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Pure Yang–Mills solutions on dS_4

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We consider pure SU(2) Yang–Mills theory on four-dimensional de Sitter space dS_4 and construct smooth and spatially homogeneous classical Yang–Mills fields. Slicing dS_4 as $\mathbb{R} \times S^3$, via an SU(2)-equivariant ansatz we reduce the Yang–Mills equations to ordinary matrix differential equations and further to Newtonian dynamics in a particular three-dimensional potential. Its classical trajectories yield spatially homogeneous Yang–Mills solutions in a very simple explicit form, depending only on de Sitter time with an exponential decay in the past and future. These configurations have not only finite energy, but their action is also finite and bounded from below. We present explicit coordinate representations of the simplest examples (for the fundamental SU(2) representation). Instantons (Yang–Mills solutions on the Wick-rotated S^4) and solutions on AdS_4 are also briefly discussed.

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1. Introduction

Minkowski space $\mathbb{R}^{3,1}$ seems to admit finite-action solutions of the Yang–Mills equations only in the presence of Higgs fields (for a review on classical Yang–Mills configurations, see [1]). However, our universe appears to be asymptotically de Sitter at very large and very early times. This is a good argument for searching finite-energy solutions in pure Yang–Mills theory on four-dimensional de Sitter space dS$_4$. In this talk we shall present the result of such an attempt, which produced a family of classical smooth analytic finite-action pure Yang–Mills configurations on a non-dynamical de Sitter background [2]. (We do not consider dynamical spacetime, i.e. the Einstein equations.) The main idea is to employ the conformal invariance of Yang–Mills theory in four dimensions to map the problem to a finite cylinder over a three-sphere, whose description as SU(2) group manifold allows for elegant and powerful geometrical tools to solve the equations using a highly symmetrical ansatz [3, 4, 5]. In the end we shall briefly discuss also the related instantons on dS$_4$ (i.e. solutions on S^4) and what can be done on AdS$_4$.

2. Description of de Sitter space

Four-dimensional de Sitter space dS$_4$ is a one-sheeted hyperboloid in $\mathbb{R}^{4,1}$ via

$$\delta_{ij}y^iy^j-(y^5)^2 \;=\; R^2 \qquad \text{where} \quad i,j=1,\ldots,4\,. \tag{2.1}$$

Topologically, dS$_4 \simeq \mathbb{R}\times S^3$. Closed-slicing global coordinates (τ,χ,θ,ϕ) are obtained by

$$y^i=R\,\omega^i\cosh\tau\,, \qquad y^5=R\sinh\tau \qquad \text{with} \quad \tau\in\mathbb{R} \qquad \text{and} \qquad \delta_{ij}\,\omega^i\omega^j=1\,, \tag{2.2}$$

where $\omega^i=\omega^i(\chi,\theta,\phi)$ embeds a unit $S^3\simeq \mathrm{SU}(2)$ with metric $\mathrm{d}\Omega_3^2$ into $\mathbb{R}^4$. The induced metric reads

$$\mathrm{d}s^2 \;=\; R^2\left(-\mathrm{d}\tau^2+\cosh^2\tau\,\mathrm{d}\Omega_3^2\right)\,. \tag{2.3}$$

We introduce an orthonormal basis $\{e^a\}$, $a=1,2,3$, of SU(2) left-invariant one-forms via

$$e^a \;=\; -\eta^a_{ij}\,\omega^i\mathrm{d}\omega^j \qquad \Rightarrow \qquad \mathrm{d}e^a+\varepsilon^a_{bc}\,e^b\wedge e^c \;=\; 0 \tag{2.4}$$

with self-dual 't Hooft symbols η^a_{ij}. They simplify the S^3 metric to $\mathrm{d}\Omega_3^2 \;=\; (e^1)^2+(e^2)^2+(e^3)^2$.

Four-dimensional de Sitter space is conformally equivalent to a finite Lorentzian cylinder $\mathscr{I}\times S^3$ via conformal time

$$t \;=\; \arctan(\sinh\tau) \;=\; 2\arctan(\tanh\tfrac{\tau}{2}) \qquad \Leftrightarrow \qquad \frac{\mathrm{d}\tau}{\mathrm{d}t} \;=\; \cosh\tau \;=\; \frac{1}{\cos t}\,, \tag{2.5}$$

$$\text{with a range} \qquad \tau\in\mathbb{R} \quad \Leftrightarrow \quad t\in\mathscr{I}=(-\tfrac{\pi}{2},+\tfrac{\pi}{2}) \tag{2.6}$$

being an open interval. In conformal coordinates, the metric takes the form

$$\mathrm{d}s^2 \;=\; \frac{R^2}{\cos^2 t}\left(-\mathrm{d}t^2+\delta_{ab}e^ae^b\right) \;=\; \frac{R^2}{\cos^2 t}\,\mathrm{d}s^2_{\mathrm{cyl}}\,. \tag{2.7}$$

PoS(CORFU2017)204

3. Reduction of Yang–Mills to matrix equations

We consider rank-N hermitian vector bundles over the cylinder $\mathcal{I}\times S^3$ conformally equivalent to dS_4. Since de Sitter space has a (conformal) boundary, we apply the standard procedure of framing the gauge bundle over this boundary [6], i.e. gauge-group elements g are restricted to

$$g(\partial\mathrm{dS}_4) \;=\; \mathrm{Id} \qquad \text{on} \quad \partial\mathrm{dS}_4 = S^3_{t=+\frac{\pi}{2}} \cup S^3_{t=-\frac{\pi}{2}}\ . \tag{3.1}$$

The gauge potential $\mathcal{A}$ and gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ are taken to lie in $su(N)$, and we pick the temporal gauge $\mathcal{A}_0 = 0$. Respecting the manifest SO(4) symmetry of our cylinder, we choose an SU(2)-equivariant ansatz,

$$\mathcal{A} \;=\; X_a(t)\,e^a \qquad \text{with} \quad X_a \in su(N)\ , \tag{3.2}$$

resulting in a gauge field

$$\mathcal{F} \;=\; \mathcal{F}_{0a}\,e^0\wedge e^a + \tfrac12 \mathcal{F}_{bc}\,e^b\wedge e^c \;=\; \dot X_a\,e^0\wedge e^a + \tfrac12\big(-2\varepsilon^a_{bc}X_a + [X_b,X_c]\big)\,e^b\wedge e^c \tag{3.3}$$

with $e^0 := \mathrm{d}t$ and $\dot X_a := \mathrm{d}X_a/\mathrm{d}t$. The vacuum Yang–Mills equations may be directly specialized to this ansatz or obtained from varying the Yang–Mills action after inserting (3.3). Both ways, one arrives at three coupled ordinary differential equations and "Gauß-law" condition for three $N{\times}N$ matrix functions $X_a(t)$,

$$\ddot X_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b,X_c] - \big[X_b,[X_a,X_b]\big] \qquad \text{and} \qquad [\dot X_a, X_a] \;=\; 0\ . \tag{3.4}$$

4. Further reduction to quintuple-well dynamics

In our context, the most natural and simple choice of a gauge group is SU(2). So let us restrict X_a to some $su(2)\subset su(N)$ by embedding a spin-j representation of $su(2)$ into $su(2j{+}1)$ for $j=\frac12(N{-}1)$. We normalize the three SU(2)-generators I_a via

$$[I_b,I_c] \;=\; 2\,\varepsilon^a_{bc}I_a \qquad \text{and} \qquad \mathrm{tr}(I_aI_b) \;=\; -4C(j)\,\delta_{ab} \quad \text{for} \quad C(j)=\tfrac13\,j(j{+}1)(2j{+}1)\ , \tag{4.1}$$

where $C(j)$ denotes the second-order Dynkin index of the representation. The simplest (but by no means general) choice for the matrices X_a is

$$X_1=\Psi_1 I_1\ ,\quad X_2=\Psi_2 I_2\ ,\quad X_3=\Psi_3 I_3 \qquad \text{with} \quad \Psi_a=\Psi_a(t)\in\mathbb{R}\ , \tag{4.2}$$

automatically obeying the Gauß-law constraint in (3.4). As a result, the Yang–Mills Lagrangian density simplifies to

$$\begin{aligned}\mathcal{L} \;&=\; \tfrac18\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;=\; -\tfrac14\,\mathrm{tr}\mathcal{F}_{0a}\mathcal{F}_{0a} \;+\; \tfrac18\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}_{ab} \\ &=\; 4C(j)\,\big\{\tfrac14\dot\Psi_a\dot\Psi_a - (\Psi_1-\Psi_2\Psi_3)^2 - (\Psi_2-\Psi_3\Psi_1)^2 - (\Psi_3-\Psi_1\Psi_2)^2\big\}\ ,\end{aligned} \tag{4.3}$$

which describes a Newtonian particle with coordinates Ψ_a in $\mathbb{R}^3$, subject to a conservative force from a potential

$$\tfrac12 V(\Psi) \;=\; (\Psi_1-\Psi_2\Psi_3)^2 \;+\; (\Psi_2-\Psi_3\Psi_1)^2 \;+\; (\Psi_3-\Psi_1\Psi_2)^2\ . \tag{4.4}$$

This three-dimensional generalization of a double-well potential has critical points $(\hat\Psi_1,\hat\Psi_2,\hat\Psi_3)$ at

$$(0,0,0)\ =\text{minimum}\quad,\qquad (\pm\tfrac12,\pm\tfrac12,\pm\tfrac12)\ =\text{saddle}\quad,\qquad (\pm1,\pm1,\pm1)\ =\text{minima}\ , \tag{4.5}$$

with

$$V(\text{minima})=0\qquad\qquad\text{and}\qquad\qquad V(\text{saddle})=\tfrac38\ , \tag{4.6}$$

where the number of minus signs in each triple must be even.

The Euler-Langrange equations

$$\begin{aligned}\tfrac14\ddot\Psi_1 &= -\Psi_1+3\,\Psi_2\Psi_3-\Psi_1(\Psi_2^2+\Psi_3^2)\ ,\\ \tfrac14\ddot\Psi_2 &= -\Psi_2+3\,\Psi_3\Psi_1-\Psi_2(\Psi_3^2+\Psi_1^2)\ ,\\ \tfrac14\ddot\Psi_3 &= -\Psi_3+3\,\Psi_1\Psi_2-\Psi_3(\Psi_1^2+\Psi_2^2)\end{aligned} \tag{4.7}$$

are still too hard to solve analytically in general, but their invariance under a tetrahedral S_4 symmetry helps finding two special solutions:

$$\begin{aligned}&\text{abelian}:\qquad \Psi_1=\Psi_2=0\ ,\quad \Psi_3=:\xi\\ &\Rightarrow\qquad V_\xi=2\,\xi^2\qquad\text{and}\qquad \ddot\xi\ =\ -4\,\xi\qquad\Rightarrow\qquad \xi(t)\ =\ -\tfrac12\,\gamma\cos2(t-t_0)\ ,\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\text{nonabelian}:\quad \Psi_1=\Psi_2=\Psi_3=:\tfrac12(1+\psi)\\ &\Rightarrow\qquad V_\psi=\tfrac12\,(1-\psi^2)^2\quad\text{and}\quad \ddot\psi\ =\ 2\,\psi\,(1-\psi^2)\quad\Rightarrow\quad \psi(t)\ =\ \text{elliptic function}\ .\end{aligned} \tag{4.9}$$

The abelian solutions describe harmonic oscillations around the central minimum, while the nonabelian ones contain the vacuum $\psi(t)\equiv\pm1$, the unstable saddle point $\psi(t)\equiv0$,[1] and the bounce

$$\psi(t)\ =\ \sqrt2\,\text{sech}\big(\sqrt2(t-t_0)\big)\ =\ \frac{\sqrt2}{\cosh\big(\sqrt2(t-t_0)\big)}\ , \tag{4.10}$$

among the generic nonlinear oscillations. The 'energy' of this Newtonian dynamics is conserved and determined by the value V_0 of the potential at the turning points, hence

$$\tfrac12\,\dot\psi^2=V_0-V_\psi(\psi)=V_0-\tfrac12(1-\psi^2)^2\ . \tag{4.11}$$

5. Yang–Mills configurations on de Sitter space

Let us translate the nonabelian double-well solutions $\psi(t)$ back to Yang–Mills fields on dS$_4$. Firstly, on the Lorentzian cylinder (with conformal time), the substitution yields

$$\mathscr{A}\ =\ \tfrac12\,(1+\psi)\,e^aI_a\qquad\text{and}\qquad \mathscr{F}\ =\ \big(\tfrac12\,\dot\psi\,e^0\wedge e^a-\tfrac14\,(1-\psi^2)\,\varepsilon^a_{bc}\,e^b\wedge e^c\big)I_a\ , \tag{5.1}$$

providing SU(2) color electric and magnetic fields

$$E_a\ =\ \mathscr{F}_{0a}\ =\ \tfrac12\,\dot\psi\,I_a\qquad\text{and}\qquad B_a\ =\ \tfrac12\varepsilon_{abc}\mathscr{F}_{bc}\ =\ -\tfrac12\,(1-\psi^2)\,I_a\ . \tag{5.2}$$

[1] The solution ψ=1 implies $\mathscr{A}=I_ae^a=g^{-1}\mathrm{d}g$. The solution ψ=0 yields $\mathscr{A}=\frac12 I_ae^a=\frac12 g^{-1}\mathrm{d}g$, reminiscent of a meron [7].

Their energy densities read

$$\rho_e \;=\; -\tfrac{1}{4}\mathrm{tr}\,E_aE_a \;=\; \tfrac{3}{4}C(j)\,\dot\psi^2 \qquad \text{and} \qquad \rho_m \;=\; -\tfrac{1}{4}\mathrm{tr}\,B_aB_a \;=\; \tfrac{3}{4}C(j)\,(1-\psi^2)^2\;, \tag{5.3}$$

and the total field energy becomes

$$\mathscr{E}_t \;=\; \int_{S^3} e^1\wedge e^2\wedge e^3\;(\rho_e+\rho_m) \;=\; \tfrac{3}{4}C(j)\,\mathrm{vol}(S^3)\,\big(\dot\psi^2+(1-\psi^2)^2\big) \;=\; 3\pi^2 C(j)\,V_0\;. \tag{5.4}$$

The action functional computes to

$$\begin{aligned} S \;&=\; \tfrac{1}{8}\int_{\mathscr{I}\times S^3} e^0\wedge e^1\wedge e^2\wedge e^3\;\mathrm{tr}(-2\mathscr{F}_{0a}\mathscr{F}_{0a}+\mathscr{F}_{ab}\mathscr{F}_{ab}) \;=\; \int_{\mathscr{I}}\mathrm{d}t\;\mathrm{vol}(S^3)\,(\rho_e-\rho_m) \\ &=\; \tfrac{3}{2}\pi^2 C(j)\int_{-\pi/2}^{\pi/2}\mathrm{d}t\;\big(\dot\psi^2-(1-\psi^2)^2\big) \;=\; 3\pi^3 C(j)\,V_0 \;-\; 6\pi^2 C(j)\int_{-\pi/2}^{\pi/2}\mathrm{d}t\;V_\psi\big(\psi(t)\big) \end{aligned} \tag{5.5}$$

Secondly, for de Sitter space the time variable is $\tilde\tau = R\,\tau$ thus

$$\mathscr{E}_{\tilde\tau} \;=\; \frac{\mathrm{d}t}{\mathrm{d}\tilde\tau}\,\mathscr{E}_t \;=\; \frac{1}{R}\,\frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathscr{E}_t \;=\; \frac{1}{R\cosh\tau}\,\mathscr{E}_t \;=\; \frac{3\pi^2 C(j)\,V_0}{R\cosh\tau} \tag{5.6}$$

which is not only finite but decays exponentially at early and late times. To evaluate the action on de Sitter space, we need to relate the field components to the appropriate orthonormal basis,

$$\mathscr{A} = \widetilde{\mathscr{A}_a}\,\tilde e^a \quad\text{and}\quad \mathscr{F} = \widetilde{\mathscr{F}_{0a}}\,\tilde e^0\wedge\tilde e^a + \tfrac{1}{2}\widetilde{\mathscr{F}_{bc}}\,\tilde e^b\wedge\tilde e^c \quad\text{with}\quad \tilde e^0 := R\,\mathrm{d}\tau \quad\text{and}\quad \tilde e^a := R\cosh\tau\,e^a$$

$$\Rightarrow \qquad \mathscr{A}_a = R\cosh\tau\,\widetilde{\mathscr{A}_a}\;,\quad \mathscr{F}_{bc} = R^2\cosh^2\tau\,\widetilde{\mathscr{F}_{bc}}\;,\quad \mathscr{F}_{0a} = \partial_t\mathscr{A}_a = R^2\cosh^2\tau\,\partial_{\tilde\tau}\widetilde{\mathscr{A}_a}\;. \tag{5.7}$$

The result

$$S \;=\; \tfrac{1}{8}\int_{\mathrm{dS}_4}\tilde e^0\wedge\tilde e^1\wedge\tilde e^2\wedge\tilde e^3\;\mathrm{tr}(-2\widetilde{\mathscr{F}_{0a}}\widetilde{\mathscr{F}_{0a}}+\widetilde{\mathscr{F}_{ab}}\widetilde{\mathscr{F}_{ab}}) \;=\; \int_{\mathbb{R}}\mathrm{d}\tau\;\mathrm{vol}(S^3)\,\frac{\rho_e-\rho_m}{\cosh\tau} \tag{5.8}$$

agrees with the value (5.5) on Lorentzian cylinder. Remarkably, despite the infinite spacetime volume of de Sitter space, it is finite and bounded from below.

For a very explicit representation, we pick some coordinates on S^3,

$$\omega^1 = \sin\chi\,\sin\theta\,\sin\phi\;,\quad \omega_2 = \sin\chi\,\sin\theta\,\cos\phi\;,\quad \omega^3 = \sin\chi\,\cos\theta\;,\quad \omega^4 = \cos\chi\;, \tag{5.9}$$

and spell out the corresponding left-invariant one-forms,

$$\begin{aligned} e^1 &= \sin\theta\sin\phi\,\mathrm{d}\chi + \sin\chi\cos\chi\,(\tan\chi\cos\phi+\cos\theta\sin\phi)\,\mathrm{d}\theta + \sin^2\!\chi\sin\theta\,(\cot\chi\cos\phi-\cos\theta\sin\phi)\,\mathrm{d}\phi\;,\\ e^2 &= \sin\theta\cos\phi\,\mathrm{d}\chi - \sin\chi\cos\chi\,(\tan\chi\sin\phi-\cos\theta\cos\phi)\,\mathrm{d}\theta - \sin^2\!\chi\sin\theta\,(\cot\chi\sin\phi+\cos\theta\cos\phi)\,\mathrm{d}\phi\;,\\ e^3 &= \cos\theta\,\mathrm{d}\chi - \sin\chi\cos\chi\sin\theta\,\mathrm{d}\theta + \sin^2\!\chi\sin^2\!\theta\,\mathrm{d}\phi\;. \end{aligned}$$

Next, let us define three matrices I_* by decomposing

$$e^a I_a \;=:\; \mathrm{d}\chi\,I_\chi + \mathrm{d}\theta\,I_\theta + \mathrm{d}\phi\,I_\phi\;. \tag{5.10}$$

In the fundamental (spin $j{=}\frac{1}{2}$) representation of $su(2)$ these read

$$
\begin{aligned}
I_\chi &= -\mathrm{i}\begin{pmatrix} \cos\theta & -\mathrm{i}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & -\cos\theta \end{pmatrix}, \\
I_\theta &= -\mathrm{i}\sin\chi\cos\chi \begin{pmatrix} -\sin\theta & (\tan\chi-\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\tan\chi+\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & \sin\theta \end{pmatrix}, \\
I_\phi &= -\mathrm{i}\sin^2\chi\sin\theta \begin{pmatrix} \sin\theta & (\cot\chi+\mathrm{i}\cos\theta)\mathrm{e}^{\mathrm{i}\phi} \\ (\cot\chi-\mathrm{i}\cos\theta)\mathrm{e}^{-\mathrm{i}\phi} & -\sin\theta \end{pmatrix},
\end{aligned}
\tag{5.11}
$$

and the field-strength components in these angular coordinates (χ,θ,ϕ) are succintly expressed as

$$
\begin{aligned}
E_\chi &= \tfrac12 \tfrac{\mathrm{d}\psi}{\mathrm{d}\tau} I_\chi , & E_\theta &= \tfrac12 \tfrac{\mathrm{d}\psi}{\mathrm{d}\tau} I_\theta , & E_\phi &= \tfrac12 \tfrac{\mathrm{d}\psi}{\mathrm{d}\tau} I_\phi , \\
B_\chi &= -\tfrac12 (1-\psi^2) I_\chi , & B_\theta &= -\tfrac12 (1-\psi^2) I_\theta , & B_\phi &= -\tfrac12 (1-\psi^2) I_\phi .
\end{aligned}
\tag{5.12}
$$

The SU(2) equivariance in our ansatz (3.2) guarantees that all fields are spatially homogeneous over the three-sphere and only varying with time.

6. Explicit examples

Let us contemplate a few prominent sample solutions for ψ and ξ and the properties of the ensuing Yang–Mills fields. The potential minima $\psi\equiv\pm1$ just yield $\mathscr{F}=0$, which is uninteresting. The local maximum $\psi\equiv 0$ is more enlightening:

$$
\mathscr{A} = \tfrac12\, e^a I_a = \frac{\cos t}{2R}\,\tilde{e}^a I_a = \frac{1}{2R\cosh\tau}\,\tilde{e}^a I_a , \tag{6.1a}
$$

$$
\mathscr{F} = -\tfrac14\,\varepsilon^a_{bc}\, e^b\wedge e^c I_a = -\frac{\cos^2 t}{4R^2}\,\varepsilon^a_{bc}\,\tilde{e}^b\wedge\tilde{e}^c I_a = -\frac{1}{4R^2\cosh^2\tau}\,\varepsilon^a_{bc}\,\tilde{e}^b\wedge\tilde{e}^c I_a , \tag{6.1b}
$$

thus producing a purely magnetic and spatially homogeneous configuration with

$$
\widetilde{E}_a = \widetilde{\mathscr{F}}_{0a} = 0 \qquad\text{and}\qquad \widetilde{B}_a = \tfrac12\varepsilon_{abc}\widetilde{\mathscr{F}}_{bc} = -\frac{\cos^2 t}{2R^2} I_a = -\frac{1}{2R^2\cosh^2\tau} I_a . \tag{6.2}
$$

With $V_0{=}\frac12$, its energy and action are readily computed to be

$$
\mathscr{E}_{\tilde\tau} = \tfrac{3\pi^2 C(j)}{2R\cosh\tau} \qquad\text{and}\qquad S = -\tfrac32\pi^3 C(j) , \tag{6.3}
$$

respectively. We conjecture the latter to be the lowest possible stationary value.

Color electric components require a time-dependent double-well solution $\psi(t)$. The simplest such configuration comes from the famous bounce (4.10), which yields

$$
\mathscr{A} = \frac{\cos t}{2R}\left\{1 + \frac{\sqrt2}{\cosh\big(\sqrt2(t-t_0)\big)}\right\}\tilde{e}^a I_a , \tag{6.4a}
$$

$$
\mathscr{F} = -\frac{\cos^2 t}{4R^2}\left\{4\frac{\sinh\big(\sqrt2(t-t_0)\big)}{\cosh^2\big(\sqrt2(t-t_0)\big)}\,\tilde{e}^0\wedge\tilde{e}^a + \frac{\sinh^2\big(\sqrt2(t-t_0)\big)-1}{\cosh^2\big(\sqrt2(t-t_0)\big)}\,\varepsilon^a_{bc}\,\tilde{e}^b\wedge\tilde{e}^c\right\} I_a . \tag{6.4b}
$$

Since the turning point is the same as for the local maximum solution $\psi\equiv 0$, they have the same energy, but the value of the action is given by

$$\begin{aligned}\frac{S}{C(j)} &= -\tfrac{3}{2}\pi^3+12\,\pi^2\int_{-\pi/2}^{\pi/2}\mathrm{d}t\,\frac{\sinh^2\bigl(\sqrt{2}(t-t_0)\bigr)}{\cosh^4\bigl(\sqrt{2}(t-t_0)\bigr)} \\ &= -\tfrac{3}{2}\pi^3+\sqrt{8}\pi^2\bigl(\tanh^3(\tfrac{\pi}{\sqrt{2}}+\delta)+\tanh^3(\tfrac{\pi}{\sqrt{2}}-\delta)\bigr)\ ,\end{aligned} \tag{6.5}$$

depending on the bounce modulus $\delta=\sqrt{2}t_0$ because of the finite t-interval $\mathscr{I}=(-\frac{\pi}{2},\frac{\pi}{2})\neq\mathbb{R}$, so that only some length-π segment (depending on δ) of the bounce profile is captured.

Let us also take a quick look at the abelian solutions. Inserting (4.8) into (4.2) and (3.3) gives

$$\mathscr{A}=-\tfrac{1}{2}\gamma\cos 2(t-t_0)\,e^3\,I_3 \;=\; -\tfrac{\gamma}{2R}\cos t\,\cos 2(t-t_0)\,\tilde{e}^3\,I_3\ , \tag{6.6a}$$

$$\mathscr{F}=\mathrm{d}\mathscr{A} \;=\; \tfrac{\gamma}{R^2}\cos^2 t\,\Bigl\{\sin 2(t-t_0)\,\tilde{e}^0\wedge\tilde{e}^3\;+\;\cos 2(t-t_0)\,\tilde{e}^1\wedge\tilde{e}^2\Bigr\}\,I_3\ , \tag{6.6b}$$

thus

$$\widetilde{E}_3 \;=\; \tfrac{\gamma}{R^2}\cos^2 t\,\sin 2(t-t_0)\,I_3 \qquad\text{and}\qquad \widetilde{B}_3 \;=\; \tfrac{\gamma}{R^2}\cos^2 t\,\cos 2(t-t_0)\,I_3\ , \tag{6.7}$$

producing

$$\rho_e \;=\; \gamma^2 C(j)\,\sin^2 2(t-t_0) \qquad\text{and}\qquad \rho_m \;=\; \gamma^2 C(j)\,\cos^2 2(t-t_0)\ . \tag{6.8}$$

The amplitude γ is a free parameter. Finally, we present the energy and the action,

$$\mathscr{E}_{\tilde{\tau}} \;=\; \frac{\mathrm{d}t}{\mathrm{d}\tilde{\tau}}\int_{S^3} e^1\wedge e^2\wedge e^3\,(\rho_e+\rho_m) \;=\; \frac{2\pi^2\gamma^2\,C(j)}{R\cosh\tau}\ , \tag{6.9}$$

$$S \;=\; \int_{\mathscr{I}}\mathrm{d}t\ \mathrm{vol}(S^3)\,(\rho_e-\rho_m) \;=\; 2\pi^2\,\gamma^2\,C(j)\int_{\mathscr{I}}\mathrm{d}t\,\bigl(\sin^2 2(t-t_0)-\cos^2 2(t-t_0)\bigr) \;=\; 0\ . \tag{6.10}$$

The vanishing is in tune with the limit of small nonabelian oscillations ($V_0\to 0$) around $\psi=\pm 1$.

7. Instantons on de Sitter space

In quantum considerations it is of interest to also know the Yang–Mills solutions on the Wick-rotated spacetime with Eucliean signature. A prominent class of such solutions are self-dual configurations known as instantons. In order to construct these on de Sitter space, we Wick-rotate the latter to the four-sphere S^4 according to the following scheme,

$$\mathrm{dS}_4 \quad\xrightarrow{\text{Wick rotation}}\quad S^4 \quad\xrightarrow{\text{conf. equiv.}}\quad \mathbb{R}\times S^3 \tag{7.1}$$

$$(\tau,\chi,\theta,\phi) \quad\longrightarrow\quad (\varphi,\chi,\theta,\phi) \quad\longrightarrow\quad ({}^{r}_{T},\chi,\theta,\phi)\ , \tag{7.2}$$

with the coordinate relations

$$\tau=\mathrm{i}\,(\varphi-\tfrac{\pi}{2})\ ,\qquad \varphi=2\arctan\tfrac{r}{R}\ ,\qquad \tfrac{r}{R}=\mathrm{e}^T \qquad\Rightarrow\qquad \sin\varphi=\tfrac{1}{\cosh T}\ . \tag{7.3}$$

The S^4 metric in different coordinates reads

$$\mathrm{d}s^2 \;=\; R^2(\mathrm{d}\varphi^2+\sin^2\varphi\,\mathrm{d}\Omega_3^2) \;=\; \frac{4R^4}{(r^2+R^2)^2}\,(\mathrm{d}r^2+r^2\,\mathrm{d}\Omega_3^2) \;=\; \frac{R^2}{\cosh^2 T}\,(\mathrm{d}T^2+\mathrm{d}\Omega_3^2)\ . \tag{7.4}$$

PoS(CORFU2017)204

Clearly, Euclidean dS_4 is conformally equivalent to Euclidean cylinder over S^3, with Euclidean conformal time T. The radial variable $r = R\,\mathrm{e}^T$ together with the S^3 angles constitute just the standard stereographic coordinates of $S^4 \simeq \mathbb{R}^4 \cup \{\infty\}$,

$$x^i = r\,\omega^i(\chi,\theta,\psi) \qquad \Rightarrow \qquad \mathrm{d}s^2 \;=\; \frac{4\,R^4}{(r^2{+}R^2)^2}\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j\;. \tag{7.5}$$

Our goal is to solve the instanton (or self-duality) equation $\mathscr{F}_{ij} = \frac12\sqrt{\det g}\,\varepsilon_{ijkl}\mathscr{F}^{kl}$. To this end, we employ the Euclidean cylinder with $\mathrm{d}s^2_{\mathrm{cyl}} = \mathrm{d}T^2 + \mathrm{d}\Omega_3^2$ and the gauge $\mathscr{A}_T = 0$. Our trusted SU(2)-equivariant ansatz

$$\mathscr{A} \;=\; X_a(T)\,e^a \qquad \Rightarrow \qquad \mathscr{F}_{4a} \;=\; \frac{\mathrm{d}X_a}{\mathrm{d}T} \qquad \text{and} \qquad \mathscr{F}_{ab} \;=\; -2\,\varepsilon_{abc}\,X_c \;+\; [X_a,X_b] \tag{7.6}$$

with $X_a \in su(N)$ reduces the instanton equation to a generalized Nahm equation,

$$\frac{\mathrm{d}X_a}{\mathrm{d}T} \;=\; 2X_a \;-\; \tfrac12\,\varepsilon_{abc}\,[X_b,X_c]\;. \tag{7.7}$$

Taking again

$$X_1 = \Psi_1 I_1\;, \quad X_2 = \Psi_2 I_2\;, \quad X_3 = \Psi_3 I_3 \qquad \text{with} \quad \Psi_a = \Psi_a(T) \in \mathbb{R} \tag{7.8}$$

yields Wick-rotated Newtonian dynamics in $\mathbb{R}^3$, or $V(\Psi) \to -V(\Psi)$, and thus

$$\ddot\Psi_a \;=\; +\frac{\partial V}{\partial \Psi_a} \qquad \Leftarrow \qquad \dot\Psi_a \;=\; \frac{\partial U}{\partial \Psi_a} \qquad \text{with} \qquad V \;=\; \tfrac12\,\frac{\partial U}{\partial \Psi_a}\frac{\partial U}{\partial \Psi_a} \tag{7.9}$$

for a superpotential

$$U(\Psi) \;=\; \Psi_1^2 + \Psi_2^2 + \Psi_3^2 \;-\; 2\,\Psi_1\Psi_2\Psi_3\;. \tag{7.10}$$

The superpotential shares the tetrahedral symmetry of V, and the (inverted) double-well arises again for the restriction

$$\Psi_1 = \Psi_2 = \Psi_3 \;=\; \tfrac12\,(1+\psi) \qquad \Rightarrow \qquad U_\psi(\psi) \;=\; \psi - \tfrac13\psi^3 \qquad \text{and} \qquad \dot\psi \;=\; 1-\psi^2\;. \tag{7.11}$$

The simplest non-vacuum solution is the kink, $\psi(T) = \tanh 2(T{-}T_0)$, yielding

$$X_a(T) \;=\; \big[1+\exp(-2(T{-}T_0))\big]^{-1} I_a \;=\; \frac{r^2}{r^2{+}\Lambda^2}\,I_a \qquad \text{with} \quad \Lambda^2 = \mathrm{e}^{2T_0}R^2\;. \tag{7.12}$$

We have absorbed the collective coordinate T_0 and the de Sitter radius into the combination Λ. The resulting gauge potential and field strength become

$$\mathscr{A} \;=\; X_a\,e^a \;=\; -\frac{1}{r^2{+}\Lambda^2}\,\eta^a_{ij}\,I_a\,x^i\mathrm{d}x^j \qquad \text{and} \qquad \mathscr{F} \;=\; -\frac{\Lambda^2}{(r{+}\Lambda^2)^2}\,\eta^a_{ij}\,I_a\,\mathrm{d}x^i{\wedge}\mathrm{d}x^j\;. \tag{7.13}$$

This is nothing but the familiar BPST instanton extended from $\mathbb{R}^4$ to S^4.

8. What about *anti*-de Sitter?

Before concluding, let us comment on the other maximally symmetric case, Yang–Mills on four-dimensional anti-de Sitter space AdS_4. It is embedded in $\mathbb{R}^{3,2}$ via

$$(y^1)^2+(y^2)^2+(y^3)^2-(y^4)^2-(y^5)^2 \;=\; -R^2 \tag{8.1}$$

and conformally equivalent to $\mathscr{I}\times\mathrm{AdS}_3 \;\simeq\; \mathscr{I}\times\mathrm{PSL}(2,\mathbb{R})$, by repeating the arguments for dS_4 modulo appropriate signature flips or analytic continuations. Moreover, AdS_4 is also conformally equivalent to $S^1\times S^3_+$ (the upper hemisphere)[8]!

The construction for gauge group SU(N) is similar to the one on dS_4, but the conformal factor now depends on a *spatial* coordinate χ. Like on dS_4, it vanishes on the (conformal) boundary, so all our solutions decay to zero there. However, for the AdS_3 slicing energy and action are proportional to the volume of PSL(2,$\mathbb{R}$) and thus infinite. In contrast, for the S^3_+ slicing we can import time-periodic dS_4 solutions and restrict them from S^3 to S^3_+. Since the three-hemisphere has finite volume and the time variable is periodic, both energy and action are finite in this case! The conjectured bound (attained for the purely magnetic solution) is even identical to the one on dS_4. The Euclidean version of AdS_4 is the hyperbolic four-space H^4. Finite-action instantons also exist, but they are unstable. Passing to the universal cover $\widetilde{\mathrm{AdS}}_4$ will stabilize them though.

9. Summary and outlook

We have established the existence of pure Yang–Mills solutions with finite energy and action on non-dynamical four-dimensional de Sitter space (with radius R), without Higgs fields. Our most symmetrical configurations feature color-electric and -magnetic fields homogeneous in the spatial S^3 slices, hence SO(4) invariant, analogous to the Dirac monopole on $\mathbb{R}^3$ restricted to S^2 or the Yang monopole on $\mathbb{R}^5$ restricted to S^4 [9]. They only depend on time, in a smooth and asymtotically decaying manner. We arrived at these solutions by using a simple SU(2)-equivariant ansatz for the gauge potential, which reduced the Yang–Mills equations to ordinary matrix differential equations. Further specialization lead to the analog problem of a Newtonian particle in three space dimensions subject to a particular tetrahedric quintuple-well potential. Special analytic particle trajectories yielded explicit Yang–Mills solutions, whose field strengths decay in de Sitter time τ as $(R\cosh\tau)^{-2}$, the energy as $(R\cosh\tau)^{-1}$. Their action is finite and bounded from below by $-3\pi^3$ (for the adjoint representation). As a byproduct, we recover the BPST instanton extended to S^4. Analog configurations on AdS_4 with H^3 slicing also enjoy finite energy and action. These classical field configurations may be relevant for the Yang–Mills vacuum structure on (A)dS_4, but this requires further (e.g. stability) analysis.

Generalizations are easily possible. Firstly, one may allow for a larger gauge group and a more general matrix ansatz, which will bring us to quiver gauge theories. Secondly, the matrix dynamics may be analyzed directly, for the potential and superpotential

$$V \;=\; -\mathrm{tr}\big\{2X_aX_a \;-\; \varepsilon_{abc}X_a[X_b,X_c] \;+\; \tfrac12[X_a,X_b][X_a,X_b]\big\}\;, \tag{9.1}$$

$$U \;=\; -\mathrm{tr}\big\{X_aX_a \;-\; \tfrac16\,\varepsilon_{abc}X_a[X_b,X_c]\big\}\;, \tag{9.2}$$

respectively. And thirdly, there exist many more Yang–Mills solutions corresponding to generic three-dimensional trajectories in the analog Newtonian system, which can only be investigated numerically. We hope to come back to these questions in due time.

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