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Analytic Neutrino Oscillation Probabilities in Matter: Revisited

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We summarize our recent paper on neutrino oscillation probabilities in matter, explaining the importance, relevance and need for simple, highly accurate approximations to the neutrino oscillation probabilities in matter. Simple expressions for the neutrino mixing angles and mass squared differences in matter are given in an Appendix. Using these in the vacuum oscillation probabilities instead of the vacuum values, gives an excellent approximation to the oscillation probabilities in matter.

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1. Neutrino Propagation in Matter

The evolution of a neutrino flavor state in matter is given by

$$i\frac{d}{dx}\nu = H\nu \text{ with } \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \text{ and } H = \frac{1}{2E}\left\{ U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

where "a" is the matter potential, $a = 2\sqrt{2}G_F N_e E$.

We have developed a *single* perturbative expansion for the oscillation probabilities in constant matter that satisfies the following criteria, see [1];

1. valid and accurate for all baseline divided by neutrino energy and all values of matter potential, i.e. over the full $(L/E, Y\rho E)$ plane,

2. has the universal form for the L/E dependence of the oscillation probabilities i.e. three sin squared terms and a CP violating triple sin term,

3. since the atmospheric and solar resonance have to be dealt with in a non-perturbative fashion, we need to use $\sqrt{\cdots}$ function but will use nothing more complex[1],

4. the form of the mass eigenvalues squared in matter is particularly simple which leads to simple forms for the mixing angles in matter at zeroth order, providing an enhanced understand of oscillation probabilities in matter.

Our perturbative expansion should be compared to other perturbative expansions in the literature, see [3], which do not satisfy *all* of the above criteria.

To develop a perturbation theory for the neutrino mass squared's in matter as well as the elements of MNS in matter, we first need to take care of the resonance regions non-perturbatively. We start by splitting the Hamiltonian, H, into two pieces, the diagonal part, H_0, and the non-diagonal part H_1 such that $H = H_0 + H_1$:

$$H_0 = \frac{1}{2E}\begin{bmatrix} \lambda_a & & \\ & \lambda_b & \\ & & \lambda_c \end{bmatrix} \quad \text{where} \quad \begin{cases} \lambda_a \equiv a + s_{13}^2 \Delta m_{ee}^2 \\ \lambda_b \equiv (c_{12}^2 - s_{12}^2)\Delta m_{21}^2 \\ \lambda_c \equiv c_{13}^2 \Delta m_{ee}^2 \end{cases}, \tag{1.1}$$

where $\Delta m_{ee}^2 \equiv \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2 = \Delta m_{32}^2 + c_{12}^2 \Delta m_{21}^2$. Note, λ_a, λ_b and λ_c are the asymptotic values[2] of the mass squared eigenstates as $|a| \to \infty$, including terms of O(a^0).

The non-diagonal part, H_1, is given by

$$H_1 = s_{13}c_{13}\frac{\Delta m_{ee}^2}{2E}\begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix} + c_{13}s_{12}c_{12}\frac{\Delta m_{21}^2}{2E}\begin{bmatrix} & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} - s_{13}s_{12}c_{12}\frac{\Delta m_{21}^2}{2E}\begin{bmatrix} & & 0 \\ & 0 & 1 \\ & 1 & \end{bmatrix}. \tag{1.2}$$

Given that

$$s_{13}c_{13} \sim 0.15, \quad c_{13}s_{12}c_{12}(\Delta m_{21}^2/\Delta m_{ee}^2) \sim 0.015 \quad \text{and} \quad s_{13}s_{12}c_{12}(\Delta m_{21}^2/\Delta m_{ee}^2) \sim 0.002,$$

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[1]Compared to the exact results which involve the $\cos[\frac{1}{3}\arccos(\cdots)]$ expressions that appear in [2].

[2]The constant $s_{12}^2 \Delta m_{21}^2$ has been subtracted from all eigenvalue compared to [1].

there is a hierarchy in the size of the three terms in H_1. So it is natural to perform first a rotation in the 1-3 space followed by a rotation in the 1-2 space. After, these rotations, the level crossings that existed in H_0 of eqn 1.1, as one varied the matter potential, no longer exist in the new H_0 given by:

$$H_0 = \frac{1}{2E}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \quad \text{and} \quad H_1 = \sin(\phi - \theta_{13})s_{12}c_{12}\frac{\Delta m_{21}^2}{2E}\begin{bmatrix} 0 & 0 & -s_\psi \\ 0 & 0 & c_\psi \\ -s_\psi & c_\psi & 0 \end{bmatrix} \quad (1.3)$$

where λ_i are the square of the neutrino mass in matter and ϕ, ψ are the mixing angles θ_{13}, θ_{12} in matter. Note, in vacuum, $\sin(\phi - \theta_{13}) = 0$ so that $H_1 = 0$. Expressions for λ_i, ϕ and ψ will be given in next section.

At this point, one could perform a further rotation. For NO, if one performs an additional rotation in the 1-3 space, then the new H_1 will be proportional to $\sin(\phi - \theta_{13})c_\psi$ whose magnitude for all "a" is $< s_{13}$. Similarly for IO, if the rotation is performed in the 2-3 space and the new H_1 wiil be proportional to $\sin(\phi - \theta_{13})s_\psi$. These rotations would significantly improve the 0th order approximation especially in the region where $\sin(\phi - \theta_{13}) \sim c_{13}$.

Instead, to keep one perturbative expansion for both mass orderings, we will do perturbation theory using the results of the first two rotations, i.e. eqn 1.3.

2. A Simple, Accurate Method for Calculate Oscillation Probabilities in Matter

A simple and accurate way to evaluate oscillation probabilities, see [1], is given in this section. Details as to the why's and how's of this method are contained in this paper.

After performing a rotation in the 1-3 space:

$$\lambda_0 = \lambda_b, \quad \lambda_\pm = \frac{1}{2}\left(\lambda_c + \lambda_a \pm \text{sign}(\Delta m_{ee}^2)\sqrt{(\lambda_c - \lambda_a)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2}\right)$$

$$= \frac{1}{2}\left(\Delta m_{ee}^2 + a \pm \text{sign}(\Delta m_{ee}^2)\sqrt{(\Delta m_{ee}^2\cos 2\theta_{13} - a)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2}\right)$$

$$\sin\phi = \sqrt{\frac{\lambda_+ - \lambda_c}{\lambda_+ - \lambda_-}} \quad \text{with } 0 \leq \phi \leq \pi/2,$$

which satisfies

$$\phi(a) = \frac{\pi}{2} - \phi(2\Delta m_{ee}^2\cos\theta_{13} - a) \quad \text{and} \quad \phi_{NO}(a) = \phi_{IO}(-a). \quad (2.1)$$

ϕ is θ_{13} in matter and $(\lambda_a, \lambda_b, \lambda_c) \to (\lambda_-, \lambda_0, \lambda_+)$. When $|\lambda_c - \lambda_a| \gg 2s_{13}c_{13}|\Delta m_{ee}^2|$ then $(\lambda_-, \lambda_+) \approx (\lambda_a, \lambda_c)$ or (λ_c, λ_a). If $a = 0$, $\phi = \theta_{13}$ and $(\lambda_-, \lambda_0, \lambda_+) = (0, (c_{12}^2 - s_{12}^2)\Delta m_{21}^2, \Delta m_{ee}^2)$.

It is simple to show that

$$\sin(\phi - \theta_{13}) = \text{sign}(a\Delta m_{ee}^2)\sqrt{(\lambda_- - ac_\phi^2)/\Delta m_{ee}^2} \approx s_{13}c_{13}(a/\Delta m_{ee}^2) + \mathcal{O}[(a/\Delta m_{ee}^2)^2],$$

which ultimately will determine the size of the perturbing Hamiltonian. Note, that for a neutrino energy of 3 GeV and earth crust density $\sin(\phi - \theta_{13}) \approx 0.04$.

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Then, performing a rotation in the 1-2 space: with $\lambda_3 = \lambda_+$ and

$$\lambda_{2,1} = \frac{1}{2}\left(\lambda_0 + \lambda_- \pm \sqrt{(\lambda_0 - \lambda_-)^2 + (2\cos(\phi - \theta_{13})s_{12}c_{12}\Delta m_{21}^2)^2}\,\right)$$

$$\approx \frac{1}{2}\left(\Delta m_{21}^2 \cos 2\theta_{12} + a\, c_{13}^2 \pm \sqrt{(\Delta m_{21}^2 \cos 2\theta_{12} - a\, c_{13}^2)^2 + (2s_{12}c_{12}\Delta m_{21}^2)^2}\,\right)$$

$$\text{for } |a/\Delta m_{ee}^2| \ll 1$$

$$\sin \psi = \sqrt{\frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1}}, \quad \text{with } 0 \leq \psi \leq \pi/2.$$

which satisfies

$$\psi(a) \approx \frac{\pi}{2} - \psi(2\Delta m_{21}^2 \cos\theta_{12}/c_{13}^2 - a) \quad \text{and} \quad \phi_{NO}(a) \approx \phi_{IO}(a). \tag{2.2}$$

ψ is θ_{12} in matter and $(\lambda_-, \lambda_0, \lambda_+) \rightarrow (\lambda_1, \lambda_2, \lambda_3)$. When $|\lambda_0 - \lambda_-| \gg 2s_{12}c_{12}\Delta m_{21}^2$ then $(\lambda_1, \lambda_2) \approx (\lambda_-, \lambda_0)$ or (λ_0, λ_-). If $a = 0$, $\psi = \theta_{12}$ and $(\lambda_1, \lambda_2, \lambda_3) = (-s_{12}^2\Delta m_{21}^2, c_{21}^2\Delta m_{21}^2, \Delta m_{ee}^2)$, thus, in vacuum, $\Delta\lambda_{jk} \equiv \lambda_j - \lambda_k = \Delta m_{jk}^2$. See Appendix for further details.

To calculate the oscillation probabilities, to 0th order, use the above $\Delta\lambda_{jk}$ instead of Δm_{jk}^2 and replace the vacuum MNS matrix as follows[3]

$$U_{MNS}^0 \equiv U_{23}(\theta_{23})U_{13}(\theta_{13}, \delta)U_{12}(\theta_{12}) \Rightarrow U_{MNS}^M \equiv U_{23}(\theta_{23})U_{13}(\phi, \delta)U_{12}(\psi). \tag{2.3}$$

That is, replace

$$\Delta m_{jk}^2 \rightarrow \Delta\lambda_{jk} \quad \theta_{13} \rightarrow \phi, \quad \theta_{12} \rightarrow \psi, \tag{2.4}$$

it is that simple. θ_{23} and δ remain unchanged. Our expansion parameter is $\left|\sin(\phi - \theta_{13})\, s_{12}c_{12}\, \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right| \leq$ 0.015, which is small and vanishes in vacuum, so that our perturbation theory reproduces the vacuum oscillation probabilities exactly. A summary of the relevant expressions are given in Fig. 1 and alternative summary using a more conventional notation is given in Appendix II.

2.1 Higher Orders

If the 0th order is not accurate enough, going to 1st order is simple and gives another two orders of magnitude in accuracy. First the eigenvalues λ_j remain unchanged but the mixing matrix is modified by

$$U_{MNS}^M \Rightarrow V \equiv U_{MNS}^M(1 + W_1), \tag{2.5}$$

where the matrix W_1 is given by[4]

$$W_1 = \sin(\phi - \theta_{13})\, s_{12}c_{12}\,\Delta m_{21}^2 \begin{pmatrix} 0 & 0 & -s_\psi e^{-i\delta}/\Delta\lambda_{31} \\ 0 & 0 & +c_\psi e^{-i\delta}/\Delta\lambda_{32} \\ +s_\psi e^{+i\delta}/\Delta\lambda_{31} & -c_\psi e^{+i\delta}/\Delta\lambda_{32} & 0 \end{pmatrix}. \tag{2.6}$$

The $\Delta\lambda_{jk}$ and the V-mixing matrix can be used to calculate the oscillation probabilities and improve the accuracy so that $\Delta P < 10^{-6}$. The next highest order is also discussed in [1].

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[3]For the rest of this paper we use the standard parameterization of the MNS matrix for the reader's convenience, as oppose to the parametrization used in [1].

[4]The phase in W_1 differ from [1] because here we use the standard parameterization of U_{MNS}.

⌗ # DMP Summary: 𝜈

$$\lambda_a \equiv a + s_{13}^2 \Delta m_{ee}^2$$
$$\lambda_b \equiv (c_{12}^2 - s_{12}^2)\Delta m_{21}^2$$
$$\lambda_c \equiv c_{13}^2 \Delta m_{ee}^2$$

where
$$\Delta m_{ee}^2 \equiv \Delta m_{31}^2 - s_{12}^2 \Delta m_{21}^2$$
$$a = 2\sqrt{2}G_F N_e E_\nu$$

$\theta_{13} \Rightarrow \phi$ in matter:
$$\lambda_{\mp} = \frac{1}{2}\left[(\lambda_a + \lambda_c) \mp \text{sign}(\Delta m_{ee}^2)\sqrt{(\lambda_a - \lambda_c)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2}\right]$$
$$\lambda_0 = \lambda_b$$

$$s_\phi = \sqrt{\frac{\lambda_+ - \lambda_c}{\lambda_+ - \lambda_-}}$$

$\theta_{12} \Rightarrow \psi$ in matter:
$$\lambda_{1,2} = \frac{1}{2}\left[(\lambda_0 + \lambda_-) \mp \sqrt{(\lambda_0 - \lambda_-)^2 + (2c_{\phi-\theta_{13}}s_{12}c_{12}\Delta m_{21}^2)^2}\right]$$
$$\lambda_3 = \lambda_+$$

$$s_\psi = \sqrt{\frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1}}$$

where $\cos^2(\phi - \theta_{13}) = (\lambda_+ - as_\phi^2)/\Delta m_{ee}^2$

θ_{23}, δ are unchanged. $(\Delta m_{31}^2, \ \Delta m_{21}^2) \Rightarrow (\Delta\lambda_{31}, \ \Delta\lambda_{21})$ in matter:

Corrections are $\mathcal{O}\left(s_{\phi-\theta_{13}}s_{12}c_{12}\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) < 10^{-3}$ at DUNE

Figure 1: Summary of DMP perturbation theory, 0th order, see [1], for the mixing angles and mass squared eigenvalues in matter. Replacing $(\theta_{12}, \ \theta_{13}, \ \theta_{23}, \ \delta, \ \Delta m_{31}^2, \ \Delta m_{21}^2)$ with $(\psi, \ \phi, \ \theta_{23}, \ \delta, \ \Delta\lambda_{31}, \ \Delta\lambda_{21})$ in the vacuum oscillation expressions gives oscillation probability in matter to 0th order. Only 6 square root operations are required to go from the vacuum to matter parameters, not counting the simple $(+, -, *, /)$ operations. No other time consuming operations like sine, cosine, arcsine, arccosine etc are needed.
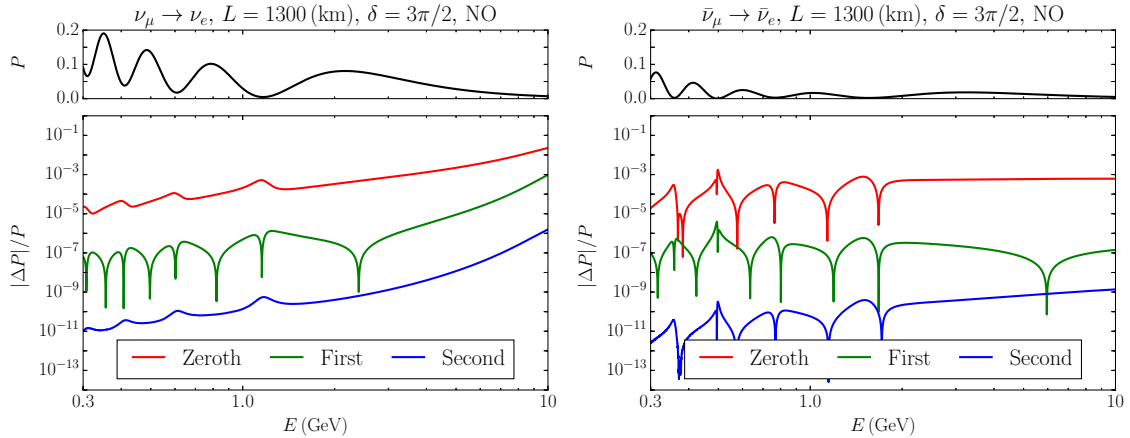

Figure 2: Dune $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ appearance probabilities, top panel. The bottom panel, shows the fractional difference between the 0th, 1st and 2nd order approximations to the exact probabilities assuming a constant matter density.

In Fig. 2 we have compared the exact oscillation probability with our approximation. One sees that the 0th order oscillation probabilities, relevant for the DUNE experiment, have a difference, from the exact calculation, $|\Delta P| < 10^{-4}$ and $|\Delta P|/P < 10^{-3}$. Higher orders are even more accurate.

3. Conclusions

We have summarized our perturbation theory in matter for the neutrino oscillation probabilities, that gives the neutrino mass eigenvalues squared in matter in simple terms only involving the $\sqrt{\cdots}$ function. We also show how the mixing angles in matter can be obtained directly once one knows the matter mass squareds. Higher orders are simple to obtain and increase the accuracy by about two orders of magnitude per order. However, the zeroth order approximations are good enough for all current and future accelerator experiments; T2K, NOνA, DUNE and T2HK/T2HKK due to the uncertainties associated with the matter density profile, height and shape, between neutrino production and detection.

4. Appendix

For NO we give approximate expressions for the λ's, in the different regions of interest;

$$\lambda_3 \approx \begin{cases} a + s_{13}^2 \Delta m_{ee}^2, & a \gg \Delta m_{ee}^2 \\ \frac{1}{2} \left(\Delta m_{ee}^2 + a \right. \\ \left. \quad + \sqrt{(\Delta m_{ee}^2 \cos 2\theta_{13} - a)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2} \right), & a \approx \Delta m_{ee}^2 \\ \Delta m_{ee}^2 + s_{13}^2 a, & |a| \ll \Delta m_{ee}^2 \\ c_{13}^2 \Delta m_{ee}^2, & -a \gg \Delta m_{ee}^2. \end{cases}$$

One can obtain similar expressions for λ_- using $\lambda_- = \Delta m_{ee}^2 + a - \lambda_3$. Also,

$$\lambda_1 \approx \begin{cases} \cos 2\theta_{12} \Delta m_{21}^2, & a \gg \Delta m_{21}^2 \\ \frac{1}{2} \left(\Delta m_{21}^2 \cos 2\theta_{12} + a\, c_{13}^2 \right. \\ \left. \quad - \sqrt{(\Delta m_{21}^2 \cos 2\theta_{12} - a\, c_{13}^2)^2 + (2s_{12}c_{12}\Delta m_{21}^2)^2} \right), & a \approx \Delta m_{21}^2 \\ -s_{12}^2 \Delta m_{21}^2 + c_{12}^2 c_{13}^2 a, & |a| \ll \Delta m_{21}^2 \\ a + s_{13}^2 \Delta m_{ee}^2, & -a \gg \Delta m_{ee}^2. \end{cases}$$

To obtain λ_2 in all regions of interest, we use $\lambda_2 = \Delta m_{ee}^2 + \cos 2\theta_{12} \Delta m_{21}^2 + a - \lambda_3 - \lambda_1$;

$$\lambda_2 \approx \begin{cases} c_{13}^2 \Delta m_{ee}^2, & a \gg \Delta m_{ee}^2 \\ \frac{1}{2} \left(\Delta m_{ee}^2 + a \right. \\ \left. \quad - \sqrt{(\Delta m_{ee}^2 \cos 2\theta_{13} - a)^2 + (2s_{13}c_{13}\Delta m_{ee}^2)^2} \right), & a \approx \Delta m_{ee}^2 \\ \frac{1}{2} \left(\Delta m_{21}^2 \cos 2\theta_{12} + a\, c_{13}^2 \right. \\ \left. \quad + \sqrt{(\Delta m_{21}^2 \cos 2\theta_{12} - a\, c_{13}^2)^2 + (2s_{12}c_{12}\Delta m_{21}^2)^2} \right), & a \approx \Delta m_{21}^2 \\ c_{12}^2 \Delta m_{21}^2 + s_{12}^2 c_{13}^2 a, & |a| \ll \Delta m_{21}^2 \\ (c_{12}^2 - s_{12}^2) \Delta m_{21}^2, & -a \gg \Delta m_{ee}^2. \end{cases}$$

Since ψ and ϕ are given in terms of the eigenvalues earlier, approximations for all variables can be easily derived from these λ's. For IO, one can give similar expressions for λ's.

5. Appendix II

We have written everything in terms of the matter eigenvalues, the λ's, since if you know the λ's you can easily calculate the matter mixing angles, $\phi \; (= \widetilde{\theta}_{13})$ and $\psi \; (= \widetilde{\theta}_{12})$, this is summarized in Fig. 1.

However, the mixing angles in matter, which we denote by a $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ here, can also be calculated in the following way, using $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, as follows[5]:

$$\cos 2\widetilde{\theta}_{13} = \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}, \tag{5.1}$$

where $\quad a \equiv 2\sqrt{2}G_F N_e E \quad$ is the standard matter potential, and

$$\cos 2\widetilde{\theta}_{12} = \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}}, \tag{5.2}$$

where $a' \equiv a\cos^2\widetilde{\theta}_{13} + \Delta m^2_{ee}\sin^2(\widetilde{\theta}_{13} - \theta_{13})$ is the 13-modified matter potential.

In these two flavor rotations, both $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ are in range $[0, \pi/2]$.

θ_{23} and δ are unchanged in matter for this approximation.

The neutrino mass squared differences in matter, i.e. the Δm^2_{jk} in matter, which we denote by $\Delta \widetilde{m^2}_{jk}$, are given by

$$\begin{aligned}
\Delta \widetilde{m^2}_{21} &= \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}\,, \\
\Delta \widetilde{m^2}_{31} &= \Delta m^2_{31} + \frac{1}{2}\left(2a - 3a' + \Delta \widetilde{m^2}_{21} - \Delta m^2_{21}\right), \\
\Delta \widetilde{m^2}_{32} &= \Delta \widetilde{m^2}_{31} - \Delta \widetilde{m^2}_{21} = \Delta m^2_{32} + \frac{1}{2}\left(2a - 3a' - \Delta \widetilde{m^2}_{21} + \Delta m^2_{21}\right)
\end{aligned} \tag{5.3}$$

Note, the same square root appears, as for the θ_{12} mixing angle in matter[6]. These expressions are valid for both NO, $\Delta m^2_{31} > 0$ and IO, $\Delta m^2_{31} < 0$. For anti-neutrinos, just change the sign of a and δ.

If $P_{\nu_\alpha \to \nu_\beta}(\Delta m^2_{31}, \Delta m^2_{21}, \theta_{13}, \theta_{12}, \theta_{23}, \delta)$ is the oscillation probability in vacuum then $P_{\nu_\alpha \to \nu_\beta}(\Delta \widetilde{m^2}_{31}, \Delta \widetilde{m^2}_{21}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$ is the oscillation probability in matter, i.e. use the same function but replace the mass squared differences and mixing angles with the matter values given in eq. 5.1-5.3. The resulting oscillation probabilities are identical to the zeroth order approximation given in Denton, Minakata and Parke, [1].

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[5]Vacuum values to be used in calculating Δm^2_{ee}.

[6]If $a = 0$, then $\widetilde{\theta}_{13} = \theta_{13}$ and since $a' = 0$ then $\widetilde{\theta}_{12} = \theta_{12}$ and both $\sqrt{\cdots} = 1$, also $\Delta \widetilde{m^2}_{jk} = \Delta m^2_{jk}$ for all (j, k) as required.

6. Acknowledgements

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