Kinetic theory of the heat conductivity in the crust of a magnetized neutron star

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The solution of Boltzmann equation for plasma in magnetic field, with arbitrarily degenerate electrons and non-degenerate nuclei, is obtained by Chapman-Enskog method. Fully ionized plasma is considered. The tensor of the heat conductivity coefficients in non-quantized magnetic field is calculated. For strongly degenerate plasma the asymptotic analytic formulas are obtained. The Lorentz approximation, with neglecting of electron-electron encounters, is asymptotically exact for strongly degenerate plasma. For strongly degenerate electrons we obtain an asymptotically exact analytical solution for the heat conductivity tensor in presence of a magnetic field. This solution has considerably more complicated dependence on the magnetic field than those in previous publications, and gives several times smaller relative value of a thermal conductivity across the magnetic field at $\omega \tau \gtrsim 0.8$.

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1. Introduction

X-ray observations of thermal emission show periodic variabilities in single neutron stars[23], indicating to the anisotropic temperature distribution. Such kind of a single neutron star variability was predicted in [4] in 1976 year. A periodic X-ray variability of a single neutron star is presented in Fig.1 from [10].
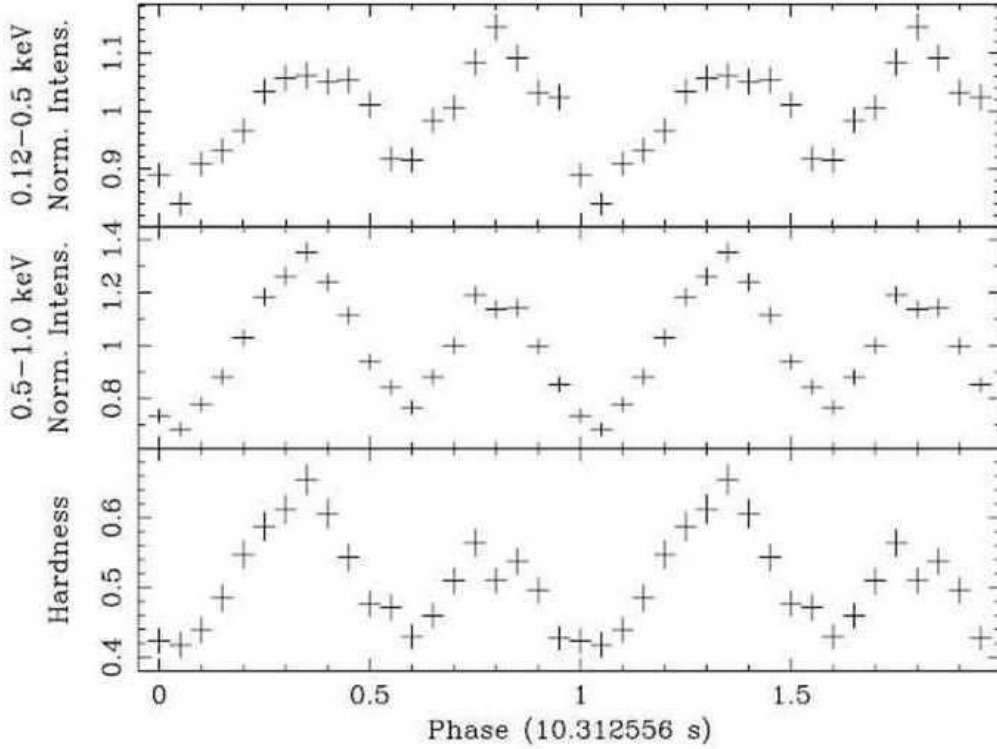

Figure 1: Pulse profile of RBS1223 in the 0.012-0.5 keV (soft) and 0.5-1 keV (hard) energy bands, together with the ratio hard/soft obtained from the EPIC-pn data of the Jan 2003 XMM observations, from [10].

The list of 7 known single neutron star, named in the literature as "magnifucent seven", with some of their properties, is presented in Fig.2, see [19].

Heat transfer in the envelopes of NS plays crucial role in many aspects of evolution of these stars. Thermal conductivity is the basic quantity needed for calculating the relationship between the internal temperature of a neutron star and its effective surface temperature. To calculate thermal conductivity we should know the transport properties of a dense matter where electrons are degenerate, and form a nearly ideal Fermi-gas [22]. The ions are usually treated as non-degenerate. They may be in a gaseous state, may form a Coulomb liquid or a Coulomb crystal [16]. Under such conditions, electrons are the most important heat carriers, and the thermal conductivity is determined by electron motion. The magnetic field limits the motion of electrons in directions perpendicular to the field lines and, since they are the main carriers of the heat transport, the thermal conductivity in these directions is suppressed, while remaining unaffected along the field lines. The conductivity of

Source, RX J	Spin Periods, s	Amplitude/2	Temperature, eV	Absorption line energy, eV
1856.5−3754	7.06	1.5%	60–62	no
0720.4−3125	8.39	11%	85–87	270
1605.3+3249 (RBS 1556)	???	-	93–96	450
1308.6+2127 (RBS 1223)	10.31	18%	102	300
2143.0+0654 (RBS 1774)	9.44	4%	102–104	700
0806.4−4123	11.37	6%	92	460
0420.0−5022	3.45	13%	45	330

Figure 2: The list of single neutron star X-ray sources (magnificent seven) with their physical characteristics, from [19]
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electrons in NS and white dwarfs in presence of a magnetic field was studied in [8, 18]. The ratio between thermal conductivity along and across magnetic field lines considered in [8] was taken as

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{1}{1+(\omega\tau)^2}.$$ (1.1)

and was used also in [18]. Here ω is electron cyclotron frequency, τ is the time between collisions. The influence of the magnetic field on the electron heat conductivity in the form (1.1) was used in subsequent papers, see [11, 21]. Classical methods of kinetic gas theory were developed by Maxwell, Boltzmann, Gilbert, Enskog and Chapman. These methods are presented in the monograph of Chapman and Cowling [7]. Here we find an analytic solution for the heat conductivity tensor of strongly degenerate electrons in a magnetic field, in the Lorentz approximation, which is asymptotically exact in this case, showing a more complicated dependence on the magnetic field strength than (1.1). The kinetic coefficients in the non-degenerate plasma, with and without magnetic field had been calculated in [14, 15, 2, 1], using Chapmen-Enskog expansion method. Coefficients of the heat conductivity tensor in a degenerate stellar cores were calculated in Lorentz approximation for a hydrogen plasma in [20]. A non-relativistic calculation, based on the quantum Lenard-Balescu transport equation for the thermal and electrical conductivities of plasma of highly degenerate, weakly coupled electrons and nondegenerate, weakly coupled ions was performaed in [12].

Here we present the solution of the Boltzmann equation for strongly degenerate electrons in non-quantized magnetic field, described in [3]. For strongly degenerate electrons we obtain an asymptotically exact analytical solution for the heat conductivity tensor in presence of a magnetic field. This solution has considerably more complicated dependence on the magnetic field than those in previous publications, and gives several times smaller relative value of a thermal conductivity

across the magnetic field.

2. Boltzmann equations and transfer equations

We use a Boltzmann equation for electrons, in a magnetic field, with an allowance of arbitrary degeneracy, and assuming them as non-relativistic. We consider the electron gas in a crystal lattice of heavy nuclei, and take into account the interaction of the electrons with a nondegenerate nuclei and with one another. The nuclear component of the matter in the crust is in a crystal state, and therefore the isotropic part of the distribution function f_{N0} may differ from the Maxwellian distribution. If the mass of the nucleus m_N is much greater, than the electron mass m_e, then to the terms $\sim m_e/m_N$ the details of the distribution function f_{N0} are unimportant, and the calculations can be made for arbitrary f_{N0}.

Boltzmann equation, which describes the time variation of the electron distribution function f in presence of the electric and magnetic fields is written as [2, 1]

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial r_i} - \frac{e}{m_e}(E_i + \frac{1}{c}\varepsilon_{ikl}c_kB_l)\frac{\partial f}{\partial c_i} + J = 0. \tag{2.1}$$

Here $(-e), m_e$ are the charge (negative) and the mass of the electron, E_i, B_i are the strength of the electric field, and magnetic induction, ε_{ikl} is the totally antisymmetric Levi-Civita tensor, c is the speed of the light, c_i is a velocity vector in the laboratory frame, r_i is a particle radius-vector in the laboratory frame. The collision integral J for arbitrary degenerate electrons, from [7, 17], is written in the form,

$$J = J_{ee} + J_{eN} = R\int[f'f_1'(1-f)(1-f_1) - ff_1(1-f')(1-f_1')] \times g_{ee}b\,db\,d\varepsilon\,dc_{1i} +$$

$$+ \int[f'f_N'(1-f) - ff_N(1-f')] \times g_{eN}b\,db\,d\varepsilon\,dc_{Ni}. \tag{2.2}$$

Here, the impact parameter b, and ε are geometrical parameters of particle collisions with relative velocities $g_{ee}, g_{eN}, R = \frac{2m_e^3}{h^3}$. The integration in electron part of the collision integral in (2.2) is performed over the phase space of the incoming particles (dc_{1i}), and the physical space of their arrival $(b\,db\,d\varepsilon)$ [7]. The velocity functions after collision are marked with touches.

The Boltzmann equation for electrons with a binary collision integral (2.2) may be applied in conditions, when the electron gas is considered as ideal, i.e. the kinetic energy of the electrons is much larger than the energy of electrostatic interactions. In the neutron stars and white dwarfs we have a plasma at very large density, when it is important to take into account the electrons degeneracy. It is known from the statistical physics, that a gas of strongly degenerate electrons becomes ideal, because large Fermi energy substitutes here the thermal energy [13]. Therefore the calculations in this paper are applied to the high density plasma with degenerate electrons. Detailed discussion of the applicability of a binary collision integral (2.2), and its modifications for high density non-degenerate gases may be found in [7]

The collision integral similar to J_{ee} from (2.2) for strongly degenerate neutrons in nuclear matter was written in [17], see also [5]. It was found that that in the presence of non-degenerate heavy nuclei and strongly degenerate neutron, the input of collisions between them in the heat

transfer and diffusion coefficients is negligibly small, in comparison with neutron-nuclei collisions. The same situation we have for the strongly degenerate electrons. Therefore for strongly degenerate electrons the Lorentz approximation, with account of collisions between light and heavy particles only, is asymptotically exact. So for our consideration we can neglect J_{ee} in comparison with J_{eN}, and we can equate $J = J_{eN}$ in the (2.2).

Lets introduce the thermal velocity of electrons, $v_i = c_i - c_{0i}$, where c_{0i} is the mass-average velocity. So we can write the Boltzmann equation with respect to the thermal velocity in the form [1]

$$\frac{df}{dt} + v_i\frac{\partial f}{\partial r_i} - \left[\frac{e}{m_e}(E_i + \frac{1}{c}\varepsilon_{ikl}v_kB_l) + \frac{dc_{0i}}{dt}\right]\frac{\partial f}{\partial v_i} \tag{2.3}$$

$$-\frac{e}{m_ec}\varepsilon_{ikl}v_kB_l\frac{\partial f}{\partial v_i} - \frac{\partial f}{\partial v_i}v_k\frac{\partial c_{0i}}{\partial r_k} + J = 0. \tag{2.4}$$

The transfer equations for the electron concentration, total momentum, and electron energy, in the two-component mixture of electrons and nuclei, can be obtained in a usual manner from the Boltzmann equation in a quasi-neutral plasma [7, 15, 2, 1].

3. Heat conductivity of strongly degenerate electrons in presence of magnetic field: Lorentz approximation

The Boltzmann equation can be solved by Chapmen-Enskog method of successive approximation [7]. This method is used here for conditions, when distribution functions are close to their values in thermodynamic equilibrium, and deviations are considered in a linear approximation. Equation for second order deviation from the equilibrium distribution function had been derived in [6] for a simple gas, see also [7]. The complexity of this equation, and rather narrow region where second order corrections could be important, strongly restricted the application of this approach.

The zeroth approximation to the electron distribution function is a Fermi-Dirac distribution, which is found by equating to zero of the collision integral J_{ee} from (2.2)

$$f_0 = [1 + \exp\left(\frac{m_ev^2 - 2\mu}{2kT}\right)]^{-1}, \quad R\int f_0dv_i = n_e. \tag{3.1}$$

Here, μ is a chemical potential of electrons, k is Boltzmann's constant, T is the temperature. The nuclear distribution function in the zeroth approximation f_{N0} is assumed to be isotropic with respect to the velocities and to depend on the local thermodynamic parameters.

In the first approximation, we seek for the function f in the form: $f = f_0[1 + \chi(1 - f_0)]$.

The function χ admits representation of the solution in the form:

$$\chi = -A_i\frac{\partial \ln T}{\partial r_i} - n_eD_id_i\frac{G_{5/2}}{G_{3/2}}, \tag{3.2}$$

$$d_i = \frac{\rho_N}{\rho}\frac{\partial \ln P_e}{\partial r_i} - \frac{\rho_e}{P_e}\frac{1}{\rho}\frac{\partial P_N}{\partial r_i} + \frac{e}{kT}(E_i + \frac{1}{c}\varepsilon_{ikl}c_{0k}B_l). \tag{3.3}$$

The plasma is supposed to be quasineutral with a zero charge density. The functions A_i and D_i determine the heat transfer and diffusion. Substituting (3.2) in the equation for χ we obtain equations for A_i, D_i [7]. It was shown in [2],[1], that in presence of a magnetic field B_i, the polar vector A_i (and similarly D_i) may be searched for in the form:

$$A_i = A^{(1)}v_i + A^{(2)}\varepsilon_{ijk}v_jB_k + A^{(3)}B_i(v_jB_j), \tag{3.4}$$

Introducing a function: $\xi = A^{(1)} + iBA^{(2)}$, and dimensionless velocity: $u_i = \sqrt{\frac{m_e}{2kT}}v_i$, we obtain the system for ξ as

$$f_0(1-f_0)(u^2 - \frac{5G_{5/2}}{2G_{3/2}}) = -iBf_0(1-f_0)\frac{e\xi}{m_ec}u_i \tag{3.5}$$

$$+ \int f_0 f_{N0}(1-f_0')(\xi u_i - \xi' u_i')g_{eN}bdbd\varepsilon dc_{Ni}.$$

Using relations $f_0' = f_0$, $\xi' = \xi$, $u_i' = u_i cos\theta$, and making integration over dc_{Ni}, the equation for the function ξ may be written in the form:

$$f_0(1-f_0)(u^2 - \frac{5G_{5/2}}{2G_{3/2}}) = -iBf_0(1-f_0)\frac{e\xi}{m_ec}u_i \tag{3.6}$$

$$+ f_0(1-f_0)n_N\xi \int (1 - \cos\theta)g_{eN}bdbd\varepsilon.$$

The function ξ is defined by expression

$$\xi = \frac{u^2 - \frac{5}{2}\frac{G_{5/2}}{G_{3/2}}}{2\pi n_N \int_0^\infty (1 - \cos\theta)gbdb - i\omega}. \tag{3.7}$$

We obtain in Lorenz approximation, with $g_{12} = v$,

$$\int_0^\infty (1 - \cos\theta)gbdb = 2\frac{e^4Z^2}{m_e^2v^3}\Lambda, \quad \xi = \frac{u^2 - \frac{5}{2}\frac{G_{5/2}}{G_{3/2}}}{4\pi n_N \left(\frac{m_e}{2kT}\right)^{3/2}\frac{e^4Z^2}{m_e^2u^3}\Lambda - i\omega}. \tag{3.8}$$

According to the complex form of ξ we have:

$$A^{(1)} = \frac{(u^2 - \frac{5}{2}\frac{G_{5/2}}{G_{3/2}})4\pi n_N \left(\frac{m_e}{2kT}\right)^{3/2}\frac{e^4Z^2}{m_e^2u^3}\Lambda}{\left[4\pi n_N \left(\frac{m_e}{2kT}\right)^{3/2}\frac{e^4Z^2}{m_e^2u^3}\Lambda\right]^2 + \omega^2},$$

$$A^{(2)} = \frac{\omega}{B}\frac{u^2 - \frac{5}{2}\frac{G_{5/2}}{G_{3/2}}}{\left[4\pi n_N \left(\frac{m_e}{2kT}\right)^{3/2}\frac{e^4Z^2}{m_e^2u^3}\Lambda\right]^2 + \omega^2}. \tag{3.9}$$

$$A^{(3)} = A^{(1)}(B=0) - A^{(1)}. \tag{3.10}$$

The expression for the heat flux is written as:

$$q_i = -\frac{2\pi}{3}\frac{m_e^4}{h^3 T}\left(\frac{2kT}{m_e}\right)^{7/2}\left[\delta_{ij}\int_0^\infty f_0(1-f_0)A^{(1)}x^{5/2}dx\right. \tag{3.11}$$

$$\left.-\varepsilon_{ijk}B_k\int_0^\infty f_0(1-f_0)A^{(2)}x^{5/2}dx + B_iB_j\int_0^\infty f_0(1-f_0)A^{(3)}x^{5/2}dx\right]\frac{\partial T}{\partial x_j} \tag{3.12}$$

$$= -\lambda^{(1)}\frac{\partial T}{\partial x_i} - \varepsilon_{ijk}B_k\lambda^{(2)}\frac{\partial T}{\partial x_j} - B_iB_j\lambda^{(3)}\frac{\partial T}{\partial x_j} \tag{3.13}$$

$$= q_i^{(1)} + q_i^{(2)} + q_i^{(3)}, \quad x = u^2 \qquad . \tag{3.14}$$

For strongly degenerate electrons at $x_0 \gg 1$ the integrals in (3.11) with $A^{(1)}$, $A^{(2)}$ $A^{(3)}$ from (3.9)-(3.10) are expressed analytically, using expansion formula [13]

$$\int_0^\infty \frac{f(x)dx}{e^{x-x_0}+1} = \int_0^{x_0} f(x)dx + \frac{\pi^2}{6}f'(x_0) + ... \tag{3.15}$$

The average frequency of electron-ion collisions ν_{ei} is written in [9] in the form

$$\nu_{ei} = \frac{4}{3}\sqrt{\frac{2\pi}{m_e}}\frac{Z^2e^4n_N\Lambda}{(kT)^{3/2}G_{3/2}}\frac{1}{1+e^{-x_0}}. \tag{3.16}$$

For strongly degenerate electrons it is expressed as

$$\nu_{ei} = \frac{32\pi^2}{3}m_e\frac{Z^2e^4\Lambda n_N}{h^3n_e}. \tag{3.17}$$

Using (3.9), and writing the formula using τ_d as an inverse value of ν_{ei} from (3.17), we write the heat conductivity coefficients in the form

$$\lambda^{(1)} = \frac{5\pi^2}{6}\frac{k^2Tn_e}{m_e}\tau_d\left\{\frac{1}{1+\omega^2\tau_d^2} - \frac{6}{5}\frac{\omega^2\tau_d^2}{(1+\omega^2\tau_d^2)^2} - \frac{\pi^2}{10}\left[\frac{1}{1+\omega^2\tau_d^2\left(\frac{x^3}{x_0^3}\right)}\right]''|_{x=x_0}\right\}, \tag{3.18}$$

$$\lambda^{(2)} = -\frac{4\pi^2}{3}\frac{k^2Tn_e}{m_e}\frac{\tau_d^2\omega}{B}\left\{\frac{1}{1+\omega^2\tau_d^2} - \frac{3}{4}\frac{\omega^2\tau_d^2}{(1+\omega^2\tau_d^2)^2} - \frac{\pi^2}{16}\left[\frac{1}{1+\omega^2\tau_d^2\left(\frac{x^3}{x_0^3}\right)}\right]''|_{x=x_0}\right\}. \tag{3.19}$$

$$B^2\lambda^{(3)} = \lambda^{(1)}(B=0) - \lambda^{(1)}. \tag{3.20}$$

In the case of strongly degenerate electrons the equations (3.9)-(3.11),(3.18),(3.19) give an asymptotically exact solution for the heat conductivity coefficients, because collisions between electrons can be neglected in this case. The difference between the exact $[\lambda^{(1)}]/[\lambda^{(1)}(B=0)]$ from (3.18), and phenomenological account of the magnetic field influence on the heat conductivity coefficients (1.1) is presented in Fig. 1. Here the ratios between the values, which are perpendicular and parallel to magnetic field, are plotted for $kT = 0.09E_f$. At $\omega\tau = 1.5$ the exact value of this ratio is 4 times smaller than the phenomenological one.
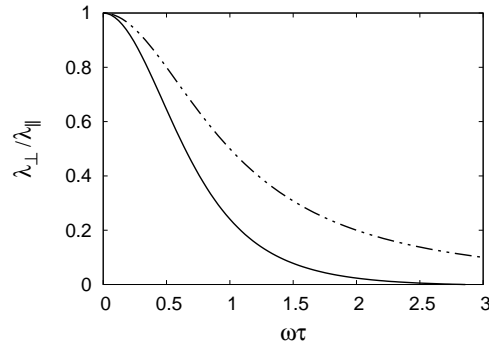
Figure 3: The plots of the ratio $\lambda_\perp/\lambda_\parallel$ as a function of $\omega\tau$ are presented for phenomenologically obtained heat conductivity (dash-dot line) for comparison with heat conductivity obtained by the solution of Boltzmann equation in Lorentz appoximation (solid line) with $kT = 0.09E_f$.

4. Discussion

In this paper a thermal conductivity tensor is found for degenerate non-relativistic electrons in presence of a non-quantizing magnetic field. The heat conductivity coefficients for strongly degenerate electrons, in presence of magnetic field, are obtained asymptotically exactly in Lorentz approximation, when the electron-electron collision may be neglected in comparison with electron-nuclei collisions at nondegenerate nuclei.

In most works considering the heat conductivity in astrophysical objects, in the neutron stars in particular, following Flowers and Itoh [8], the influence of the magnetic field on the heat flux was taken into account phenomenologically using the coefficient $1/(1 + \omega^2\tau^2)$, which decreases the heat flux in the direction perpendicular to the direction of a magnetic field. Our results, obtained by the solution of Boltzmann equation show, that the influence of the magnetic field on the coefficients of heat conductivity is stronger, and has a more complicated character Fig. 1.

The new coefficients can be used for calculation of temperature distribution in white dwarfs, near the surface, and in the crust of magnetized neutron star. The temperature distribution over the surface of NS is important for understanding of the geometry of magnetic field inside the neutron star and near its surface.

Acknowledgments

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