pi_{\alpha 2}$ where $\pi_{\alpha 2}$ is the projection onto the second factor in the product $U_\alpha \times F$. Since in general $\phi_{\alpha*} \neq \operatorname{Ker}\pi_{\alpha 2}$, we again see that the Ehresmann connection on a fibre bundle can be *locally* distinguished from the corresponding product space.

5.3 Connections on principal bundles and Yang-Mills theory

We can define Ehresmann connections as before on principal bundles. However, because principal G-bundles admit a right G-action, we are mostly interested in those connections which are compatible with the G-action. Recall, that given a principal G-bundle $\pi : P \longrightarrow B$, we have a right action $R_g : P \longrightarrow P$ for all $g \in G$ which satisfies

$$\pi \circ R_g = \pi \,. \tag{5.23}$$

Since the vertical subbundle is defined by the push-forward of π to TP, this immediately implies that the vertical subbundle is G-invariant:

$$R_{g*}V_uP = V_{ug} \ \forall u \in P, g \in G \,. \tag{5.24}$$

We will demand the same G-invariance for the connection.

Definition: A **principal connection** (or **principal G-connection**) is an Ehresmann connection on a principal G-bundle $\pi : P \longrightarrow B$ such that

$$H_{ug}P = R_{g*}H_uP \ \forall u \in P, g \in G \,, \tag{5.25}$$

where R_g denotes the right action of $g \in G$ on P and R_{g*} is its pull-back to the tangent bundle TP.

This condition relates the horizontal subspaces at two points in P using the G-action.We can also summarise the compatibility condition with the right action as the following commutative diagram.

$$\begin{array}{ccc} HP & \xrightarrow{R_{g*}} & HP \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ P & \xrightarrow{R_g} & P \end{array}$$

It is useful to introduce a linear condition which determines the horizontal subbundle of P at each point, i.e. we want a map at each point $u \in P$, $\omega_u : T_uP \longrightarrow T_uP$ such that the horizontal subspace H_uP is the kernel of ω_u. Since the vertical subspace V_uP at each point $u \in P$ is tangent to the fibre G it has rank $\dim G$. Therefore, we need ω_u to consist of $\dim G$ linear equations to define H_uP. We will do this by introducing a connection 1-form, which is closely related to the way you are probably used to thinking of connections.

Before we can introduce this 1-form, we first need the follwoing definition.

Definition: Given a principal G-bundle $\pi : P \longrightarrow B$ with $\mathfrak{g}$ the Lie algebra of G, a **fundamental vector field** is a smooth map

$$\# : \mathfrak{g} \longrightarrow \Gamma(TP)\,, \tag{5.26}$$

such that at each point $u \in P$

$$\#_u : \mathfrak{g} \longrightarrow T_uP\,, \tag{5.27}$$

given by

$$\#_u(A)(f) = \frac{d}{dt} f\left(u \exp(tA)\right)|_{t=0}\ \forall A \in \mathfrak{g}\,. \tag{5.28}$$

The fundamental vector field corresponding to $A \in \mathfrak{g}$ is often also denoted as

$$A^\#_u \equiv \#_u(A)\,. \tag{5.29}$$

Exercise 5.6: Show that $\#_u : \mathfrak{g} \longrightarrow V_uP$ for all $u \in P$, i.e.

$$\pi_*\left(A^\#\right) = 0\ \forall A \in \mathfrak{g}\,. \tag{5.30}$$

Since $\operatorname{rank} \pi_* = \dim G$, $\#_u : \mathfrak{g} \longrightarrow V_uP$ for $u \in P$ is in fact an isomorphism. By the same argument, and using that $\Gamma(TP)$ is a vector space and $\# : \mathfrak{g} \longrightarrow \Gamma(TP)$ is a linear map, we have that $\# : \mathfrak{g} \longrightarrow \Gamma(VP)$ is an isomorphism. Finally, this construction shows that VP is parallelisable, hence $VP = P \times \mathfrak{g}$.

Exercise 5.7: Show that $\#$ is compatible with the Lie bracket, i.e. for any $A, B \in \mathfrak{g}$,

$$\left[A^\#, B^\#\right] = \left[A, B\right]^\#\,. \tag{5.31}$$

Exercise 5.8: Using the isomorphism $\# : \mathfrak{g} \longrightarrow \Gamma(VP)$ show that for any $X \in \Gamma(VP)$ and $Y \in \Gamma(HP)$,

$$[X, Y] \in \Gamma(HP)\,. \tag{5.32}$$

This is the infinitesimal version of the statement that HP is G-invariant since $\mathfrak{g}$ generates the infitesimal right G-action.

Now we are ready to introduce the connection 1-form.

Definition: A **connection 1-form** on a principle G-bundle is a $\mathfrak{g}$-valued 1-form on P,

$$\omega \in \Omega^1(P) \otimes \mathfrak{g} \,, \tag{5.33}$$

satisfying

(i) $\omega(A^\#) = A \;\forall A \in \mathfrak{g}$,

(ii) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$.

Explicitly, the second condition states that $\forall X \in T_u P$,

$$\begin{aligned} R_g^* \omega_{ug}(X) &= \omega_{ug}(R_{g*} X) \\ &= g^{-1} \omega_u(X)\, g \,, \end{aligned} \tag{5.34}$$

which ensures that the $\mathfrak{g}$-valued 1-form is G-invariant.

Now we can define the horizontal subspace associated to the connection 1-form as its Kernel.

Definition: Given a principal G-bundle P with connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, the **horizontal subspace associated to the connection 1-form** is $\mathrm{Ker}\,\omega$,

$$H_u P = \{ X \in T_u P \mid \omega_u(X) = 0 \} \,. \tag{5.35}$$

As an exercise, you should show that the horizontal subspace defined via the connection 1-form above satisfies the compatibiltiy condition with the G-action on the prinipcal G-bundle.

Exercise 5.9: Show that the horizontal subspace defined via the connection 1-form is G-invariant, i.e.

$$R_{g*} H_u P = H_{ug} P \,. \tag{5.36}$$

5.3.1 Parallel transport in a principal bundle

We saw previously that a connection on a fibre bundle allows us to define a notion of horizontal lift of a curve $\gamma : I \longrightarrow B$ in the base, which is a curve $\tilde{\gamma} : I \longrightarrow P$ in the principal bundle such that $\pi \circ \tilde{\gamma} = \gamma$ and the tangent vectors to $\tilde{\gamma}$ are everywhere horizontal. However, we noted that in general the horizontal lift can only be guaranteed for small $t \in I$ and generically the horizontal lifts cannot be extended to the full curve $\tilde{\gamma} : I \longrightarrow P$.

Now we can see another benefit of having the right G-action on a principal G-bundle $\pi : P \longrightarrow B$. This allows us to show that horizontal lifts always exist in principal bundles.

Theorem 5.5: Given a path $\gamma : I \longrightarrow B$ and a principal bundle $\pi : P \longrightarrow B$ with principal connection HP, then the horizontal lift $\tilde{\gamma}_{u_0}$ of γ through $u_0 \in \pi^{-1}(\gamma(0))$ exists and is unique.

We will not present a proof here but will mention that the crucial difference to general fibre bundles is that the right G-action allows us to extend the horizontal lift to arbitrary $t \in I$. This is precisely the same reason as why the exponential map $\exp : TG \longrightarrow G$ is well-defined on group manifolds. For a more complete proof, see for example Nakahara section 10.1.4.

Corollary: Let $\pi : P \longrightarrow B$ be a principal G-bundle with principal connection HP, $\gamma : I \longrightarrow B$ be a curve with horizontal lift $\tilde{\gamma}_{u_0} : I \longrightarrow P$ through u_0 and $\tilde{\gamma}'$ another horizontal lift of γ through $u_0\, g$ for some $g \in G$. Then

$$\tilde{\gamma}'(t) = \tilde{\gamma} g \;\forall t \in I\,. \tag{5.37}$$

Proof. Consider the curve

$$\tilde{\gamma}_g = \tilde{\gamma} g : I \longrightarrow P\,, \tag{5.38}$$

i.e. $\tilde{\gamma}_g = R_{g*}\tilde{\gamma}$ is the push-forward by the right G-action $R_g : P \longrightarrow P$. It then follows from the right-invariance of the horizontal subspace $R_{g*}H_uP = H_{ug}P$ that $\tilde{\gamma}_g$ is horizontal. Finally, since $\tilde{\gamma}_g(0) = u_0\, g$, we have by uniqueness that

$$\tilde{\gamma}' = \tilde{\gamma}_g = \tilde{\gamma} g\,. \tag{5.39}$$

□

A nice consequence of the existence and uniqueness of horizontal lifts is that we can define parallel transport for principal bundles.

Definition: Let $\pi : P \longrightarrow B$ be a principal bundle with principal connection HE and $\gamma : I \longrightarrow B$ a path. The **parallel transport** along γ is the map

$$\Gamma(\gamma) : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1))\,, \tag{5.40}$$

with

$$\Gamma(\gamma)(u_0) = \tilde{\gamma}_{u_0}(1) \in \pi^{-1}(\gamma(1)) \;\forall u_0 \in \pi^{-1}(\gamma(0))\,, \tag{5.41}$$

where $\tilde{\gamma}_{u_0}$ is the horizontal lift of γ through $u_0 \in \pi^{-1}(\gamma(0))$. $\Gamma(\gamma)(u_0)$ is called the **parallel transport of u_0 along γ.**

A very important fact of parallel transport is that, in general, $u_1 \in \pi^{-1}(\gamma(1))$ depends not only on $u_0 \in \pi^{-1}(\gamma(0))$ but also on the principal connection HP and on the path $\gamma : I \longrightarrow B$ that is chosen. Therefore, it is, in general, meaningless to speak of the parallel transport of a point $u_0 \in P$ without specifying the principal connection and the path $\gamma : I \longrightarrow B$ along which the parallel transport is performed.

Exercise 5.10: Using the result (5.38), show that for all $g \in G$

$$R_g\Gamma(\gamma) = \Gamma(\gamma) R_g\,, \tag{5.42}$$

i.e. for all $u_0 \in P$,

$$R_g\Gamma(\gamma)(u_0) = \Gamma(\gamma)(u_0\, g)\,. \tag{5.43}$$

Note that since the fibre is G, we can view parallel transport as a map $G \longrightarrow G$ which by the result of exercise 5.10 is a homomorphism.

Exercise 5.11: Let $\pi : P \longrightarrow B$ be a principal G-bundle with principal connection HP, $\gamma : I \longrightarrow B$ a path with inverse path $\gamma^{-1} : I \longrightarrow B$. Show that

$$\Gamma\left(\gamma^{-1}\right) = \Gamma(\gamma)^{-1} . \tag{5.44}$$

This shows that in fact parallel transport along $\gamma : I \longrightarrow B$ is an isomorphism of $\pi^{-1}(\gamma(0)) \simeq \pi^{-1}(\gamma(1))$.

Exercise 5.12: Show that for any paths $\gamma : I \longrightarrow B$ and $\sigma : I \longrightarrow B$ with $\gamma(1) = \sigma(0)$,

$$\Gamma(\gamma \star \sigma) = \Gamma(\sigma) \circ \Gamma(\gamma) . \tag{5.45}$$

That is, composition of parallel transport respects path multiplication.

It is particularly interesting to consider parallel transport along a loop $\gamma : S^1 \longrightarrow B$. The horizontal lift $\tilde{\gamma}$ through $u_0 \in \pi^{-1}(\gamma(0))$ is in general no longer a loop because $\Gamma(\gamma)(u_0) = u_1 \in \pi^{-1}(\gamma(0))$, with $u_1 \neq u_0$ in general. Instead, the loop γ generates a map $\Gamma(\gamma) : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(0))$ which is compatible with the right-action on G as you have shown in exercise 5.10.

Definition: Let $\pi : P \longrightarrow B$ be a principal G-bundle with principal connection HP. For each point $u \in P$, we can define the **holonomy group** at u as the set

$$\Phi_u = \left\{ g \in G \,|\, \Gamma(\gamma)(u) = u\, g\,,\ \gamma \in C_{\pi(u)} B \right\} . \tag{5.46}$$

Exercise 5.13: Show that Φ_u is a subgroup of G.

Sometimes it is also useful to only consider loops that are homotopic to the constant loop. Let $C^0_p(B)$ be the space of loops at $p \in B$ that are homotopic to the constant loop. Then,

Definition: Let $\pi : P \longrightarrow B$ be a principal G-bundle with principal connection HP. For each point $u \in P$, we can define the **restricted holonomy group** at u as the set

$$\Phi^0_u = \left\{ g \in G \,|\, \Gamma(\gamma)(u) = u\, g\,,\ \gamma \in C^0_{\pi(u)} B \right\} . \tag{5.47}$$

Note that the holonomy group depends on the principal connection HP used. It makes no sense to speak of the holonomy of a principal bundle without specifying the principal connection. You may have come across the term holonomy before in either the context of general relativity or perhaps string theory. Usually, there the holonomy refers to a specific connection which is torsion-free and metric. We will not have time in these lectures to explore this further.

5.3.2 Local connection 1-form

We now return to the connection 1-form and investigate its relation to objects we know gauge theories. In gauge theories we use the gauge potential, which is only locally a 1-form because between different open subsets of the spacetime manifold B it may be shifted by a gauge transformation. By contrast the connection 1-form we defined above is globally a 1-form on the principal bundle $\pi : P \longrightarrow B$.

To obtain a 1-form on B we have to pull-back ω from P to B using a smooth map

$$\sigma : B \longrightarrow P\,. \tag{5.48}$$

However, (if $\pi \circ \sigma = \mathbf{1}_B$) this would mean that σ is a section of P which as we saw at the start of this chapter only exists when P is trivial. More generally, we can define a *local* 1-form on open subsets $\{U_\alpha\} \subset B$ using *local* sections

$$\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}\left(U_\alpha\right)\,. \tag{5.49}$$

Definition: Given a principal G-bundle $\pi : P \longrightarrow B$ with connection 1-form $\omega \in \Omega^1(TP) \otimes \mathfrak{g}$, the **local connection 1-form** on B associated to the local sections $\{U_\alpha,\, \sigma_\alpha\}$ is defined as

$$\mathscr{A}_\alpha = \sigma_\alpha^* \omega \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\,. \tag{5.50}$$

In fact, given a set of local $\mathfrak{g}$-valued 1-forms $\mathscr{A}_\alpha$ for the local sections $\{U_\alpha, \sigma_\alpha\}$ we can reconstruct the connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ whose pull-back gives $\mathscr{A}_\alpha$. To do this, we use the canonical local trivialisations associated to the local sections $\{U_\alpha, \sigma_\alpha\}$, to express the connection 1-form $\omega \in \Omega^(P) \otimes \mathfrak{g}$ in terms of the local connection 1-form $\mathscr{A}_\alpha$.

Proposition:

$$\omega_\alpha \equiv \omega|_{U_\alpha} = g_\alpha^{-1} \pi^* \mathscr{A}_\alpha g_\alpha + g_\alpha^{-1} d_P g_\alpha\,, \tag{5.51}$$

where d_P denotes the exterior derivative on P.

Proof. The proof of this can be found in a textbook, for example in Nakahara subsection 10.1.3. □

The gauge potential we use in physics transforms by gauge transformations on the intersection of two open subsets of spacetime. The following theorem shows that this follows from the above definitions.

Theorem 5.6: Given a principal G-bundle $\pi : P \longrightarrow B$ with connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ and local connection 1-forms $\mathscr{A}_\alpha$ associated to the local sections $\{U_\alpha, \sigma_\alpha\}$, then on an overlap $U_\alpha \cap U_\beta \neq \emptyset$, the local connection 1-forms are related by

$$\mathscr{A}_\beta = t_{\alpha\beta}^{-1} \mathscr{A}_\alpha t_{\alpha\beta} + t_{\alpha\beta}^{-1} dt_{\alpha\beta}\,, \tag{5.52}$$

where $t_{\alpha\beta}$ are the transition functions of P.

Proof. See a textbook, for example Nakahara 10.1.3. □

Remember the transition functions are maps $t_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$. Now you should recognise (5.52) as a Yang-Mills gauge transformation of the gauge potential $\mathscr{A}$.

Similarly, in the following exercise you will show that gauge transformations of the gauge potential are generated by choosing different local sections.

Exercise 5.14: Let $\pi : P \longrightarrow G$ be a principal G-bundle with connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ and local connection 1-form $\mathscr{A}_\alpha$ associated to the local section $\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$. Let$\sigma'_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$ be another local section with

$$\sigma'_\alpha = \sigma_\alpha g\,, \tag{5.53}$$

for some $g : U_\alpha \longrightarrow G$.
Show that the local connection 1-form A'_α associated to σ' is given by

$$\mathscr{A}'_\alpha = g^{-1}\mathscr{A}_\alpha g + g^{-1} dg\,. \tag{5.54}$$

This gives us a geometric understanding of the Yang-Mills gauge potential and its gauge transformations. The gauge potential is in general not a globally well-defined 1-form on spacetime, B, because the principal G-bundle is not in general trivial and hence does not admit global sections. Instead the gauge potential is defined using local sections and on the overlap of open subsets $U_\alpha, U_\beta \subset B$ with $U_\alpha \cap U_\beta \neq \emptyset$, the gauge potential is patched using gauge transformations arising from changing the local sections via the transition functions of the principal G-bundle. Furthermore, gauge transformation of the gauge potential arise when we change our choice of local sections as you have shown in the exercise above.

Let us now show how the general concept of curvature of an Ehresmann connection reduces in the case of principal connections. We will see that we recover the usual notions of curvature (or field strength) of Yang-Mills gauge potentials.

To do this, we will use the connection 1-form to define a curvature 2-form of a principal connection and show that it agrees with the general definition 5.2.2. To do this, we first introduce the exterior covariant derivative operator of $\mathfrak{g}$-valued n-forms.

Definition: Let $\pi : P \longrightarrow B$ be a principal G-bundle with connection HE. Then the **exterior covariant derivative** of a $\mathfrak{g}$-valued n-form $\omega \in \Omega^n(P) \otimes \mathfrak{g}$, is given by

$$D\omega(X_1, \ldots, X_{n+1}) = \hat{d}_P\omega(X_1^H, \ldots, X_{n+1}^H)\ \forall X_1, \ldots X_{n+1} \in TP\,, \tag{5.55}$$

where X_i^H denotes the horizontal part of the vector field X_i and $\hat{d}_P\omega$ can be defined using a basis of $\mathfrak{g}$, $(t_a)\,, a = 1, \ldots, \mathrm{rank}(\mathfrak{g})$. Expanding $\omega = \omega^a t_a$ in this basis, with $\omega \in \Omega^n(P)$ we have

$$\hat{d}_P\omega \equiv (d_P\omega^a)\, t_a\,, \tag{5.56}$$

where d_P is the usual exterior derivative on P.

Note that $\hat{d}_P\omega$ is defined to coincide with the way you are used to differentiating Lie algebra-valued 1-forms from physics: you simply “ignore the Lie algebra index of the 1-form”.

We can use the exterior covariant derivative to define a curvature 2-form in terms of the connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$.

Definition: Given a principal G-bundle P with connection 1-form ω, the **curvature 2-form** $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ is given by

$$\Omega = D\omega \,. \tag{5.57}$$

Exercise 5.15: Show that the curvature 2-form Ω satisfies

$$R_g^* \Omega = g \Omega g^{-1} \; \forall g \in G \,. \tag{5.58}$$

where R_g denotes the right-action of $g \in G$ on P.

Let us now show that the definition (5.57) is compatible with (5.22). We can use (3.21) to evaluate the curvature of two vector fields $X, Y \in TP$

$$\begin{aligned} \Omega(X,Y) &= D\omega(X,Y) \\ &= \hat{d}_P \omega(X_H, Y_H) \\ &= \left(d_P \omega(X_H, Y_H)^a\right) t_a \\ &= \left(X_H \omega(Y_H)^a - Y_H \omega(X_H)^a - \omega\left([X_H, Y_H]\right)^a\right) t_a \\ &= -\omega\left([X_H, Y_H]\right) . \end{aligned} \tag{5.59}$$

Finally, we use the fact that $\mathfrak{g} \simeq \Gamma(VP)$ to see that $\omega\left([X_H, Y_H]\right)$ maps onto the vertical subbundle VP. This shows that (5.57) and (5.22) agree.

There is another formula for Ω that is often useful in applications. First we define the commutator between $\mathfrak{g}$-valued 1-forms using a basis t_a of $\mathfrak{g}$, so that $\omega_1 = \omega_1^a t_a$ and $\omega_2^a t_a$. Then we let

$$[\omega_1,\, \omega_2] \equiv (\omega_1 \wedge \omega_2)^a \otimes [t_a,\, t_b] \,. \tag{5.60}$$

Using this definition we find the following.

Theorem 5.7: Let $\pi : P \longrightarrow B$ be a principal G-bundle with connection 1-form $\omega \in \Omega^1(P) \times \mathfrak{g}$. Then the curvature 2-form satisfies **Cartan's structure equation**:

$$\Omega(X,Y) = d_P \omega(X,Y) + [\omega(X),\, \omega(Y)] \,, \tag{5.61}$$

for all $X, Y \in \Gamma(TP)$.

Exercise 5.16*: Prove the above by evaluating the left- and right-hand sides for the three cases that

(i) X, Y are both horizontal.

(ii) X is horizontal but Y is vertical.

(iii) X and Y are both vertical.

Hint: Recall that exercise 5.8 showed that $[X, Y] \in \Gamma(HP)$ for $X \in \Gamma(VP)$ and $Y \in \Gamma(HP)$.

Given a principal G-bundle $\pi : P \longrightarrow B$ with local connection 1-form ω, we can use a local section $\sigma_\alpha : U_\alpha \longrightarrow \pi^{-1}(U_\alpha)$ to define a local curvature 2-form $\mathscr{F} \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$ in the same way as we did for the connection 1-form.

Definition: Let $\pi : P \longrightarrow B$ be a principal G-bundle with local connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ and curvature 2-form $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$. Then, we can define the **local curvature 2-form** F associated to to the local sections $\{U_\alpha, \sigma_\alpha\}$ as

$$\mathscr{F} = \sigma_\alpha^* \Omega \in \Omega^2(U_\alpha) \otimes \mathfrak{g}\,. \tag{5.62}$$

Exercise 5.17: Using Cartan's structure equation (5.61), show that

$$\mathscr{F}_\alpha(X,Y) = d\mathscr{A}_\alpha(X,Y) + [\mathscr{A}_\alpha(X), \mathscr{A}_\alpha(Y)] \;\; \forall X, Y \in \Gamma(TM)\,, \tag{5.63}$$

and hence

$$\mathscr{F}_\alpha = d\mathscr{A}_\alpha + [\mathscr{A}_\alpha, \mathscr{A}_\alpha]\,. \tag{5.64}$$

The above exercise shows that the local curvature 2-form of a principal G-connection agrees with the defintion we are used to from Yang-Mills theory. The following theorem also shows that the local curvature 2-form is gauge covariant.

Theorem 5.8: Let $\pi : P \longrightarrow G$ be a principal G-bundle with connection 1-form ω, curvature 2-form Ω and local curvature 2-forms $\mathscr{F}_\alpha$ associated to the local sections $\{U_\alpha, \sigma_\alpha\}$, then on an overlap $U_\alpha \cap U_\beta \neq \emptyset$, the local connection 1-forms are related by

$$\mathscr{F}_\beta = t_{\alpha\beta}^{-1} \mathscr{F}_\alpha t_{\alpha\beta}\,, \tag{5.65}$$

where $t_{\alpha\beta}$ are the transition functions of P.

Proof. See a textbook, for example Nakahara 10.3.4. □

We see that the objects we know from gauge theory, the gauge potential and its field strength, have a mathematical interpretation as local pull-backs via local sections of connection 1-forms and curvature 2-forms on principal bundles. There are many good sources to learn more about the relationship between fibre bundles and gauge theories, as well as gravitational theories, such as Nakahara's "Topology, Geometry and Physics", in particular chapters 10 - 13.

5.4 Applications in physics

As we have seen, fibre bundles are the mathematical setting of gauge theories. For example, let us use fibre bundles to understand the Dirac monopole that we encountered in section 3.6. Since we are working with electromagnetism we want to interpret the Dirac monopole as a $U(1)$-principal bundle. For the point-like magnetic monopole, we have $dF = 0$ everywhere except at one point and thus we see that the base of the fibre bundle must be $\mathbb{R}^3 - \{pt\} \sim S^2$. Therefore, the Dirac monopole is described by a $U(1)$-principal bundle $\pi : P \longrightarrow S^2$. We saw in example 4.4 that these are Lens spaces, with the Hopf fibration as a particular example.

Furthermore, since $F \in H^2(S^2)$, we know that $F \propto vol_2$ the volume form on S^2. We expect that F is the local 2-form curvature obtained by pulling back the curvature of a principal connection on P. However, we can also use the projection map $\pi : P \longrightarrow S^2$ to pull-back $F \in \Omega^2(S^2)$ to the curvature 2-form on P, $\pi^* F \in \Omega^2(P)$. Because we are dealing with a $U(1)$-bundle equation (5.61) tells us that $\pi^* F = d_P \omega$ with ω the connection 1-form on P and hence $\pi^* F$ must be exact. This can easily be checked for the case of the Hopf fibration $\pi : S^3 \longrightarrow S^2$ where $H^2(S^3) = 0$ and hence $\pi^* F$ is indeed necessarily exact.

Another important application of fibre bundles is in the study of instantons. These are finite-action solutions of Euclidean Yang-Mills action $S = \int d^d x \frac{1}{2} \mathrm{tr}\,(F \wedge \star F)$ with gauge group G. If we consider a non-compact space, such as $\mathbb{R}^4$, then having finite action requires the gauge potential to be pure gauge at infinity. Hence these instantons are defined by a map $g : S^3 \longrightarrow G$, i.e. $g \in \pi_3(G)$, in a way that is analogous to defects in condensed matter theory (here the S^3 is the "sphere at infinity"). Only when $\pi_3(G)$ is non-zero are there instantons. Furthermore, these are stable because they are in topologically distinct sectors from the vacuum configuration.

Now let us connect this to a description of instantons in terms of fibre bundles. We use the fact that F vanishes at infinity to take the 1-point compactification of $\mathbb{R}^4$, i.e. we introduce the "point at infinity". This compactification takes $\mathbb{R}^4 \cup \{pt\} \longrightarrow S^4$. We can now take charts on the northern and southern hemisphere, U_N and U_S, of S^4 which intersect in $S^3 \times I \sim S^3$, the equatorial S^3. Therefore the transition functions of the fibre bundle are maps $t_{NS} : S^3 \longrightarrow G$ i.e. $t_{NS} \in \pi_3(G)$. As a result, we once again see that the bundle is classified by $\pi_3(G)$. For example, using that $\pi_3(SU(2)) = \mathbb{Z}$, we see that $SU(2)$ Yang-Mills theory has instanton configurations in $\mathbb{R}^4$ labelled by an integer winding number.

In fact, the winding number can be obtained from the integral $\int_{S^4} F \wedge F$. This is analogous to how the monopole charge was obtained from the integral $\int_{S^2} F$. These integrals are special because they capture the topological information of the fibre bundle, i.e. they are independent of a choice of connection on the fibre bundle. To understand why this happens one needs to study the theory of characteristic classes, for which we do not have time here. However, interested readers may refer to chapter 11 of Nakahara.

Exercise 5.18:

(i) Using Euler angles (θ, ϕ, ψ) on S^3, defined by

$$z_1 = \exp\left(i\frac{\phi+\psi}{2}\right)\cos\frac{\theta}{2}\,, \qquad z_2 = \exp\left(i\frac{\psi-\phi}{2}\right)\sin\frac{\theta}{2}\,, \tag{5.66}$$

show that the Hopf map, equation (4.34), becomes

$$z_0 = e^{i\phi}\sin\theta\,, \qquad x = \cos\theta\,, \tag{5.67}$$

with θ and ϕ the usual spherical coordinates on S^2, and ψ the local coordinate along the $\mathrm{U}(1)$ fibre.

(ii) Show that

$$\omega = -g\left(d\psi + \cos\theta\, d\phi\right)\,, \tag{5.68}$$

is a connection 1-form on S^3.

(iii) Using the sections $\sigma_N : U_N \longrightarrow \pi^{-1}(U_N)$ and $\sigma_S : U_S \longrightarrow \pi^{-1}(U_S)$ defined by

$$\begin{aligned} \sigma_N(\theta,\phi) &= (\psi,\theta,\phi) \text{ with } \psi = \phi\,, \\ \sigma_S(\theta,\phi) &= (\psi,\theta,\phi) \text{ with } \psi = -\phi\,, \end{aligned} \tag{5.69}$$

show that the local connection 1-form on S^2 is given by

$$\begin{aligned} A_N &= -g\left(1+\cos\theta\right)d\phi\,, \\ A_S &= g\left(1-\cos\theta\right)d\phi\,. \end{aligned} \tag{5.70}$$

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