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No-π Theorem for Euclidean Massless Correlators

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We provide the reader with a (very) short review of recent advances in our understanding of the π-dependent terms in massless (Euclidean) 2-point functions as well as in generic anomalous dimensions and β-functions. We extend the considerations of [1] by one more loop, that is for the case of 6-loop correlators and 7-loop renormalization group (RG) functions.

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1. Introduction and Preliminaries

Since the seminal calculation of the Adler function at order α_s^3 [2] it has been known that p-functions demonstrate striking regularities in terms proportional to π^{2n}, with n being positive integer. Here by p-functions we understand ($\overline{\text{MS}}$-renormalized) Euclidean Green functions[1] or 2-point correlators or even some combination thereof, expressible in terms of massless propagator-like Feynman integrals (to be named p-integrals below).

To describe these regularities we need to introduce a few notations and conventions. (In what follows we limit ourselves by the case of QCD considered in the Landau gauge). Let

$$F_n(a,\ell_\mu) = 1 + \sum_{1\le i\le n}^{0\le j\le i} g_{i,j}\,(\ell_\mu)^j\, a^i \tag{1.1}$$

be a p-function, where $a = \frac{\alpha_s(\mu)}{4\pi}$, $\ell_\mu = \ln\frac{\mu^2}{Q^2}$ and Q is an (Euclidean) external momentum. The integer n stands for the (maximal) power of α_s appearing in the p-integrals contributing to F_n. The F without n will stand as a shortcut for a formal series F_∞. In terms of bare quantities[2]

$$F = Z F_B(a_B,\ell_\mu), \qquad\qquad Z = 1 + \sum_{i\ge 1}^{1\le j\le i} Z_{i,j}\,\frac{a^i}{\varepsilon^j}, \tag{1.2}$$

with the bare coupling constant and the corresponding renormalization constant being

$$a_B = \mu^{2\varepsilon} Z_a\, a, \qquad\qquad Z_a = 1 + \sum_{i\ge 1}^{1\le j\le i} \Big(Z_a\Big)_{i,j}\frac{a^i}{\varepsilon^j}, \tag{1.3}$$

$$\Big(\frac{\partial}{\partial \ell_\mu} + \beta\, a\,\frac{\partial}{\partial a}\Big) F = \gamma F, \tag{1.4}$$

with the anomalous dimension (AD)

$$\gamma(a) = \sum_{i\ge 1}\gamma_i a^i, \ \ \gamma_i = -iZ_{i,1}. \tag{1.5}$$

The coefficients of the β-function β_i are related to Z_a in the standard way:

$$\beta_i = i\,(Z_a)_{i,1}\,. \tag{1.6}$$

A p-function F is called scale-independent if the corresponding AD $\gamma\equiv 0$. If $\gamma\neq 0$ then one can always construct a scale-invariant object from F and γ, namely:

$$F^{\rm si}_{n+1}(a,\ell_\mu) = \frac{\partial}{\partial \ell_\mu}(\ln F)_{n+1} \equiv \left(\frac{\Big(\gamma(a) - \beta(a) a\frac{\partial}{\partial a}\Big)F_n}{F_n}\right)_{n+1}. \tag{1.7}$$

Note that $F^{si}_{n+1}(a,\ell_\mu)$ starts from the first power of the coupling constant a and is formally composed from $\mathscr{O}(\alpha_s^{n+1})$ Feynman diagrams. In the same time is can be completely restored from F_n and the $(n+1)$-loop AD γ.

An (incomplete) list of the currently known regularities[3] includes the following cases.

[1]Like quark-quark-qluon vertex in QCD with the external gluon line carrying no momentum.

[2]We assume the use of the dimensional regularization with the space-time dimension $D = 4-2\varepsilon$.

[3]For discussion of particular examples of π-dependent contributions into various p-functions we refer to works [3, 4, 5, 6].

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1. Scale-independent p-functions F_n and F_n^{si} with $n \le 4$ are free from π-dependent terms.

2. Scale-independent p-functions F_5^{si} are free from π^6 and π^2 but do depend on π^4.

3. The QCD β-function starts to depend on π at 5 loops only [7, 8, 9] (via $\zeta_4 = \pi^4/90$). In addition, there exits a remarkable identity [1]

$$\beta_5^{\zeta_4} = \frac{9}{8}\beta_1\,\beta_4^{\zeta_3}, \quad \text{with} \quad F^{\zeta_i} = \lim_{\zeta_i \to 0} \frac{\partial}{\partial \zeta_i} F.$$

4. If we change the $\overline{\text{MS}}$-renormalization scheme as follows:

$$a = \bar{a}\,(1 + c_1\,\bar{a} + c_2\,\bar{a}^2 + c_3\,\bar{a}^3 + \frac{1}{3}\,\frac{\beta_5}{\beta_1}\,\bar{a}^4), \tag{1.8}$$

   with c_1, c_2 and c_3 being any rational numbers, then the function $\hat{F}_5^{si}(\bar{a}, \ell_\mu)$ and the (5-loop) β-function $\bar{\beta}(\bar{a})$ both loose any dependence on π. This remarkable fact was discovered in [3].

It should be stressed that eventually every separate diagram contributing to F_n and F_{n+1} contains the following set of irrational numbers: $\zeta_3, \zeta_4, \zeta_5, \zeta_6$ and ζ_7 for $n = 4$, ζ_3, ζ_4 and ζ_5 for $n = 3$. Thus, the regularities listed above are quite nontrivial and for sure can not be explained by pure coincidence.

2. Hatted representation of p-integrals and its implications

The full understanding and a generic proof of points 1,2 and 3 above have been recently achieved in our work [1]. The main tool of the work was the so-called "hatted" representation of transcendental objects contributing to a given set of p-integrals. Let us reformulate the main results of [1] in an abstract form.

We will call the set of all L-loop p-integrals $\mathscr{P}_L$ a π-safe one if the following is true.

(i) All p-integrals from the set can be expressed in terms of $(M+1)$ mutually independent (and ε-independent) transcendental generators

$$\mathscr{T} = \{t_1, t_2, \ldots, t_{M+1}\} \;\text{ with } t_{M+1} = \pi. \tag{2.1}$$

This means that any p-integral $F(\varepsilon)$ from $\mathscr{P}_L$ can be uniquely[4] presented as follows

$$F(\varepsilon) = F(\varepsilon, t_1, t_2, \ldots, \pi) + \mathscr{O}(\varepsilon), \tag{2.2}$$

where by F we understand the *exact* value of the p-integral F while the combination $\varepsilon^L F(\varepsilon, \hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M, \pi)$ should be a rational polynomial[5] in $\varepsilon, t_1 \ldots, t_M, \pi$. Every such polynomial is a sum of monomials T_i of the generic form

$$\sum_\alpha r_\alpha T_\alpha, \;\; T_\alpha = \varepsilon^n \prod_{i=1,M+1} t_i^{n_i}, \tag{2.3}$$

[4] We assume that $F(\varepsilon, t_1, t_2, \ldots, \pi)$ does not contain terms proportional to ε^n with $n \ge 1$.

[5] That is a polynomial having rational coefficients.

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with $n \leq L$, n_i and r_α being some non-negative integers and rational numbers respectively. A monomial T_α will be called *π-dependent* and denoted as $T_{\pi,\alpha}$ if $n_{M+1} > 0$. Note that a generator t_i with $i \leq M$ may still include explictly the constant π in its definition, see below.

(ii) For every t_i with $i \leq M$ let us define its hatted counterpart as follows:

$$\hat{t}_i = t_i + \sum_{j=1,M} h_j(\varepsilon)\, T_{\pi,j}, \tag{2.4}$$

with $\{h_j\}$ being rational polynomials in ε vanishing in the limit of $\varepsilon = 0$ and $T_{\pi,j}$ are *all* π-dependent monomials as defined in (2.3). Then there should exist a choice of both a basis $\mathscr{T}$ and polynomials $\{h_j\}$ such that for every L-loop p-integral $F(\varepsilon, t_i)$ the following equality holds:

$$F(\varepsilon, t_1, t_2, \ldots, t_{M+1}) = F(\varepsilon, \hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M, 0) + \mathscr{O}(\varepsilon). \tag{2.5}$$

We will call *π-free* any polynomial (with possibly ε-dependent coefficients) in $\{t_i,\ i = 1, \ldots, M\}$.

As we will discuss below the sets $\mathscr{P}_i$ with $i = 3,4,5$ are for sure π-safe while $\mathscr{P}_6$ highly likely shares the property. In what follows we will always assume that every (renormalized) L-loop p-function as well as (L+1)-loop $\overline{\text{MS}}$ β-functions and anomalous dimensions are all expressed in terms of the generators $t_1, t_2, \ldots, t_{M+1}$.

Moreover, for any polynomial $P(t_1, t_2, \ldots, t_{M+1})$ we define its *hatted* version as

$$\hat{P}(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M) := P(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_M, 0).$$

Let F_L is a (renormalized, with ε set to zero) p-function, γ_L and β_L are the corresponding anomalous dimension and the β-function (all taken in the L-loop approximation). The following statements have been proved in [1] *under the condition that the set $\mathscr{P}_L$ is π-safe and that both the set $\mathscr{T}$ and the polynomilas $\{h_i(\varepsilon)\}$ are fixed.*

1. No-π Theorem

(a) F_L is p-free in any (massless) renormalization scheme for which corresponding β-function and AD γ are both π-free at least at the level of $L+1$ loops.

(b) The scale-invariant combination F^{si}_{L+1} is π-free in any (massless) renormalization scheme provided the β-function is π-independent at least at the level of $L+1$ loops.

2. π-dependence of L-loop p-functions

If F_L is renormalized in $\overline{\text{MS}}$-scheme, then all its π-dependent contributions can be expressed in terms of $\hat{F}_{L-1}|_{\varepsilon=0}$, $\hat{\beta}_{L-1}|_{\varepsilon=0}$ and $\hat{\gamma}_{L-1}|_{\varepsilon=0}$.

3. π-dependence of L-loop β-functions and AD

If β_L and γ_L are given in the $\overline{\text{MS}}$-scheme, then all their π-dependent contributions can be expressed in terms of $\hat{\beta}_{L-1}|_{\varepsilon=0}$ and $\hat{\beta}_{L-1}|_{\varepsilon=0}$, $\hat{\gamma}_{L-1}|_{\varepsilon=0}$ correspondingly.

3. π-structure of 3,4,5 and 6-loop p-integrals

A hatted representation of p-integrals is known for loop numbers $L = 3$ [10], $L = 4$ [11] and $L = 5$ [12]. In all three cases it was constructed by looking for such a basis $\mathscr{T}$ as well as polynomials $h_j(\varepsilon)$ (see eq. (2.4)) that eq. (2.5) would be valid for sufficiently large subset of $\mathscr{P}_L$.

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In principle, the strategy requires the knowledge of all (or almost all) L-loop master integrals. On the other hand, if we *assume* the π-safeness of the set $\mathscr{P}_6$ we could try to fix polynomials $h_j(\varepsilon)$ by considering some limited subset of $\mathscr{P}_6$.

Actually, we do have at our disposal a subset of $\mathscr{P}_6$ due to work [13] where all 4-loop master integrals have been computed up to the transcendental weight 12 in their ε expansion. As every particular 4-loop p-integral divided by ε^n can be considered as a $(4+n)$ loop p-integral we have tried this subset for n=2. Our results are given below (we use even the zetas $\zeta_4 = \pi^2/90$, $\zeta_6 = \pi^6/945, \zeta_8 = \pi^8/9450$ and $\zeta_{10} = \pi^{10}/93555$ instead of the corresponding even powers of π).

$$\underbrace{\hat{\zeta}_3 := \boxed{\zeta_3} + \frac{3\varepsilon}{2}\zeta_4}_{L=3} \quad \underbrace{-\frac{5\varepsilon^3}{2}\zeta_6}_{\delta(L=4)} \quad \underbrace{+\frac{21\varepsilon^5}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-\frac{153\varepsilon^7}{2}\zeta_{10}}_{\delta(L=6)}, \tag{3.1}$$

$$\underbrace{\hat{\zeta}_5 := \boxed{\zeta_5} + \frac{5\varepsilon}{2}\zeta_6}_{(L=4)} \quad \underbrace{-\frac{35\varepsilon^3}{4}\zeta_8}_{\delta(L=5)} \quad \underbrace{+63\varepsilon^5\zeta_{10}}_{\delta(L=6)}, \tag{3.2}$$

$$\underbrace{\hat{\zeta}_7 := \boxed{\zeta_7}}_{L=4} \quad \underbrace{+\frac{7\varepsilon}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-21\varepsilon^3\zeta_{10}}_{\delta(L=6)}, \tag{3.3}$$

$$\underbrace{\hat{\varphi} := \boxed{\varphi} - 3\varepsilon\,\zeta_4\,\zeta_5 + \frac{5\varepsilon}{2}\zeta_3\,\zeta_6}_{L=5} \quad \underbrace{-\frac{24\,\varepsilon^2}{47}\zeta_{10} + \varepsilon^3\,(-\frac{35}{4}\zeta_3\zeta_8 + 5\zeta_5\zeta_6)}_{\delta(L=6)}, \tag{3.4}$$

$$\underbrace{\hat{\zeta}_9 := \boxed{\zeta_9}}_{L=5} \quad \underbrace{+\frac{9}{2}\varepsilon\,\zeta_{10}}_{\delta(L=6)}, \tag{3.5}$$

$$\underbrace{\hat{\zeta}_{7,3} := \boxed{\zeta_{7,3} - \tfrac{793}{94}\zeta_{10}} + 3\varepsilon(-7\zeta_4\zeta_7 - 5\zeta_5\zeta_6)}_{L=6}, \tag{3.6}$$

$$\underbrace{\hat{\zeta}_{11} := \boxed{\zeta_{11}}}_{L=6}, \tag{3.7}$$

$$\underbrace{\hat{\zeta}_{5,3,3} := \boxed{\zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \tfrac{5}{2}\zeta_5\zeta_6}}_{L=6}. \tag{3.8}$$

Here

$$\varphi := \frac{3}{5}\,\zeta_{5,3} + \zeta_3\,\zeta_5 - \frac{29}{20}\,\zeta_8 = \zeta_{6,2} - \zeta_{3,5} \approx -0.1868414 \tag{3.9}$$

and multiple zeta values are defined as [14]

$$\zeta_{n_1,n_2} := \sum_{i>j>0} \frac{1}{i^{n_1}\,j^{n_2}}, \quad \zeta_{n_1,n_2,n_3} := \sum_{i>j>k>0} \frac{1}{i^{n_1}\,j^{n_2}k^{n_3}}. \tag{3.10}$$

Some comments on these eqs. are in order.

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- The boxed entries form a set of π-independent (by definition!) generators for the cases of $L=3$ (eq. (3.1)), $L=4$ (eqs. (3.1—3.3), $L=5$ (eqs. (3.1–3.5) and $L=6$ (eqs. (3.1—3.8).
- For $L=5$ we recover the hatted representation for the set $\mathscr{P}_5$ first found in [12].
- We do not claim that the generators

$$\zeta_3, \zeta_5, \zeta_7, \phi, \zeta_9, \hat{\zeta}_{7,3}|_{\varepsilon=0}, \hat{\zeta}_{5,3,3} \text{ and } \pi \tag{3.11}$$

are sufficient to present the pole and finite parts of every 6-loop p-integral. In fact, it is not true [15, 16, 17]. However we believe that it is safe to assume that all missing irrational constants can be associated with the values of some convergent 6-loop p-integrals at $\varepsilon=0$.

4. π-dependence of 7-loop β-functions and AD

Using the approach of [1] and the hatted representation of the irrational generators (3.11) as described by eqs. (3.1)-(3.8) we can straightforwardly predict the π-dependent terms in the β-function and the anomalous dimensions in the case of *any* 1-charge minimally renormalized field model at the level of 7 loops.

Our results read (the combination $F^{t_{\alpha_1}t_{\alpha_2}\cdots t_{\alpha_n}}$ stands for the coefficient of the monomial $(t_{\alpha_1}t_{\alpha_2}\ldots t_{\alpha_n})$ in F; in addition, by $F^{(1)}$ we understand F with every generator t_i from $\{t_1,t_2,\ldots,t_{M+1}\}$ set to zero).

$$\gamma_4^{\zeta_4} = -\frac{1}{2}\beta_3^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_1\gamma_3^{\zeta_3}, \tag{4.1}$$

$$\gamma_5^{\zeta_4} = -\frac{3}{8}\beta_4^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_2\gamma_3^{\zeta_3} - \beta_3^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_1\gamma_4^{\zeta_3}, \tag{4.2}$$

$$\gamma_5^{\zeta_6} = -\frac{5}{8}\beta_4^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_1\gamma_4^{\zeta_5}, \tag{4.3}$$

$$\gamma_5^{\zeta_3\zeta_4} = 0, \tag{4.4}$$

$$\gamma_6^{\zeta_4} = \frac{3}{2}\beta_3^{(1)}\gamma_3^{\zeta_3} - \frac{3}{10}\beta_5^{\zeta_3}\gamma_1 - \frac{3}{4}\beta_4^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_2\gamma_4^{\zeta_3} - \frac{3}{2}\beta_3^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_1\gamma_5^{\zeta_3}, \tag{4.5}$$

$$\gamma_6^{\zeta_6} = -\frac{1}{2}\beta_5^{\zeta_5}\gamma_1 - \frac{5}{4}\beta_4^{\zeta_5}\gamma_2 + \frac{5}{2}\beta_2\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_5^{\zeta_5} + \frac{3}{2}\beta_1^2\beta_3^{\zeta_3}\gamma_1 - \frac{5}{2}\beta_1^3\gamma_3^{\zeta_3}, \tag{4.6}$$

$$\gamma_6^{\zeta_3\zeta_4} = -\frac{3}{5}\beta_5^{\zeta_3^2}\gamma_1 + 3\beta_1\gamma_5^{\zeta_3^2}, \tag{4.7}$$

$$\gamma_6^{\zeta_8} = -\frac{7}{10}\beta_5^{\zeta_7}\gamma_1 + \frac{7}{2}\beta_1\gamma_5^{\zeta_7}, \tag{4.8}$$

$$\gamma_6^{\zeta_3\zeta_6} = \gamma_6^{\zeta_4\zeta_5} = 0, \tag{4.9}$$

$$\begin{aligned}\gamma_7^{\zeta_4} = &-\frac{1}{4}\beta_6^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_3^{(1)}\gamma_4^{\zeta_3} + \frac{3}{2}\beta_4^{(1)}\gamma_3^{\zeta_3} - \frac{3}{5}\beta_5^{\zeta_3}\gamma_2 \\ &-\frac{9}{8}\beta_4^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_2\gamma_5^{\zeta_3} - 2\beta_3^{\zeta_3}\gamma_4^{(1)} + \frac{3}{2}\beta_1\gamma_6^{\zeta_3},\end{aligned} \tag{4.10}$$

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$$\gamma_7^{\zeta_6} = -\frac{5}{12}\beta_6^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_3^{(1)}\gamma_4^{\zeta_5} - \beta_5^{\zeta_5}\gamma_2 - \frac{15}{8}\beta_4^{\zeta_5}\gamma_3^{(1)} + \frac{5}{2}\beta_2\gamma_5^{\zeta_5} + \frac{5}{2}\beta_1\gamma_6^{\zeta_5}$$
$$+\frac{5}{2}\beta_1\beta_3^{\zeta_3}\beta_2\gamma_1 + \frac{5}{4}\beta_1^2\beta_4^{\zeta_3}\gamma_1 - \frac{15}{2}\beta_1^2\beta_2\gamma_3^{\zeta_3} + 3\beta_1^2\beta_3^{\zeta_3}\gamma_2 - \frac{5}{2}\beta_1^3\gamma_4^{\zeta_3}, \tag{4.11}$$

$$\gamma_7^{\zeta_3\zeta_4} = -\frac{1}{2}\beta_6^{\zeta_3^2}\gamma_1 - \frac{6}{5}\beta_5^{\zeta_3^2}\gamma_2 + \frac{3}{8}\beta_4^{\zeta_3}\gamma_3^{\zeta_3} + 3\beta_2\gamma_5^{\zeta_3^2} - \frac{1}{2}\beta_3^{\zeta_3}\gamma_4^{\zeta_3} + 3\beta_1\gamma_6^{\zeta_3^2}, \tag{4.12}$$

$$\gamma_7^{\zeta_8} = -\frac{7}{12}\beta_6^{\zeta_7}\gamma_1 - \frac{7}{5}\beta_5^{\zeta_7}\gamma_2 + \frac{7}{2}\beta_2\gamma_5^{\zeta_7} + \frac{7}{12}(\beta_3^{\zeta_3})^2\gamma_1 + \frac{7}{2}\beta_1\gamma_6^{\zeta_7} - \frac{7}{8}\beta_1\beta_5^{\zeta_3^2}\gamma_1$$
$$-\frac{7}{8}\beta_1\beta_3^{\zeta_3}\gamma_3^{\zeta_3} + \frac{21}{8}\beta_1^2\gamma_5^{\zeta_3^2} + \frac{35}{8}\beta_1^2\beta_4^{\zeta_5}\gamma_1 - \frac{35}{4}\beta_1^3\gamma_4^{\zeta_5}, \tag{4.13}$$

$$\gamma_7^{\zeta_3\zeta_6} = -\frac{5}{12}\beta_6^{\zeta_3\zeta_5}\gamma_1 - \frac{5}{12}\beta_6^{\phi}\gamma_1 - \frac{15}{8}\beta_4^{\zeta_5}\gamma_3^{\zeta_3} + \frac{5}{2}\beta_3^{\zeta_3}\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_6^{\zeta_3\zeta_5} + \frac{5}{2}\beta_1\gamma_6^{\phi}, \tag{4.14}$$

$$\gamma_7^{\zeta_4\zeta_5} = -\frac{1}{4}\beta_6^{\zeta_3\zeta_5}\gamma_1 + \frac{1}{2}\beta_6^{\phi}\gamma_1 + \frac{3}{2}\beta_4^{\zeta_5}\gamma_3^{\zeta_3} - 2\beta_3^{\zeta_3}\gamma_4^{\zeta_5} + \frac{3}{2}\beta_1\gamma_6^{\zeta_3\zeta_5} - 3\beta_1\gamma_6^{\phi}, \tag{4.15}$$

$$\gamma_7^{\zeta_{10}} = -\frac{3}{4}\beta_6^{\zeta_9}\gamma_1 + \frac{9}{2}\beta_1\gamma_6^{\zeta_9}, \tag{4.16}$$

$$\gamma_7^{\zeta_4\zeta_3^2} = -\frac{3}{4}\beta_6^{\zeta_3^3}\gamma_1 + \frac{9}{2}\beta_1\gamma_6^{\zeta_3^3}, \tag{4.17}$$

$$\gamma_7^{\zeta_4\zeta_7} = \gamma_7^{\zeta_5\zeta_6} = \gamma_7^{\zeta_3\zeta_8} = 0. \tag{4.18}$$

The results for π-dependent contributions to a β-function are obtained from the above eqs. by a formal replacement of γ by β in every term. For instance, the 7-loop π-dependent contributions read:

$$\beta_7^{\zeta_4} = \frac{3}{8}\beta_4^{\zeta_3}\beta_3^{(1)} + \frac{9}{10}\beta_2\beta_5^{\zeta_3} - \frac{1}{2}\beta_3^{\zeta_3}\beta_4^{(1)} + \frac{5}{4}\beta_1\beta_6^{\zeta_3}, \tag{4.19}$$

$$\beta_7^{\zeta_6} = \frac{5}{8}\beta_4^{\zeta_5}\beta_3^{(1)} + \frac{3}{2}\beta_2\beta_5^{\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\zeta_5} - 2\beta_1^2\beta_3^{\zeta_3}\beta_2 - \frac{5}{4}\beta_1^3\beta_4^{\zeta_3}, \tag{4.20}$$

$$\beta_7^{\zeta_3\zeta_4} = \frac{9}{5}\beta_2\beta_5^{\zeta_3^2} - \frac{1}{8}\beta_3^{\zeta_3}\beta_4^{\zeta_3} + \frac{5}{2}\beta_1\beta_6^{\zeta_3^2}, \tag{4.21}$$

$$\beta_7^{\zeta_8} = \frac{21}{10}\beta_2\beta_5^{\zeta_7} + \frac{35}{12}\beta_1\beta_6^{\zeta_7} - \frac{7}{24}\beta_1(\beta_3^{\zeta_3})^2 + \frac{7}{4}\beta_1^2\beta_5^{\zeta_3^2} - \frac{35}{8}\beta_1^3\beta_4^{\zeta_5}, \tag{4.22}$$

$$\beta_7^{\zeta_3\zeta_6} = \frac{5}{8}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\zeta_3\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\phi}, \tag{4.23}$$

$$\beta_7^{\zeta_4\zeta_5} = -\frac{1}{2}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{5}{4}\beta_1\beta_6^{\zeta_3\zeta_5} - \frac{5}{2}\beta_1\beta_6^{\phi}, \tag{4.24}$$

$$\beta_7^{\zeta_{10}} = \frac{15}{4}\beta_1\beta_6^{\zeta_9}, \tag{4.25}$$

$$\beta_7^{\zeta_4\zeta_3^2} = \frac{15}{4}\beta_1\beta_6^{\zeta_3^3}, \tag{4.26}$$

$$\beta_7^{\zeta_4\zeta_7} = \beta_7^{\zeta_5\zeta_6} = \beta_7^{\zeta_3\zeta_8} = 0. \tag{4.27}$$

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4.1 Tests

With eqs. (4.1)–(4.27) we have been able to reproduce successfully all π-dependent constants appearing in the β-function and anomalous dimensions γ_m and γ_2 of the $O(n)$ φ^4 model which all are known at 7 loops from [17]. In addition, we have checked that the π-dependent contributions to the terms of order $n_f^6 \alpha_s^7$ in the the QCD β-function as well as to the terms of order $n_f^6 \alpha_s^7$ and of order $n_f^5 \alpha_s^7$ contributing to the quark mass AD (all computed in [18, 19, 20]) are in agreement with constraints (4.19)–(4.27) and (4.10)–(4.18) respectively.

Numerous successful tests at 4,5 and 6 loops have been presented in [1].

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