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Loop–tree duality at two loops

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We derive the loop-tree duality (LTD) theorem at two loops, and outline the extension of the four-dimensional unsubtraction (FDU) scheme for the computation of physical observables to NNLO.

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1. The Loop–Tree Duality at two-loops

The loop–tree duality (LTD) [1, 2, 3] transforms any loop integral or loop scattering amplitude into a sum of tree-level like objects that are constructed by setting on-shell a number of internal propagators equal to the number of loops. Explicitly, LTD is realised by modifying the $\imath 0$ prescription of the Feynman propagators that remain off-shell

$$G_F(q_j) = \frac{1}{q_j^2 - m_j^2 + \imath 0} \qquad \to \qquad G_D(q_i;q_j) = \left. \frac{1}{q_j^2 - m_j^2 - \imath 0\, \eta k_{ji}} \right|_{G_F(q_i)\ \text{on-shell}} , \tag{1.1}$$

with $k_{ji} = q_j - q_i$, and η^μ an arbitrary future-like vector. The most convenient choice is $\eta^\mu = (1, \mathbf{0})$, which is equivalent to integrate out the loop energy components of the loop momenta through the Cauchy residue theorem. The left-over integration is then restricted to the Euclidean space of the loop three-momenta. The dual prescription (indeed, only the sign matters) can hence be either $-\imath 0\, \eta k_{ji} = -\imath 0$ for some dual propagators or $-\imath 0\, \eta k_{ji} = +\imath 0$ for the others, and encodes in a compact and elegant way the contribution of the multiple cuts that are introduced by the Feynman tree theorem [4]. The on-shell condition is given by $\tilde{\delta}(q_i) = \imath 2\pi\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$, and determines that the loop integration is restricted to the positive energy modes, $q_{i,0} > 0$, of the on-shell hyperboloids (light-cones for massless particles) of the internal propagators. We also introduce the short-hand notation for the loop integration measure

$$\int_{\ell_i} \bullet = -\imath \int \frac{d^d \ell_i}{(2\pi)^d} \bullet \ . \tag{1.2}$$

In order to generalise LTD to higher orders [2], we need to introduce the following functions

$$G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) , \qquad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j \in \alpha_k \\ j \neq i}} G_D(q_i;q_j) , \tag{1.3}$$

where α_k labels all the propagators, Feynman or dual, of a given subset. An interesting identity fulfilled by these functions is the following

$$G_D(\alpha_i \cup \alpha_j) = G_D(\alpha_i)\, G_D(\alpha_j) + G_D(\alpha_i)\, G_F(\alpha_j) + G_F(\alpha_i)\, G_D(\alpha_j) , \tag{1.4}$$

involving the union of two subsets α_i and α_j. These are all the ingredients necessary to iteratively extend LTD to two loops and beyond. For example, at one loop, the Feynman and the dual representations of a N-leg scattering amplitude are

$$\mathscr{A}_N^{(1)} = \int_{\ell_1} \mathscr{N}(\ell_1, \{p_i\}_N)\, G_F(\alpha_1) = -\int_{\ell_1} \mathscr{N}(\ell_1, \{p_i\}_N) \otimes G_D(\alpha_1) , \tag{1.5}$$

respectively, where $\mathscr{N}(\ell_1, \{p_i\}_N)$ is the numerator that depends on the loop momentum ℓ_1 and the four-momenta of the N external partons $\{p_i\}_N$. In the absence of multiple powers of the Feynman propagators, the numerator is not altered by the application of the Cauchy theorem. However, the calculation of the residues of multiple poles to obtain the corresponding LTD representation requires the participation of the numerator. This is represented in Eq. (1.5) by the symbol $\otimes$.

At two loops all the internal propagators can be classified into three different subsets (e.g. those depending on ℓ_1, ℓ_2 and their sum $\ell_1 + \ell_2$, as shown in Fig. 1). Starting from the Feynman representation of a two-loop scattering amplitude

$$\mathscr{A}_N^{(2)} = \int_{\ell_1} \int_{\ell_2} \mathscr{N}(\ell_1, \ell_2, \{p_i\}_N) \, G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \,, \tag{1.6}$$

we obtain in a first step by applying LTD to one of the loops (Eq. (1.5)):

$$\mathscr{A}_N^{(2)} = -\int_{\ell_1} \int_{\ell_2} \mathscr{N}(\ell_1, \ell_2, \{p_i\}_N) \, G_F(\alpha_1) \, G_D(\alpha_2 \cup \alpha_3) \,. \tag{1.7}$$

Before applying LTD to the second loop, it is necessary to use Eq. (1.4) to express the dual function $G_D(\alpha_2 \cup \alpha_3)$ in a suitable form. The identity in Eq. (1.4) splits the dual integrand into a first term that contains two dual functions, and therefore two internal lines on-shell, and two more terms with a single dual function and Feynman propagators involving the other two sets of propagators, to which we can recursively apply LTD. The final dual representation of the two-loop amplitude in Eq. (1.6) is

$$\begin{aligned} \mathscr{A}_N^{(2)} = & \int_{\ell_1} \int_{\ell_2} \mathscr{N}(\ell_1, \ell_2, \{p_i\}_N) \otimes \Big\{ G_D(\alpha_2) \, G_D(\alpha_1 \cup \alpha_3) \\ & + G_D(-\alpha_2 \cup \alpha_1) \, G_D(\alpha_3) - G_F(\alpha_1) \, G_D(\alpha_2) \, G_D(\alpha_3) \Big\} \,. \end{aligned} \tag{1.8}$$

In Eq. (1.8), it is necessary to take into account that the momentum flow in the loop formed by the union of α_1 and α_2 occurs in opposite directions. Therefore, it is compulsory to change the direction of the momentum flow in one of the two sets. This is represented by adding a sign in front of e.g. α_2, namely, we have written

$$\int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) \, G_F(\alpha_2) = -\int_{\ell_1} \int_{\ell_2} G_D(-\alpha_2 \cup \alpha_1) \,. \tag{1.9}$$

Changing the momentum flow is equivalent to select the negative energy modes. For the internal momenta in the set α_2, this means

$$\tilde{\delta}(-q_j) = \frac{\imath \pi}{q_{j,0}^{(+)}} \, \delta(q_{j,0} + q_{j,0}^{(+)}) \,, \qquad j \in \alpha_2 \,. \tag{1.10}$$

The dual representation gets its simplest form if the Feynman representation contains only single powers of the Feynman propagators. This restriction cannot be avoided anymore at two-loops where, for example, selfenergy insertions in internal lines lead automatically to double powers of one propagator. However, all the double poles can be included with a clever labelling of the internal momenta in the set α_1, exclusively, which is not integrated in the first instance. Therefore, we have assumed that the numerator in Eq. (1.7) is not affected by the application of LTD. The final dual representation in Eq. (1.8) depends, in general, on the explicit form of the numerator. Again, this is represented by the symbol $\otimes$.

The number of independent double cuts in Eq. (1.8) per Feynman diagram is

$$N(\alpha_2 \times (\alpha_1 + \alpha_3) + (\alpha_1 + \alpha_2) \times \alpha_3) \,. \tag{1.11}$$

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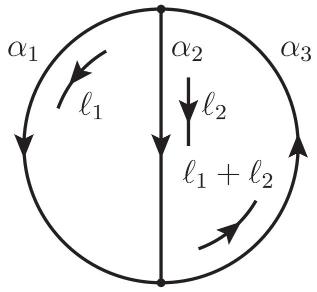

Figure 1: Momentum flow of a two-loop Feynman diagram. An arbitrary number of external legs (not shown) are attached to each loop line α_i.

Therefore, it is convenient to have α_2 as the set with the smallest number of propagators. For planar diagrams, the set α_2 will contain one single propagator.

It is interesting to note that although the integration over the loop three-momenta is unrestricted, after analysing the singular behaviour of the loop integrand one realises that thanks to a partial cancellation of singularities among different dual components, all the physical threshold and IR singularities remain confined to a compact region of the loop three-momentum [5, 6]. This relevant fact allows to construct mappings between the virtual and real kinematics, which are based on the factorisation properties of QCD, to implement the summation over degenerate soft and collinear states for physical observables in the four-dimensional unsubtraction (FDU) scheme, as explained in Section 2.

2. Four-Dimensional Unsubtraction

An alternative approach to the subtraction methods [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] is introduced by the four-dimensional unsubtraction (FDU) [20, 21, 22], which is based in the loop–tree duality (LTD). The idea behind FDU is to exploit suitable mappings of momenta between the virtual and real kinematics in such a way that the summation over the degenerate soft and collinear quantum states is performed locally at integrand level without the necessity to introduce infrared (IR) subtractions. Suitable counter-terms are used to cancel, also locally, the UV singularities, in such a way that calculations can be performed without altering the dimensions of the space-time. The method should improve the efficiency of Monte Carlo event generators because it is designed for integrating simultaneously the real and virtual contributions.

As usual, the NLO cross-section is constructed in FDU from the one-loop virtual correction with N external partons and the exclusive real cross-section with $N+1$ partons

$$\sigma^{\rm NLO} = \int_N d\sigma_{\rm V}^{(1,{\rm R})} + \int_{N+1} d\sigma_{\rm R}^{(1)} \,, \tag{2.1}$$

integrated over the corresponding phase-space, $\int_N$ and $\int_{N+1}$ respectively. The virtual contribution is obtained from its LTD representation

$$\int_N d\sigma_{\rm V}^{(1,{\rm R})} = \int_N \int_{\vec{\ell}_1} 2\,{\rm Re}\,\langle \mathcal{M}_N^{(0)} | \left(\sum_i \mathcal{M}_N^{(1)}(\tilde{\delta}(q_i)) \right) - \mathcal{M}_{\rm UV}^{(1)}(\tilde{\delta}(q_{\rm UV})) \rangle \, \hat{\mathcal{O}}(\{p_k\}_N) \,. \tag{2.2}$$

In Eq. (2.2), $\mathcal{M}_N^{(0)}$ is the N-leg scattering amplitude at LO, and $\mathcal{M}_N^{(1)}(\tilde{\delta}(q_i))$ is the dual representation of the unrenormalised one-loop scattering amplitude with the internal momentum q_i set on-shell. The integral is weighted with the function $\hat{\mathcal{O}}(\{p_k\}_N)$ that defines a given observable, for example the jet cross-section in the k_T-algorithm. The expression (2.2) includes appropriate counter-terms, $\mathcal{M}_{\rm UV}^{(1)}(\tilde{\delta}(q_{\rm UV}))$, that implement renormalization by subtracting the UV singularities locally, as discussed in Ref. [21], including UV singularities of degree higher than logarithmic that integrate to zero.

By means of appropriate mappings between the real and virtual kinematics[22, 21]:

$$\{p'_j\}_{N+1} \to (q_i, \{p_k\}_N) \ , \tag{2.3}$$

the real phase-space is rewritten in terms of the virtual phase-space and the loop three-momentum

$$\int_{N+1} = \int_N \int_{\vec{\ell}_1} \sum_i \mathcal{J}_i(q_i) \mathcal{R}_i(\{p'_j\}_{N+1}) \ , \tag{2.4}$$

where $\mathcal{J}_i(q_i)$ is the Jacobian of the transformation with q_i on-shell, and $\mathcal{R}_i(\{p'_j\}_{N+1})$ defines a complete partition of the real phase-space

$$\sum_i \mathcal{R}_i(\{p'_j\}_{N+1}) = 1 \ . \tag{2.5}$$

In this way, the NLO cross-section can be cast into a single integral in the Born/virtual phase-space and the loop three momentum

$$\begin{aligned}\sigma^{\rm NLO} &= \int_N \int_{\vec{\ell}_1} \bigg[2\,{\rm Re}\, \langle \mathcal{M}_N^{(0)} | \Big(\sum_i \mathcal{M}_N^{(1)}(\tilde{\delta}(q_i)) \Big) - \mathcal{M}_{\rm UV}^{(1)}(\tilde{\delta}(q_{\rm UV})) \rangle \, \hat{\mathcal{O}}(\{p_k\}_N) \\ &\quad + \sum_i \mathcal{J}_i(q_i) \mathcal{R}_i(\{p'_j\}_{N+1}) \, |\mathcal{M}_{N+1}^{(0)}(\{p'_j\}_{N+1})|^2 \, \hat{\mathcal{O}}(\{p'_j\}_{N+1}) \bigg] \ . \end{aligned} \tag{2.6}$$

The NLO cross-section defined in Eq. (2.6) has a smooth four-dimensional limit and can be evaluated directly in four space-time dimensions. DREG is only necessary to fix the UV renormalisation counter-terms in order to define the cross-section in e.g. the $\overline{MS}$ scheme, the rest of the calculation is feasible directly at $d = 4$. The Eq. (2.6) exhibits also an smooth massless limit for massive partons if the mappings in Eq. (2.3) map conveniently the quasicollinear configurations [21]. This is another advantage of the formalism because it allows to describe with a single implementation the same process with either massless or massive partons.

Once we have obtained the dual representation of the two-loop scattering amplitude, we can outline how to extend FDU at NNLO and higher orders. Analogously to the NLO case, the total cross-section at NNLO consists of three contributions

$$\sigma^{\rm NNLO} = \int_N d\sigma_{\rm VV}^{(2)} + \int_{N+1} d\sigma_{\rm VR}^{(2)} + \int_{N+2} d\sigma_{\rm RR}^{(2)} \ , \tag{2.7}$$

where the double virtual cross-section $d\sigma_{\rm VV}^{(2)}$ receives contributions from the interference of the two-loop with the Born scattering amplitudes, and the square of the one-loop scattering amplitude with N external partons, the virtual-real cross-section $d\sigma_{\rm VR}^{(2)}$ includes the contributions from the interference of one-loop and tree-level scattering amplitudes with one extra external particle,

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and the double real cross-section $d\sigma_{\rm RR}^{(2)}$ are tree-level contributions with emission of two extra particles. The LTD representation of the two-loop scattering amplitude is obtained by setting two internal lines on-shell [2], as described in Section 1. It leads to the two-loop dual components $\langle \mathscr{M}_N^{(0)} | \mathscr{M}_N^{(2)}(\tilde\delta(q_i,q_j))\rangle$, while the two-loop momenta of the squared one-loop amplitude are independent and generate dual contributions of the type $\langle \mathscr{M}_N^{(1)}(\tilde\delta(q_i)) | \mathscr{M}_N^{(1)}(\tilde\delta(q_j))\rangle$. In both cases, there are two independent loop three-momenta and N external momenta, from where we can reconstruct the kinematics of the tree-level corrections entering $d\sigma_{\rm RR}^{(2)}$ and the one-loop corrections in $d\sigma_{\rm VR}^{(2)}$:

$$\{p_r''\}_{N+2} \to (q_i,q_j,\{p_k\}_N)~, \qquad (q_l', \{p_s'\}_{N+1}) \to (q_i,q_j,\{p_k\}_N)~. \tag{2.8}$$

3. Applications and outline

We showed in a recent paper [23] that the amplitudes of the Higgs boson production through gluon fusion and the Higgs boson decay to two photons exhibit remarkable properties in the LTD representation. The dual contributions obtained for different internal particles – charged electroweak gauge bosons, top quarks or charged scalars – featured the exact same functional form, and could be written in an universal way by using flavour dependent scalar parameters depending only on the space-time dimension d and the scales and masses of the particles involved in the process. We also achieved a pure four-dimensional ($d = 4$) representation of the loop amplitude by introducing a local renormalization of the UV singularities of the integrand. Therefore, it is natural to extend the analysis at two-loops as benchmark calculation in LTD [24]. It is worth to stress that the classical approach of integration-by-parts (IBP) [25] and reduction to master integrals is not suitable in this framework as it would alter the local IR and UV behaviour.

Preliminary results for the two-loop $H \to \gamma\gamma$ amplitude show that the dual representation can be written in terms of order 20 flavour dependent parameters. The integrand expressions of the dual amplitudes are therefore quite compact and are obtained by aplying an algebraic reduction of the dual amplitudes to dual integrals that involve both positive and negative powers of dual propagator denominators. We also proof that, as it happens at one-loop, there is partial cancellation of the integrand singularities in such a way that the numerical integration remains stable with less contour deformations. These results will soon be published [24].

4. Conclusions

The bottleneck in higher order perturbative calculations for scattering processes at high energies is not only the evaluation of multi-loop Feynman diagrams, but also the gathering of all the quantum corrections from different loop orders (and thus different number of final-state partons). In order to match the expected experimental accuracy at the LHC, particularly in the high luminosity phase, and at future colliders new theoretical efforts are still needed to overcome the current precision frontier. The LTD/FDU formalism offers an alternative approach with potential advantages. New interesting results within this framework will be published soon.

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