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Weak Gravity Conjecture and Black Holes in $\mathcal{N} = 2$ Supergravity

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In a previous paper [1], we used the entropy function formalism to argue that loops of charged particles can give unexpected contributions to the entropy of extremal black holes. Here, we show that similar results hold for loops of BPS particles in $\mathcal{N} = 2$ supergravity, potentially connecting these results to familiar string setups. However, we will argue that string theory avoids these corrections altogether with the help of the Weak Gravity Conjecture. Finally, we discuss the relationship between these results and the semi-classical analysis of Antonov [2, 3], who showed that sub-extremal particles suffer a 'gravo-thermal catastrophe' in the Newtonian limit.

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1. Introduction

String theory is widely believed to provide a consistent theory of quantum gravity. Nonetheless, it has proven difficult to develop a convenient off-shell formulation, thus leaving many important problems intractable with current technology. For instance, despite the phenomenal progress in understanding various dualities amongst supersymmetric theories, we still have no definitive understanding of the dynamics governing the early universe, nor how vacuum selection within the string landscape is achieved. Even finding controlled, semi-realistic models of supersymmetry breaking in a fully string theoretical setting has proven rather difficult.

In practice, when confronting these problems, one almost always works within the framework of effective field theory, assuming that only a few important degrees of freedom are relevant. Unfortunately, in some cases, and especially in cosmology, such assumptions are questionable. Furthermore, current technology does not allow us to derive the desired effective theory from first principles. Instead, model builders must argue that in the vast landscape of string theory there must be at least some corner where the properties that one is after may be found. However, proceeding too far down this route is tantamount to forgetting about the constraints coming from string theory altogether. There is simply no guarantee that an arbitrary low-energy model *must* have a consistent ultra-violet completion within string theory.

Given this situation, it is highly desirable to have tools that allow us to distinguish between theories which possibly permit an embedding in string theory and those which definitely do not. The set of theories which do not admit a UV embedding has been dubbed the 'swampland' [4] and the attempt to map out its bounds has become an active area of research in recent years [5, 6, 7, 8, 9, 10]. Equally active, has been the endeavor to draw model independent lessons from the criteria of UV completability, especially in applications to cosmology [11, 12, 13, 14, 15, 16, 17, 18, 19].

Several loosely related criteria for UV completability have been presented in the literature [5, 6]. To summarize, the criteria that are relevant for us, we have:

- No global symmetries are allowed.
- The spectrum of electric and magnetic charges forms a complete set, consistent with Dirac quantization.
- Gravity is the weakest force.
- The low energy effective field theory containing a $U(1)$ gauge theory of charge g should have a cutoff at a scale $\Lambda \sim gM_P$.

There are two main sources of motivation for these conjectures. One is purely empirical; all known solutions in string theory seem to satisfy the conjectures above. For example, the lack of dimensionless parameters is a well-known property of string theory, while the lack of global symmetries is strongly suggested by simple world-sheet arguments [33].

While the evidence offered by stringy examples is certainly compelling, such arguments do not leave one with any sense of *why* these conjectures ought to be true. Moreover, while the swampland conjectures may be strongly motivated in string theory, this still falls short of motivating a UV completability criteria in general. It is thus useful to seek a rational for completability via another means.

A very natural place to seek such a rational is in terms of entropy bounds, such as the Covariant Entropy Bound [20]. To see this, let us consider Einstein-Maxwell theory with a single particle of mass $m \neq 0$ and charge q. Notice that if $q = 0$ then the particle number becomes a global charge in violation of the second conjecture above. Now, consider a collection of N such particles at fixed binding energy $E < 0$ and in the Newtonian limit. A classical result of Antonov[1] [2, 3] shows that semi-classically, the microcanonical entropy of this system is unbounded for sufficiently large $|E|$. This phenomenon goes by the name of 'gravo-thermal catastrophe'. Here, we have seen that the presence of a global charge and the catastrophe are related - at least in the Newtonian limit[2].

We can make a stronger statement if we allow the charge to be non-zero. Again going to the semi-classical Newtonian limit, one finds that there are divergences in the microcanonical entropy whenever $\Delta m^2 \equiv m^2 - 2q^2 M_P^2 > 0$. However, the catastrophe is avoided if we adopt the hypothesis that 'gravity is the weakest force', whose precise statement in this context is just $\Delta m^2 < 0$. Thus, the Weak Gravity Conjecture (WGC) seems to effectively protect against an entropic runaway.

A divergence in the entropy, if real, would undermine the consistency of the theory. For instance, one would have to worry about spontaneous creation of high-entropy bound states in almost any process. However, the semi-classical, Newtonian analysis of [2, 3] leaves much to be desired. Ideally, we would like to re-do the entropy calculation including general relativity and quantum theory. Unfortunately, the conjecture we are investigating says precisely that there is no consistent framework with which to do this calculation when $\Delta m^2 > 0$. Even if a consistent framework were available, it is evident that an actual calculation of the microcanonical entropy would be hindered by the presence of horizons.

Given such an impasse, we are forced to temper our ambitions somewhat and seek other probes of the breakdown of EFT. It is therefore interesting to consider Reissner-Nordstrom black holes representing bound states of charged particles and to calculate the *horizon entanglement entropy*. A priori, there is no reason to expect that the entanglement entropy of these black holes should equal the microcanonical entropy that we are really after. Although the equivalence may be shown under some mild assumptions regarding the asymptotics [26], the maximal extension of a Reissner-Nordstrom black hole is not sufficiently well-behaved for these results to apply. We should therefore view entanglement entropy as an independent probe of the health of an effective field theory, though not as strong as the one we would like.

In order to compute the entanglement entropy we utilize the 'entropy function formalism' of Sen and collaborators [30]. The basic idea is as follows. The Wald entropy [31] provides a systematic means for computing the entropy of a horizon given an arbitrary local lagrangian. The formalism of Sen et al then instructs us to apply the Wald entropy formula to the quantum corrected 1PI effective action. This action is no longer local and so it is not immediately obvious how Wald's formula should be applied. However, assuming the near horizon geometry approaches $AdS_2 \times X$ and that the background fields approach constants, one can rewrite Wald's formula in terms of a

[1]The proof requires that the number of particles be conserved, and therefore does not apply to neutrons or neutrinos which are not expected to carry a protected quantum number.

[2]We see no reason why deforming from the Newtonian theory to general relativity should cause a large decrease in the microcanonical entropy, but we have not rigorously excluded this possibility.

legendre transform of the near-horizon lagrangian density. This formula may then be extended to non-local lagrangians and thus applied to the 1PI effective action.

The goal of the current note is to describe how the presence of elementary[3] charged particles affects the entanglement entropy of extremal black holes in $\mathcal{N}=2$ supergravity. Our basic technical result is that, for black holes whose curvature (inverse radius) lies between the WGC cutoff and the Planck scale, integrating out charged matter multiplets leads to unexpected corrections. For larger black holes we note that there is a pole in the fermionic spectral density which inevitably leads to even more dramatic modifications, hard to reconcile with the classical geometry. Our previous paper [1] established similar results for $\mathcal{N}=0,1$ quantum fields in an extremal Reissner-Nordstrom background with minimal couplings. However, one might worry that the problematic one-loop corrections arise due to the low degree of supersymmetry or due to our failure to incorporate the non-minimal couplings present in supergravity (SUGRA). To address this concern, here we demonstrate that even in the context of $4d$, $\mathcal{N}=2$ supergravity, BPS hypermultiplets can induce large corrections on the background geometry unless the WGC cutoff is applied.

This raises a number of questions as to the proper interpretation of these results. Intuitively, one does not expect that loops of very massive particles should drastically alter the background geometry of macroscopic black holes. However, extremal black holes are somewhat special [32] and perhaps should be thought of as elementary quantum particles in their own right. Moreover, as we will see, charged extremal particles can be effectively massless in the background of an extremal black hole, opening the door for large corrections. On the other hand, there are many results regarding the entropy of supersymmetric ($\mathcal{N}\geq 2$) black holes in string theory which must constrain our interpretation. As we discuss further in the Section 4, the resolution of the conifold singularity [25] hints that loops of extremal particles should not be considered to begin with! This dovetails nicely with the so-called 'magnetic Weak Gravity Conjecture', which is just the statement that the UV cutoff should be at the scale of extremal particles[4]. In other words, the evidence suggest that in any UV complete theory of quantum gravity, extremal particles cannot be elementary, rather, they must be a solitons.

The plan of this note is as follows. In Section 2 we review the entropy function formalism and use this to compute the entanglement entropy for a theory with minimally coupled particles. We also show how these results may be extended to a theory with multiple $U(1)$'s and dyonic charges. Next, in Section 3 we extend this to supergravity theories with non-minimal couplings using the results of [22]. We also describe briefly how to compute the corrections in the general $\mathcal{N}=2$ supergravity background starting from the prepotential. Finally, we offer some concluding remarks in Section 4.

2. Loop Calculations

2.1 Classical entropy function

Let us begin by considering an Einstein-Maxwell theory containing a single charged particle, which may be either a boson or fermion. We will think of this as an effective theory regulated by

[3] By 'elementary' we essentially mean the the particle is below the cutoff and may run in loops.

[4] We are defining the extremal bound in terms of the limit M/Q, as $Q\to\infty$ in the limit of large charge.

a short distance cutoff ε. For the time being we will not ask where such a theory could have come from. For the time being, we will not make any assumptions about the relationship between ε and M_P.

In our conventions, the action is:

$$\mathscr{S} = \int d^4x\sqrt{-g}\left(\mathscr{L}^{(0)}_{GR+EM} + \mathscr{L}_{matter}\right) \tag{2.1}$$

$$\mathscr{L}^{(0)}_{GR+EM} = \frac{M_P^2}{2}\mathscr{R} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathscr{L}_{matter} = \begin{cases}\mathscr{L}_{scalar} = -(D_\mu\phi)^\star D^\mu\phi - m^2|\phi|^2, \\ \mathscr{L}_{fermion} = \overline{\psi}\left(\not{D} - m\right)\psi,\end{cases}$$

Classically, this theory admits Reissner-Nordstrom black hole solutions and our task is now to determine how these are modified in the quantum theory. To do this, we will compute the 1PI action obtained by integrating out the the particles using the heat-kernel formalism [23]. We are particularly interested in how black holes in this 1-loop corrected theory depend upon the parameters q and m.

It is a useful observation that regardless of the detailed form of the corrections, the near-horizon geometry is described by $AdS_2 \times S^2$, which we write as:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = a^2\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + b^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.2}$$
$$F = Edt\wedge dr$$

Here a and b parameterize the radii of AdS_2 and S^2, respectively, and E represents the electric field sourced by the black hole. In terms of this parameterization, the Wald entropy is now given by [30] minimizing the entropy functional, $\mathscr{E}$, defined by:

$$\mathscr{E}(Q;E,a,b) = 2\pi\left[QE - 4\pi a^2b^2\mathscr{L}_{GR+EM}(E,a,b)\right], \tag{2.3}$$

with $\mathscr{L}_{GR+EM}$ evaluated on the near-horizon geometry (2.2). More precisely, the entropy of a black hole of electric charge Q is given by $S(Q) = \min\limits_{a,b,E}\mathscr{E}$.

At tree level the entropy is straightforward to compute. First, the lagrangian density is computed on the background (2.2). One finds:

$$\mathscr{L}^{(0)}_{GR+EM} = M_P^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{E^2}{2a^4} \tag{2.4}$$

Plugging this into the expression for $\mathscr{E}$ and minimizing we immediately get the equations:

$$E_0 = \frac{Q}{4\pi} \tag{2.5}$$
$$a_0^2 = b_0^2 = \frac{Q^2}{32\pi^2M_P^2}$$

For the sake of reference, the ADM mass of the full solution is also known to be $M=\sqrt{2}QM_P$. Finally, plugging this solution back in to (2.3) we find the expected Beckenstein-Hawking formula:

$$S^{(0)} = \frac{Q^2}{4} = \frac{A}{4G_N} \tag{2.6}$$

2.2 One-Loop Correction

We now wish to repeat this procedure with the 1-loop corrections induced by the matter fields included. In general, these corrections may be calculated using the heat kernel formalism, as explained in [27]. To summarize, one first computes the heat kernel, $K(x,y;s)$, defined by:

$$(\partial_s - D)K(x,y;s) = 0 \qquad K(x,y;0) = \delta^4(x-y) \tag{2.7}$$

where D is a generalized laplacian containing the kinetic and mass terms of the field being integrated out. Once this is known, the one loop correction is:

$$\mathcal{L}^{(1)} = \frac{1}{2}\int_{\varepsilon^2}^{\infty} \frac{ds}{s} K(s) \tag{2.8}$$

and the net effective action is:

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}_{ct} \tag{2.9}$$

where $\mathcal{L}_{ct}$ is the counterterm lagrangian. The notation $K(s)$ is shorthand for $K(x,x;s)$ which is independent of x in the near-horizon geometry. In the present case, the geometry is the product space $AdS_2 \times S^2$ and so the heat kernel factorizes into $K = K_{AdS_2} \times K_{S^2}$. This fact allows one to compute the heat kernel using the techniques of [28, 29, 27]. The total K is just the sum of the result for the different matter fields considered. The results are as follows:

Charged Scalar

$$\begin{aligned} K_s(s) &= \frac{e^{-s\Delta m^2}}{4\pi^2 a^2 b^2} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} d\lambda\, \lambda\, \rho_s(\lambda) e^{-s\left[\left(\lambda^2+\frac{1}{4}\right)/a^2 + l(l+1)/b^2\right]} \\ \rho_s(\lambda) &= \frac{\sinh(2\pi\lambda)}{\cosh(2\pi\lambda)+\cosh(2\pi qE)} \end{aligned} \tag{2.10}$$

Chiral Fermion

$$\begin{aligned} K_f(s) &= \frac{e^{-s\Delta m^2}}{4\pi^2 a^2 b^2} \sum_{l=0}^{\infty} (2l+2) \int_0^{\infty} d\lambda\, \lambda\, \rho_f(\lambda)\, e^{-s\left[\lambda^2/a^2 + (l+1)^2/b^2\right]} \\ \rho_f(\lambda) &= \frac{\sinh(2\pi\lambda)}{\cosh(2\pi qE)-\cosh(2\pi\lambda)} \end{aligned} \tag{2.11}$$

where we have defined:

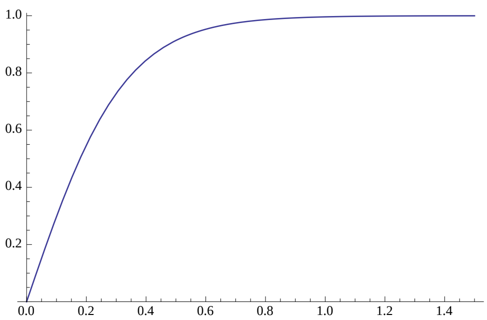
$$\begin{aligned} \Delta m^2 &= m^2 - \frac{q^2E^2}{a^2} \\ &\to m^2 - 2q^2M_P^2 \end{aligned} \tag{2.12}$$

and the arrow above indicates the classical value.

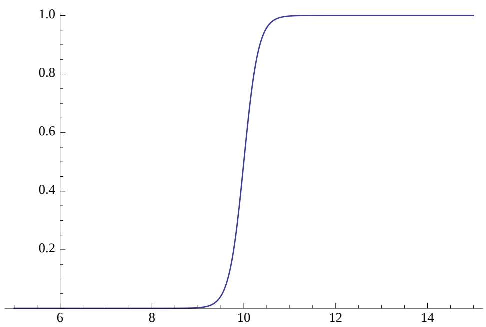
There are two crucial features of the density of states presented here. The first is that the fermionic spectral density has a pole at $\lambda = qE$, corresponding to a physical energy of $\lambda/a =$ Energy $\leq \sqrt{2}qM_P$. Note that this is precisely the magnetic WGC cutoff. In order to avoid a physically divergent one loop correction one must impose a UV cutoff on the λ integral, in addition to the cutoff on the s integral[5]. By working perturbatively in qE one may temporarily ignore this cutoff, though an expansion in qE should at best be interpreted as an asymptotic series.

The second important feature is that for large qE, the spectral density of modes with $\lambda \leq qE$ are suppressed by $e^{-2\pi qE}$ relative to flat space. Thus, if the cutoff is kept below the WGC scale then the entire correction will be highly suppressed. However, the counterterms, which are designed to cancel flat space divergences, are unsuppressed. Thus, the counterterms will no longer cancel the leading terms in the one loop corrections. As a result, the corrections will be large, though not divergent since the cutoff is intrinsically constrained by the scale of extremal particles[6]. This effect may be dubbed 'un-renormalization' since the background undoes the one-loop correction, though the counterterms are still present.

Now, in order to evaluate (2.8), much care is needed in performing the appropriate expansion of the heat kernels. In general we want to require $E \gg 1$ so that the black hole is very large relative to the Planck scale and quantum gravity effects can be neglected. Under this assumption, two different regimes can be distinguished: *large* black holes $qE \gg 1$ and *intermediate* black holes $qE \ll 1$, where E is classicaly related to the size via $E = \sqrt{2}aM_p$. The physical interpretation of the two regimes are as follows: *intermediate* black holes are those whose curvature scale (the inverse radius) lies above the WGC cutoff $\Lambda_{WGC} = qM_p$, that is: $qM_p \ll 1/a \ll M_p$. Likewise, *large* black holes are those whose curvature is smaller even than the WGC scale. Note furthermore that for intermediate black holes, extremal particles are effectively massless relative to the curvature scale, even though the black holes may be very large.



(a) $qE = .1$



(b) $qE = 10$

Figure 1: Spectral densities for charged scalars in intermediate (a) and large (b) black holes.

For intermediate black holes, the particles mass is negligible relative to the curvatures scale and so one may expand the spectral density in a manner similar to what was used for massless neutral particles [27, 1], treating the curvature as an IR cutoff. This expansion, described further

[5]One may also consider the possibility of interpreting the pole as an IR cutoff, though, in this case, we would find unacceptably large corrections once the standard counterterms are added to the action.

[6]I.e., we are assuming that a suitable regularization procedure imposes $\lambda/a \leq \sqrt{2}qM_P$ and $\varepsilon^{-1} \leq \sqrt{2}qM_P$.

in [1], leads to a series in terms of qE which is formally divergent but nevertheless useful as an asymptotic series. On the other hand, this expansion certainly fails to be useful for large black holes. For scalars, one may nevertheless proceed by approximating the spectral density as a step function plus corrections (see Figure 1). Note that this is similar to what we do for *fermionic* systems with chemical potential $\mu = 2qE/a$ and $T = 1/2\pi a$. In contrast, for the fermionic spectrum we encounter something similar to a *bosonic* spectral density. However, an important distinction is that the pole in the usual bosonic spectral density is at negative energy and hence unphysical unless the chemical potential is zero, in which case the pole corresponds to Bose-Einstein condensation (BEC). Here, given the *positive* chemical potential, the pole occurs at the UV energy scale $\sqrt{2}qM_P$. We thus find something akin to BEC unless the WGC cutoff is imposed.

2.3 Intermediate Black Holes

We now turn to intermediate black holes, which admit an expansion in large Q (or equivalently large a) and in small qQ. In this limit, the entropy to leading order in $1/Q$ and qQ for bosons and fermions are [1]:

$$S_s \approx \mathscr{E}(Q;E_0,a_0,b_0) = \frac{Q^2}{4} + \left(\frac{m^4Q^4}{(8\pi)^4M_P^4} + \frac{q^2Q^2}{192\pi^2}\right)\ln\left(\frac{\Delta m^2}{m^2}\right) + \text{corrections} \tag{2.13}$$

$$S_f \approx \mathscr{E}(Q;E_0,a_0,b_0) = \frac{Q^2}{4} - \left(\frac{m^4Q^4}{(8\pi)^4M_P^4} - \frac{q^2Q^2}{96\pi^2}\right)\ln\left(\frac{\Delta m^2}{m^2}\right) + \text{corrections} \tag{2.14}$$

The term of order Q^4 may be thought of as a residual effect of cosmological constant renormalization. One can check that it begins to compete with the classical term only when $m \geq M_P/qQ \gg M_P$. This correction may be tamed by considering supersymmetric multiplets. For example, if we consider an $\mathcal{N}=1$ multiplet, then the Q^4 term cancels between the boson and fermion. However, the correction of order $q^2Q^2\ln(\Delta m/m)$ still arises in the intermediate black hole regime. In the extremal limit, $\Delta m \to 0$ and the correction is enhanced. In this case, the form of the result is modified via $\ln(\Delta m/m) \to -\ln(ma)$ as explained in [1]. The net modification to the entropy in the $\mathcal{N}=1$ case is therefore:

$$S_{\mathcal{N}=1} = \frac{Q^2}{4} - \left(\frac{q^2Q^2}{64\pi^2} + \frac{1}{24}\right)\ln(q^2Q^2) \tag{2.15}$$

In the regime of intermediate black holes the logarithmic correction of order $(qQ)^2\ln(qQ)$ is always small relative to the classical term.

2.4 Large Black Holes

Let us now ask what happens in the regime of large black holes. We can at least make progress in the scalar sector by treating the spectral density as a step-function with corrections. The derivative of this step function is approximately gaussian, so we can reduce the λ integral to an approximately gaussian form after integration by parts. Concretely:

$$\begin{aligned}\int_0^\infty d\lambda\,\lambda\rho_s(\lambda)e^{-s\lambda^2/a^2} &= \frac{a^2}{2s}\int_0^\infty d\lambda\,\lambda\rho_s'(\lambda)e^{-s\lambda^2/a^2} \\ &= \int_0^\infty e^{\ln\rho_s'(qE)+(\lambda-qE)(\ln\rho_s')'+\frac{(\lambda-qE)^2}{2}(\ln\rho_s')''+\ldots-s\lambda^2/a^2}\end{aligned} \tag{2.16}$$

One can check that if we expand around $\lambda = qE$ then all the odd terms in the expansion of $\ln \rho_s'$ are suppressed by factors of $e^{-2\pi qE}$ and may therefore be neglected. Working to quadratic order one gets:

$$\int_0^\infty d\lambda\, \lambda \rho_s(\lambda) e^{-s\lambda^2/a^2} = \frac{\sqrt{\pi}a^2}{4s\sqrt{1+\frac{s}{\pi^2 a^2}}} e^{-\frac{q^2E^2\pi^2 s}{s+\pi^2 a^2}} \tag{2.17}$$

If we now plug this back in to the full expression for the heat kernel (2.10) we find that the exponential suppression goes like:

$$K_s \sim e^{-s\Delta m^2 - \frac{q^2E^2\pi^2 s}{s+\pi^2 a^2}} \sim e^{-sm^2 + \mathcal{O}(s^2)} \tag{2.18}$$

Miraculously, the s integral is suppressed precisely at $s \sim m^{-2}$ even though the effective mass Δm^2 may be zero. Apparently, the density of states behaves just so as to compensate for the electric repulsion and this effect kicks in right at the transition between intermediate and large black holes. This would seem to imply that for scalars there are no large corrections coming form loops and that renormalization is working just as in flat space.

However, in the fermionic sector there is no way to proceed without imposing the UV cutoff Energy $\leq \sqrt{2}qM_P$ to avoid the 'Bose-Einstein-like' pole in the density of states. We may think of this as effectively imposing a cutoff on the Schwinger parameter, i.e., $\varepsilon^{-1} \leq \sqrt{2}qM_P$ due to the factor $e^{-s\lambda^2/a^2}$. This cutoff must apply in flat space as well in order to have a consistent description. However, once we impose this cutoff, then we conclude that the one-loop corrections are uniformly suppressed by $e^{-2\pi qE}$ relative to their flat space counterparts due to the spectral density (2.10), (2.11). However, the counterterms, which are designed to cancel the dominant (in ε^{-1}) corrections, are unsuppressed. Thus, the loop corrections are effectively negligible relative to the counterterms, leaving the counterterms to be un-cancelled.

This conclusion is somewhat perplexing since we would not expect sizable corrections coming from massive particles. In particular, we never see such corrections coming form microstate counting. On the other hand, there are no microscopic results for $\mathcal{N}=0,1$ black holes to begin with. This motivates us to repeat the calculation for $\mathcal{N}=2$ black holes, for which some microscopic results are known. Before doing so, it is useful to briefly discuss dyonic black holes, which are relevant for understanding $\mathcal{N}=2$ black holes in string theory which have macroscopic entropy.

2.5 Multiple $U(1)$'s and Dyonic Black Holes

It is not difficult to extend the results above to theories with multiple $U(1)$'s and dyonic black holes with dyonic elementary charges. Since we are ultimately interested in theories with $\mathcal{N}=2$ supersymmetry, we are particularly interested in BPS dyons in extremal black hole backgrounds. The probe dyon may be described by electric and magnetic charges $\vec{\gamma} \equiv (m^i, n_i)$ where the i indexes the $U(1)$ gauge field in a basis where the kinetic terms are canonical, n_i is the quantized electric charge, and m^i is the magnetic charge. Similarly, the background black hole may be described by $\vec{\Gamma} \equiv (M^i, N_i)$.[7]

[7] Even without referring to string theory constructions, considering dyonic black holes and multiple $U(1)$'s is important since, when $\vec{\Gamma}$ is a non-primitive vector (i.e. $\vec{\Gamma} = k\vec{\xi}$, for some integer vector $\vec{\xi}$ and $k>1$), the black hole is marginally unstable against decay into k extremal black holes of charge $\vec{\xi}$. We want to avoid this possibility by considering exactly stable primitively charged black holes, which carry necessarily multiple (possibly dyonic) charges.

In this setup, the result for dyons may be obtained by modifying equations (2.10) and (2.11) as follows. First, as shown in [21], the angular momentum quantum number l gets replaced by $l+\frac{1}{2}|\langle\vec{\gamma},\vec{\Gamma}\rangle|$, where $\langle\vec{\gamma},\vec{\Gamma}\rangle$ is the 'symplectic inner product', defined by:

$$\langle\vec{\gamma},\vec{\Gamma}\rangle \equiv \sum_i \left(n_i M^i - m^i N_i\right) \tag{2.19}$$

For later reference, we may also extend the hodge star operation to the quantized charges via $\star\vec{\gamma} \equiv (q_i^2 n_i, -(2\pi/q_i)^2 m^i)$, which gives us a natural inner product, $||\vec{\gamma}||^2 \equiv \langle\star\vec{\gamma},\vec{\gamma}\rangle$ such that $m_{BPS} = \sqrt{2}M_P||\gamma||$.

Secondly, everywhere we write qE in (2.10) and (2.11), we must replace:

$$qE \to \sum_i q_i n_i E^i + 2\pi m^i B_i/q_i \tag{2.20}$$

Finally, the effective mass is $\Delta m^2 = m^2 - \left(q_i n_i E^i + 2\pi m^i B_i/q_i\right)^2/a^2$. However, this simplifies if we consider only BPS probes in on-shell backgrounds. In this case, an easy calculation shows, $m = m_{BPS}(\vec{\gamma})$ and $a = ||\vec{\Gamma}||/(\sqrt{2}\times 16\pi^2 M_P)$. Putting all this together we find

$$a^2\Delta m^2 = \frac{1}{16\pi^2}\left(\langle\vec{\gamma},\star\vec{\gamma}\rangle\langle\vec{\Gamma},\star\vec{\Gamma}\rangle - \langle\vec{\gamma},\star\vec{\Gamma}\rangle^2\right) \tag{2.21}$$

Thus, the effective mass is just determined by the symplectic inner product. For a single $U(1)$ this reduces to $a\Delta m = \frac{1}{2}|\langle\vec{\gamma},\vec{\Gamma}\rangle|$ (no '$\star$' here) reflecting the fact that mixed electric and magnetic fields carry angular momentum energy proportional to $\langle\vec{\gamma},\vec{\Gamma}\rangle$. Also, note that the RHS is always positive and is minimized when $\vec{\gamma}$ and $\vec{\Gamma}$ are parallel, in which case $\Delta m = 0$.

Finally, observe that even if the charge vectors $\vec{\gamma}$ and $\vec{\Gamma}$ are relatively prime and $||\vec{\Gamma}|| \gg 1$, the symplectic inner product can still be of order 1 if the two vectors are only slightly misaligned. In this case, we can arrange for $\Delta m^2 \sim \frac{1}{a^2}$, in which case logarithmic corrections such as those in (2.15) for intermediate black holes, and the problems with the fermionic density of states, will still have to be addressed.

3. Loop Corrections in theories with $\mathcal{N}=2$ supersymmetry

We now turn to the problem of integrating out a hypermultiplet, Φ, in 3+1d $\mathcal{N}=2$ SUGRA by combining the results of [22] and [1][8]. We first consider the case of electric monopole charges, though the result may easily be extended to the multi-$U(1)$ dyonic case following the prescriptions of Section 2.5.

The main difficulty of working in $\mathcal{N}=2$ SUGRA rather than in Einstein-Maxwell is that there are many new interactions to consider. These induce mixing of the kinetic terms of various fields which can be quite complicated. However, we know that the physical excitations must organize themselves into reps of the supersymmetry (SUSY) algebra. Fortunately, the authors of [22] have worked out these reps in generality for various SUSY multiplets. The result may be stated in terms of reps of the isometry group of $AdS_2\times S^2$, which is $SL(2)\times SU(2)$. We may parameterize these reps by a pair of integers, (h, j), where h is the conformal weight and j is the usual angular

[8]The $\mathcal{N}=4$ case is not so interesting since the loop corrections vanish.

momentum quantum number for $SU(2)$ with degeneracy $2j+1$. For each conformal weight, h, the laplacian contributes a term to the action equivalent to an effective mass of $(am_{eff})^2 = h(h-1)$ and $(am_{eff})^2 = h(h-1)+1/4$ for bosons and fermions respectively[9]. We may now just use our previous results for this mass. In terms of the $SL(2)\times SU(2)$ reps, the degeneracies for the hypermultiplet, Φ, (including all kinetic mixing effects!) are given by:

$$\text{Hypermultiplet:} \qquad 2[(k+\frac{1}{2},k+\frac{1}{2}),\, 2(k+1,k),\, (k+\frac{3}{2},k-\frac{1}{2})] \tag{3.1}$$

where $k=0,1,2,\ldots$. Notice that for the scalar component, the spectrum is exactly what we had without mixing; we just need to multiply our previous result by 2 to go form a chiral multiplet to a hypermultiplet. This makes sense because the scalar has no extra indices to couple to the graviphoton field. The result for the scalars is thus identical to what we wrote down in equation (2.10).

On the other hand, the fermion partition function receives modifications. We see that we have two series give contributions to the mass like $(a\,\delta m)^2 = h(h-1)+1/4 = k^2$ and multiplicity $2j+1 = 2k+2$ and also two series with mass $(a\,\delta m)^2 = (k+1)^2$ and multiplicity $2k$. This leads to the following heat kernel for a (Dirac) fermion.

$$\begin{aligned} K_f^{tot} &= \frac{1}{2\pi a^2} K_A \times \sum_{k=0}^{\infty} \left(e^{-sk^2/a^2}(2k+2) + e^{-s(k+1)^2/a^2} 2k \right) \\ &= \frac{1}{\pi a^2} K_A \times \left(\sum_{k=0} e^{-s(k+1)^2}(2k+2) + 1 \right) \end{aligned} \tag{3.2}$$

where

$$K_A^f = \frac{e^{-s\Delta m^2}}{2\pi a^2} \int_0^\infty d\lambda\, \lambda \rho_f(\lambda) e^{-s\lambda^2/a^2} \tag{3.3}$$

and $\rho_f(\lambda)$ takes the same form as before, eq. (2.11). The extra '+1' in the second line of (3.2) represents the net effect of kinetic mixing while the discrete sum is due to the KK reduction on S^2 and was present previously.

It is straightforward to calculate the effect of this extra term in the regime of intermediate black holes where $qE \ll 1$. For extremal particles we find a net change in the heat kernel of:

$$\Delta K = \frac{1}{2\pi^2 a^4} \int_0^\infty d\lambda\, \lambda \rho_f e^{-s\lambda^2/a^2} = \frac{1}{2\pi^2 a^4} \left(\frac{a^2}{2s} - \left(\frac{1}{24} + \frac{(qE)^2}{2} \right) \right) + \ldots \tag{3.4}$$

Taking this into account and combining with previous results, the net entropy for intermediate black holes turns out to be:

$$S_{\mathcal{N}=2} = \frac{Q^2}{4} + \left(\frac{(qQ)^2}{16\pi^2} + \frac{1}{6} \right) \ln(qQ) \tag{3.5}$$

We remark that the net effect of the interaction is just to flip the sign of the logarithmic correction. This is identical to what was reported in [22] for neutral particles.

For large black holes we still run in to the same issue with the spectral density of the fermions.

[9] In higher dimensions we would normally identify $(am)^2 = h(h-1)$ and $(am)^2 = h(h-1)+1/4$ for bosons/fermions. However, AdS_2 is somewhat special in this regard.

3.1 Backgrounds in $\mathcal{N} = 2$ supergravity

We are now ready to connect the results above to black holes in an $\mathcal{N} = 2$ supergravity background defined by a prepotential, $F(X^I)$. Following the conventions of [24], we first define F_I and the matrix $\mathcal{N}_{IJ}$ via:

$$F_I \equiv \frac{\partial F}{\partial X^I} \equiv \mathcal{N}_{IJ} X^J \tag{3.6}$$

From this we may define a symplectic charge vector, $\Omega \equiv (X^I, F_I)$ and also the Kahler potential:

$$\mathcal{K} = -\ln K(X, \overline{X}), \qquad K(X, \overline{X}) = i\left(\overline{X}^I F_I - X^I \overline{F}_I\right) = -i\langle \Omega, \overline{\Omega} \rangle \tag{3.7}$$

The Einstein-Maxwell piece of the $\mathcal{N} = 2$ lagrangian may then be written as:

$$\mathcal{L}_{EM+GR} = \sqrt{g}\left(\frac{M_P^2}{2}\mathcal{R} + \mathrm{Im}\left(\overline{\mathcal{N}}_{IJ} \mathcal{F}^{I-} \wedge \star \mathcal{F}^{J-}\right)\right) \tag{3.8}$$

where $\mathcal{F}^{I\pm}$ are the (anti-) self-dual field strengths:

$$\mathcal{F}^{I\pm} = \frac{1}{2}\left(\mathcal{F}^I \pm i \star \mathcal{F}^I\right) \tag{3.9}$$

The usual extremal Reissner-Nordstrom geometry may be constructed as a consistent solution of the full $\mathcal{N} = 2$ supergravity. The solution is again defined by a set of electric and magnetic charges, encoded in the vector $\vec{\Gamma} = (M^I, N_I)$. Now, however, due to the change in conventions[10], the quantized charges are determined by:

$$\begin{aligned} N_I &= \frac{1}{4\pi}\int_{S^2} \mathcal{G}_{I_-} \\ M^I &= \frac{1}{4\pi}\int_{S^2} \mathcal{F}^{I-} \end{aligned} \tag{3.10}$$

where $\mathcal{G}_{I_-} \equiv \overline{\mathcal{N}}_{IJ} \mathcal{F}^{J-}$ is the dual field strength.

The moduli at the horizon of an extremal black hole are determined by a set of attractor equations.

$$N_I - \overline{\mathcal{N}}_{IJ} M^J = -2\overline{Z} e^{\mathcal{K}/2} \mathrm{Im} \mathcal{N}_{IJ} X^J \tag{3.11}$$

These are $n_V + 1$ complex equations[11] for the same number of unknowns. After solving the attractor equations we may determine the field strength using:

$$\mathcal{F}^{I-} = \frac{1}{2}\left(M^I - i(\mathrm{Im}\mathcal{N})^{IJ}\left(N_J - (\mathrm{Re}\mathcal{N})_{JK} M^K\right)\right)\left(\sin\theta d\theta \wedge d\phi - \frac{i}{|Z|^2} dt \wedge dr\right) \tag{3.12}$$

Now, write the field strength (3.12) in the form:

$$\begin{aligned} \mathcal{F}^I &= (....)dt \wedge dr + B^I \sin\theta d\theta \wedge d\phi \\ \mathcal{G}_I &= (....)dt \wedge dr + E_I \sin\theta d\theta \wedge d\phi \end{aligned} \tag{3.13}$$

[10] The kinetic term of the gauge field is no longer canonical.

[11] Here, n_V is the number of vector multiplets, excluding the graviphoton.

where the factors $(\ldots)$ are implicitly determined by $\overline{\mathcal{N}}_{IJ}$ but will not be needed. From the perspective of a probe particle we just have an extremal dyonic black hole with a net effective electric and magnetic charge. We may therefore use the results from Section (2.5) and immediately write down the result once we properly translate conventions. Since we are now working with gauge couplings contained in the $U(1)$ kinetic term, rather than equation (2.20) we must write:

$$qE \to E_{eff} + B_{eff} \tag{3.14}$$

where the effective electric and magnetic field seen by the probe are $(E_{eff}, B_{eff}) = (n_I E^I, m^I B_I)$. The effective mass is somewhat more complicated since it depends on the solutions of the attractor equations. However, we may write the result as:

$$a^2 \Delta m^2 = \frac{1}{8\pi} |Z_*(\vec{\Gamma})|^2 |Z(\vec{\gamma})|^2 - (E_{eff} + B_{eff})^2 \tag{3.15}$$

where the central charge, Z, is $Z(\vec{\gamma}) = e^{\mathcal{K}/2} \langle \Omega, \vec{\gamma} \rangle$. The '$*$' in $Z_*(\vec{\Gamma})$ indicates that is evaluated at the solution to the attractor equations (3.11) whereas $Z(\vec{\gamma})$ is evaluated at infinity. In the simplest case where $\mathcal{N}_{IJ}$ is constant equation (3.15) just reduces to (2.21).

As in section 2.5, one can consider a macroscopic extremal black hole with primitive charge vector $\vec{\Gamma}$ (and hence safely bound), and integrate out an extremal state whose charge vector $\vec{\gamma}$ is almost aligned with $\vec{\Gamma}$ so that $\Delta m^2 \sim 1/a^2$. The results of this section show that then the considerations of [1] apply to $\mathcal{N} = 2$ supergravity theories: intermediate black holes are subject to corrections of the form (3.5), while the fermionic spectral density for larger black holes indicates the necessity of imposing the WGC cutoff.

4. Interpretation

We have seen that integrating out charged matter can have a significant impact on the background metric, even in the context of $\mathcal{N} = 2$ supersymmetry, and *even* when the background and probe brane have relatively prime charges. At first this is a somewhat surprising and so it is natural to inquire further as to the interpretation of this result.

At this point, the story diverges based upon how we view the effective field theory we started with. From the perspective of exploring the realm of possible theories it is natural to simply stipulate that the Einstein-Maxwell+Matter theory we wrote down is valid up to some arbitrarily high scale, ε^{-1}. Within this framework, the fermionic spectral density diverges at the WGC cutoff, but one may nevertheless try and interpret the loop correction as an asymptotic series when $qE \ll 1$. In this case, (2.15) follows as a prediction of the entropy function formalism for intermediate black holes. For large qE, the result will depend on the details of how one regulates the fermionic spectral density, but it seems difficult to avoid large corrections in this regime.

On the other hand, if one is working specifically within string theory, we must first ask how one is to interpret extremal particles within low energy field theory to begin with. In fact, in the context of Type IIB compactified on a Calabi-Yau one can argue that we *should not* integrate out the BPS charges associated to RR $U(1)$'s, i.e. wrapped D-branes. This follows from the resolution of the conifold singularity presented in [25], where it was shown that the low energy SUGRA description behaves like a 1PI action in which the D-particles are already integrated out. Importantly, the structure of Ramond-Ramond couplings allows for loops of D-particles to contribute at the same order in perturbation theory as the tree-level action. However, the argument presented in [25] relies somewhat on the decoupling of vector- and hyper-multiplet moduli spaces specific to $\mathcal{N} = 2$. It would thus be useful to understand more generally why or why not loops of BPS states should be considered.

We may also consider compactifications with non-trivial fundamental group in which case one generally finds a tower of extremal KK particles, whose loops contribute at a higher order and must therefore be considered in addition to the tree level action. This case, however, is complicated by the dilatonic mode associated with the non-trivial cycle and so the results presented here do not immediately apply unless the dilaton is stabilized at a sufficiently high scale[12]. It would be interesting to understand this situation further.

On a final note, the disconnect between the entanglement entropy calculation and the divergent semi-classical microcanonical entropy calculation of [2, 3] teach us that the asymptotics must not satisfy the holographic boundary conditions alluded to in [26]. It is interesting to note that the possible mismatch between microcanonical and entanglement entropy is well reflected in the structure of the Penrose diagram of the extremal Reissner-Nordstrom black hole. This provides a hint that certain non-equilibrium processes may be represented in some way by the Penrose diagram. We hope to return to this intriguing possibility in the future.

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