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Four-loop results on anomalous dimensions and splitting functions in QCD

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We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop (N^3LO) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been been reached.

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1. Introduction

Up to power corrections, observables in ep and pp hard scattering can be schematically expressed as

$$O^{ep} \;=\; f_i \otimes c_i^{\rm o} \,, \quad O^{pp} \;=\; f_i \otimes f_k \otimes c_{ik}^{\rm o} \tag{1.1}$$

in terms of the respective partonic cross sections (coefficient functions) $c^{\rm o}$ and the universal parton distribution functions (PDFs) $f_i(x,\mu^2)$ of the proton at a (renormalization and factorization) scale μ of the order of a physical hard scale, e.g., M_H for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction x is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{\partial}{\partial \ln \mu^2}\, f_i(x,\mu^2) \;=\; \big[P_{ik}(\alpha_{\rm s}(\mu^2)) \otimes f_k(\mu^2)\big](x) \tag{1.2}$$

where $\otimes$ denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling $a_{\rm s} \equiv \alpha_{\rm s}(\mu^2)/(4\pi)$,

$$P \;=\; \quad a_{\rm s} P^{(0)} + a_{\rm s}^2 P^{(1)} + a_{\rm s}^3 P^{(2)} + a_{\rm s}^4 P^{(3)} + \ldots \quad , \tag{1.3}$$

$$c_a^{\rm o} \;=\; a_{\rm s}^{n_{\rm o}} \big[\, c_{\rm o}^{(0)} \;+ a_{\rm s}\, c_{\rm o}^{(1)} \;+ a_{\rm s}^2\, c_{\rm o}^{(2)} \;+ a_{\rm s}^3\, c_{\rm o}^{(3)} \;+ \ldots \big] \,. \tag{1.4}$$

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order ($\mathrm{N^2LO}$) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond $\mathrm{N^2LO}$ are of phenomenological interest where high precision is required, such as in determinations of $\alpha_{\rm s}$ from deep-inelastic scattering (DIS) (see refs. [5, 6] for the $\mathrm{N^3LO}$ corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at $\mathrm{N^3LO}$. The size and structure of the corrections beyond $\mathrm{N^2LO}$ are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop ($\mathrm{N^3LO}$) non-singlet splitting functions. We focus on the quantities $P_{\rm ns}^{\pm(3)}(x)$ for the evolution of flavour-differences $q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$ of quark and antiquark distributions; for more details see ref. [8].

2. Diagram calculations of fixed-N moments

Two methods have been applied for obtaining Mellin moments of the quantities $P^{(3)}$ in eq. (1.3). Depending on the function, both can be used to determine the same even-N *or* the odd-N moments.

In the first one calculates, via the optical theorem and a dispersion relation in x, the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with N, hence only $N \leq 6$ has been covered completely so far [14]. Much higher N can be accessed for simpler cases, e.g., values up to $N > 40$ have been reached for high-n_f parts. These were sufficient to determine the complete n_f^2 and n_f^3 parts of the non-singlet splitting functions $P_{\rm ns}^{(3)}(x)$ and the n_f^3 parts of the corresponding flavour-singlet quantities [15].

The increase of the complexity of the Feynman integrals with N is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. [16], see also ref. [17]. FORCER calculations in this framework have reached $N = 16$ for all contributions to the functions $P_{\rm ns}^{(3)}$, $N = 18$ for their n_f parts and $N = 20$ for the complete limit of a large number of colours n_c [8]. See refs. [18–21] for earlier calculations of $P_{\rm ns}^{\pm(3)}$ at $N \leq 4$.

3. Towards all-N expressions

If the anomalous dimensions $\gamma_{\rm ns}(N) = -P_{\rm ns}(N)$ at N$^{n>2}$LO are analogous to the lower orders, then they can be expressed in terms of harmonic sums $S_{\vec{w}}$ [22, 23] and denominators $D_a^k \equiv (N+a)^{-k}$ as

$$\gamma_{\rm ns}^{(n)}(N) \;=\; \sum_{w=0}^{2n+1} c_{00\vec{w}}\, S_{\vec{w}}(N) + \sum_a \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{ak\vec{w}}\, D_a^k\, S_{\vec{w}}(N)\;. \tag{3.1}$$

The denominators at the calculated values of N indicate $a = 0,1$ for $\gamma_{\rm ns}^{\pm}$, with coefficients $c_{00\vec{w}}$, $c_{ak\vec{w}}$ that are integer modulo low powers of 1/2 and 1/3. Sums up to weight $w = 2n+1$ occur at N^{n}LO.

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions D, it has been conjectured that the $\overline{\rm MS}$ functions $\gamma_{\rm ns}(N)$ are constrained by 'self-tuning' [24, 25],

$$\gamma_{\rm ns}(N) \;=\; \gamma_{\rm u}\,(N + \sigma\,\gamma_{\rm ns}(N) - \beta(a_{\rm s})/a_{\rm s})) \tag{3.2}$$

where $\beta(a_{\rm s}) = -\beta_0\, a_{\rm s}^2 - \beta_1\, a_{\rm s}^3 - \ldots$ is the beta function, for its present status see refs. [26, 27]. The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for $\sigma = -1$ and $\sigma = 1$, respectively, and the universal kernel $\gamma_{\rm u}$ is reciprocity respecting (RR), i.e., invariant under replacement $N \to (1-N)$. Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence 'only' $\gamma_{\rm u}$, which includes 2^{w-1} RR (combinations of) harmonic sums of weight w, needs to be determined at four loops.

Present information, given by the even-N (odd-N) values $N \leq 16$ (15) of $\gamma_{\rm ns}^{+(3)}(N)$ ($\gamma_{\rm ns}^{-(3)}(N)$) and endpoint constraints (see below), is insufficient to determine the $n = 3$ coefficients in eq. (3.1). However, $\gamma_{\rm ns}^{+} = \gamma_{\rm ns}^{-}$ in the large-n_c limit, hence the known even-N *and* odd-N values can be used. Moreover, alternating sums do not contribute to $\gamma_{\rm ns}^{\pm}$ in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci(w) RR sums at weight $w = 1,\ldots,7$ and a total of 87 basis functions for $n = 3$ in eq. (3.1).

Large-N and small-x limits provide more than 40 constraints on their coefficients. At large-N, the non-singlet anomalous dimensions have the form [33–35]

$$\gamma_{\rm ns}^{(n-1)}(N) \;=\; A_n \ln\widetilde{N} - B_n + N^{-1}\{C_n \ln\widetilde{N} - \widetilde{D}_n + \tfrac{1}{2}A_n\} + O(N^{-2}) \tag{3.3}$$

with $\ln\widetilde{N} \equiv \ln N + \gamma_{\rm e}$, where $\gamma_{\rm e}$ denotes the Euler-Mascheroni constant. C_n and $\widetilde{D}_n$ are given by

$$C(a_{\rm s}) \;=\; (A(a_{\rm s}))^2 \quad , \quad \widetilde{D}(a_{\rm s}) \;=\; A(a_{\rm s})\cdot(B(a_{\rm s}) - \beta(a_{\rm s})/a_{\rm s})\;, \tag{3.4}$$

in terms of lower-order information on the cusp anomalous dimension $A(a_{\rm s}) = A_1 a_{\rm s} + A_2\, a_{\rm s}^2 + \ldots$ and the quantity $B(a_{\rm s}) = B_1 a_{\rm s} + B_2\, a_{\rm s}^2 + \ldots$ sometimes called the virtual anomalous dimension.

The resummation of small-x double logarithms [28–31] provides the four-loop coefficients of $x^a \ln^b x$ at $4 \leq b \leq 6$ and all a in the large-n_c limit (in full QCD, this holds only at even a for $P_{\rm ns}^{+}(x)$ and odd a for $P_{\rm ns}^{-}(x)$). Moreover, a relation leading to a single-logarithmic resummation at $a = 0$,

$$\gamma_{\rm ns}^{+}(N)\cdot\big(\,\gamma_{\rm ns}^{+}(N) + N - \beta(a_{\rm s})/a_{\rm s}\big) \;=\; O(1)\;, \tag{3.5}$$

has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with $\zeta_2 = \pi^2/6$ that vanish in the large-n_c limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients $c_{00\vec{w}}$, $c_{ak\vec{w}}$ of $\gamma_{\rm ns}^{\pm(3)}(N)$ in the large-n_c limit that can be solved using the moments $1 \le N \le 18$, leaving the results of the diagram calculation at $N = 19, 20$ as checks.

4. All-N anomalous dimension in the large-n_c limit

The exact expressions for the new n_f^0 and n_f^1 parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the n_f^2 and n_f^3 terms see ref. [15]. The resulting large-N coefficients $A_{L,4}$ and $B_{L,4}$ – the subscript L indicates the large-n_c limit – are found to be

$$\begin{aligned} A_{L,4} = {} & C_F n_c^3 \Big(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\,\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\,\zeta_5 - 32\,\zeta_3^2 - 876\,\zeta_6 \Big) \\ & - C_F n_c^2 n_f \Big(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \Big) \\ & + C_F n_c n_f^2 \Big(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \Big) - C_F n_f^3 \Big(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \Big) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} B_{L,4} = {} & C_F n_c^3 \Big(-\frac{1379569}{5184} + \frac{24211}{27}\,\zeta_2 - \frac{9803}{162}\,\zeta_3 - \frac{9382}{9}\,\zeta_4 + \frac{838}{9}\,\zeta_2\zeta_3 + 1002\zeta_5 + \frac{16}{3}\,\zeta_3^2 \\ & \qquad + 135\zeta_6 - 80\zeta_2\zeta_5 + 32\zeta_3\zeta_4 - 560\zeta_7 \Big) \\ & + C_F n_c^2 n_f \Big(\frac{353}{3} - \frac{85175}{162}\,\zeta_2 - \frac{137}{9}\,\zeta_3 + \frac{16186}{27}\,\zeta_4 - \frac{584}{9}\,\zeta_2\zeta_3 - \frac{248}{3}\,\zeta_5 - \frac{16}{3}\,\zeta_3^2 - 144\zeta_6 \Big) \\ & - C_F n_c n_f^2 \Big(\frac{127}{18} - \frac{5036}{81}\,\zeta_2 + \frac{932}{27}\,\zeta_3 + \frac{1292}{27}\,\zeta_4 - \frac{160}{9}\,\zeta_2\zeta_3 - \frac{32}{3}\,\zeta_5 \Big) \\ & - C_F n_f^3 \Big(\frac{131}{81} - \frac{32}{81}\,\zeta_2 - \frac{304}{81}\,\zeta_3 + \frac{32}{27}\,\zeta_4 \Big) \,. \end{aligned} \tag{4.2}$$

The agreement of the four-loop cusp anomalous dimension (4.1) with the result obtained from the large-n_c photon-quark form factor [36,37] provides a further non-trivial check of the determination of the all-N expressions from the moments at $N \le 18$, and hence also of the relations (3.1) – (3.5).

The maximum-weight ζ_3^2 and ζ_6 parts of eq. (4.1) also agree with the result obtained in planar $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar $\mathcal{N} = 4$ MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ MSYM.

The all-N large-n_c limit of $\gamma_{\rm ns}^{\pm(3)}(N)$ is compared in fig. 1 with the integer-N QCD results at $N \le 16$. As illustrated in the left panel, the former are a decent approximation to the latter for the individual n_f^k contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of $n_f = 5$ light quark flavours outside the large-N / large-x region. Hence the large-n_c suppressed contributions – indicated by the subscript N below – need to be taken into account in phenomenological N^{3}LO analyses.

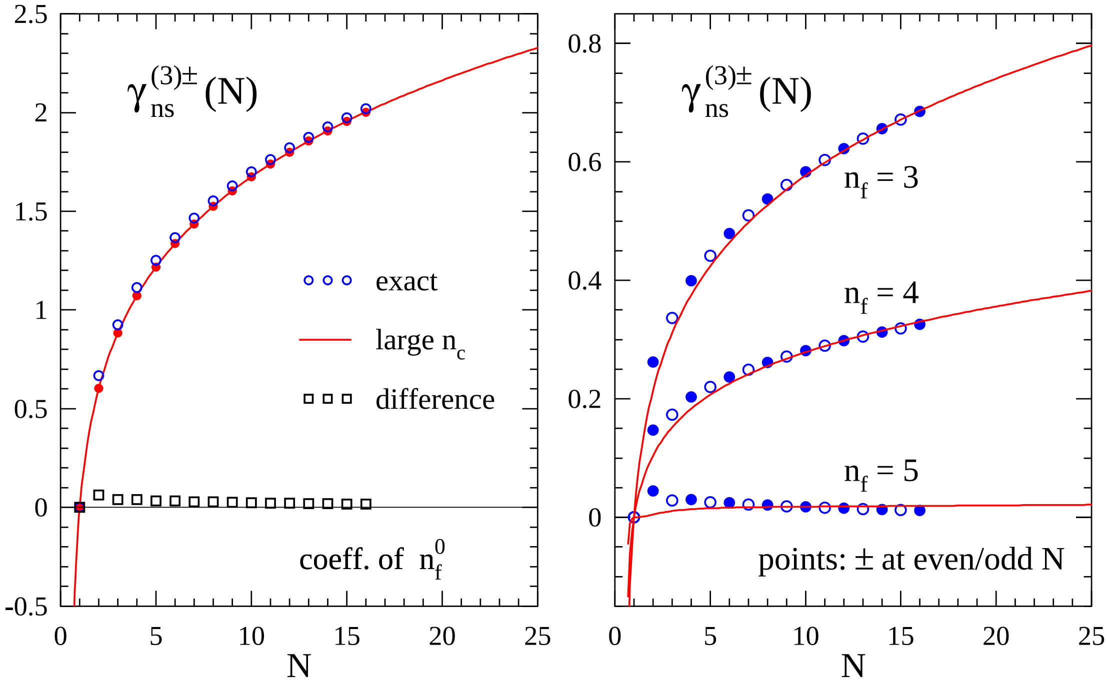

Figure 1: The large-n_c limit of the four-loop anomalous dimensions $\gamma_{\rm ns}^{\pm(3)}(N)$ (lines) compared to the QCD results for $\gamma_{\rm ns}^{+(3)}(N)$ at even N and $\gamma_{\rm ns}^{-(3)}(N)$ at odd N (points). Left: the n_f-independent contributions. Right: the results for physically relevant values of n_f. The values have been converted to an expansion in α_s.

5. x-space approximations of the large-n_c suppressed parts

With eight integer-N moments known for both $P_{\rm ns}^{+(3)}(x)$ and $P_{\rm ns}^{-(3)}(x)$ and the large-x and small-x knowledge discussed in section 2, it is possible to construct approximate x-space expressions which are analogous to (but more accurate than) those used before 2004 at N^{2}LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large-x parameters A_4 and B_4 in eq. (3.3),
- two of three suppressed large-x logs $(1-x)\ln^k(1-x)$, $k=1,2,3$,
- one of ten two-parameter polynomials in x that vanish for $x \to 1$,
- two of the three unknown small-x logarithms $\ln^k x$, $k=1,2,3$

is built for the large-n_c suppressed n_f^0 and n_f^1 parts $P_{\rm N,0/1}^{+(3)}$ of $P_{\rm ns}^{+(3)}(x)$. This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives A and B are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large-n_c parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g., $N = 18$ for $P_{\rm ns}^{+(3)}$. The residual uncertainty at this N-value is a consequence of the width of the band at large x, which in turn is correlated with the uncertainties at smaller x. If the spread of the result A and B would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of N should reveal a discrepancy.

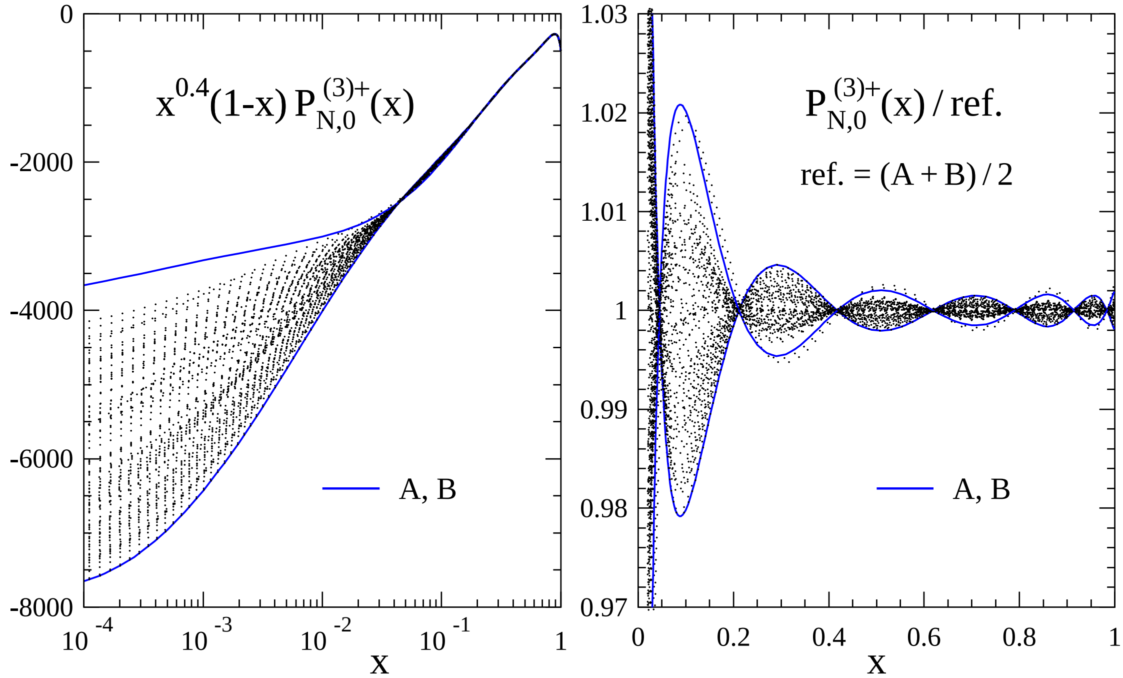

Figure 2: About 90 trial functions for the n_f-independent contribution to the large-n_c suppressed part of splitting function $P^{+(3)}_{\rm ns}(x)$, multiplied by $x^{0.4}(1-x)$. The two functions chosen to represent the remaining uncertainty are denoted by A and B and shown by solid (blue) lines. Due to the factor $(1-x)$ the contribution $A_{N,4}$ to the four-loop cusp anomalous dimension can be read off at $x=1$.

We were able to extend the diagram computations of the n_f^1 parts of $P^{(3)+}_{\rm ns}(x)$ to $N = 18$ and find

$$P^{+(3)}_{\rm N,1}(N=18) \;=\; 195.8888792_B \;<\; 195.8888857\ldots_{\rm exact} \;<\; 195.8888968_A \;. \qquad (5.1)$$

A similar check for $P^{+(3)}_{\rm N,0}$ has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at $N = 16$.

The case of $P^{-(3)}_{\rm ns}(x)$ has been treated in the same manner, but taking into account that only its leading small-x logarithm is known up to now [29]. See ref. [8] for the (large-N suppressed) additional $d^{abc}d_{abc}$ contribution $P^{s(3)}_{\rm ns}(x)$ to the splitting function for the total valence quark PDF.

6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large-n_c results, the approximations for the remaining n_f^0 and n_f^1 contributions and the complete high-n_f contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with n_f quark flavours are given by

$$A_4 \;=\; 20702(2) - 5171.9(2)\,n_f + 195.5772\,n_f^2 + 3.272344\,n_f^3 \;, \qquad (6.1)$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of α_s leads to

$$\begin{aligned}
A_q(\alpha_s, n_f=3) \;&=\; 0.42441\,\alpha_s\,(1 + 0.72657\,\alpha_s + 0.73405\,a_s^2 + 0.6647(2)\,a_s^3 + \ldots)\;,\\
A_q(\alpha_s, n_f=4) \;&=\; 0.42441\,\alpha_s\,(1 + 0.63815\,\alpha_s + 0.50998\,a_s^2 + 0.3168(2)\,a_s^3 + \ldots)\;,\\
A_q(\alpha_s, n_f=5) \;&=\; 0.42441\,\alpha_s\,(1 + 0.54973\,\alpha_s + 0.28403\,a_s^2 + 0.0133(2)\,a_s^3 + \ldots)\;. \qquad (6.2)
\end{aligned}$$

The corresponding results for the virtual anomalous dimension, i.e., the coefficient of $\delta(1-x)$ show a similarly benign expansion with

$$B_4 \;=\; 23393(10) - 5551(1)\, n_f + 193.8554\, n_f^2 + 3.014982\, n_f^3 \tag{6.3}$$

and

$$\begin{aligned}
B_q(\alpha_s, n_f\!=\!3) \;&=\; 0.31831\, \alpha_s\, (1 + 0.99712\, \alpha_s + 1.24116\, a_s^2 + 1.0791(13)\, a_s^3 + \ldots)\,, \\
B_q(\alpha_s, n_f\!=\!4) \;&=\; 0.31831\, \alpha_s\, (1 + 0.87192\, \alpha_s + 0.97833\, a_s^2 + 0.5649(13)\, a_s^3 + \ldots)\,, \\
B_q(\alpha_s, n_f\!=\!5) \;&=\; 0.31831\, \alpha_s\, (1 + 1.74672\, \alpha_s + 0.71907\, a_s^2 + 0.1085(13)\, a_s^3 + \ldots)\,.
\end{aligned} \tag{6.4}$$

Due to constraints by large-N moments, the errors of A_4 and B_4 are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for A_4 and B_4 which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

	A_4	B_4
C_F^4	0	$197. \pm 3.$
$C_F^3 C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	610.3 ± 0.3	295.6 ± 2.4
$d_R^{abcd} d_A^{abcd}/N_R$	-507.5 ± 6.0	$-996. \pm 45.$
$n_f C_F^3$	-31.00 ± 0.4	81.4 ± 2.2
$n_f C_F^2 C_A$	38.75 ± 0.2	-455.7 ± 1.1
$n_f C_F C_A^2$	-440.65 ± 0.2	-274.4 ± 1.1
$n_f d_R^{abcd} d_R^{abcd}/N_R$	-123.90 ± 0.2	-143.5 ± 1.2
$n_f^2 C_F^2$	-21.31439	-5.775288
$n_f^2 C_F C_A$	58.36737	51.03056
$n_f^3 C_F$	2.454258	2.261237

where the exactly known n_f^2 and n_f^3 coefficients have been included for completeness. Due to the constraint provided by the exact large-n_c limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases, $A_q = C_F/C_A\, A_g$, does not hold beyond three loops.

7. N^{3}LO corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale, $\dot{q}_{\rm ns}^i \equiv d \ln q_{\rm ns}^i / d \ln \mu_f^2$, at a suitably chosen reference point.

As in ref. [1], we choose the the schematic, order-independent initial conditions

$$xq_{\rm ns}^{\pm,\rm v}(x,\mu_0^2) \;=\; x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_{\rm s}(\mu_0^2) \;=\; 0.2\,. \tag{7.1}$$

For $\alpha_{\rm s}(M_Z^2) = 0.114\ldots 0.120$ this value for $\alpha_{\rm s}$ corresponds to $\mu_0^2 \simeq 25\ldots 50$ GeV^2 beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the *ep* collider HERA.

The new N^{3}LO corrections to $\dot{q}_{\rm ns}^{\,i}$ are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the N^{2}LO contributions for the standard identification $\mu_r = \mu_f \equiv \mu$ of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at $x \simeq 0.07$, the relative N^{3}LO effects are (well) below 1% for the flavour-differences $q_{\rm ns}^+$ and $q_{\rm ns}^-$ (left and middle panel). The N^{2}LO and N^{3}LO corrections are larger for the valence distribution $q_{\rm ns}^{\rm v}$ at $x < 0.07$ due to the effect of the $d^{abc}d_{abc}$ 'sea' contribution $P_{\rm ns}^{\rm s}(x)$, note the different scale of the right panel in fig. 3. Also in this case the N^{3}LO evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large-n_c limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the 'x-averaging' effect of the Mellin convolution,

$$[P_{\rm ns} \otimes q_{\rm ns}](x) \;=\; \int_x^1 \frac{dy}{y}\, P_{\rm ns}(y)\, q_{\rm ns}\!\left(\frac{x}{y}\right)\,, \tag{7.2}$$

the results of section 4 are safely applicable to lower values of x than one might expect from fig. 2.

The stability of the NLO, N^{2}LO and N^{3}LO results under variation of the renormalization scale over the range $\frac{1}{8}\mu_f^2 \le \mu_r^2 \le 8\mu_f^2$ is illustrated in fig. 4 at typical values of x. Except close to the sign change of $\dot{q}_{\rm ns}^{+}$, the variation is well below 1% for the conventional interval $\frac{1}{2}\mu_f \le \mu_r \le 2\mu_f$.

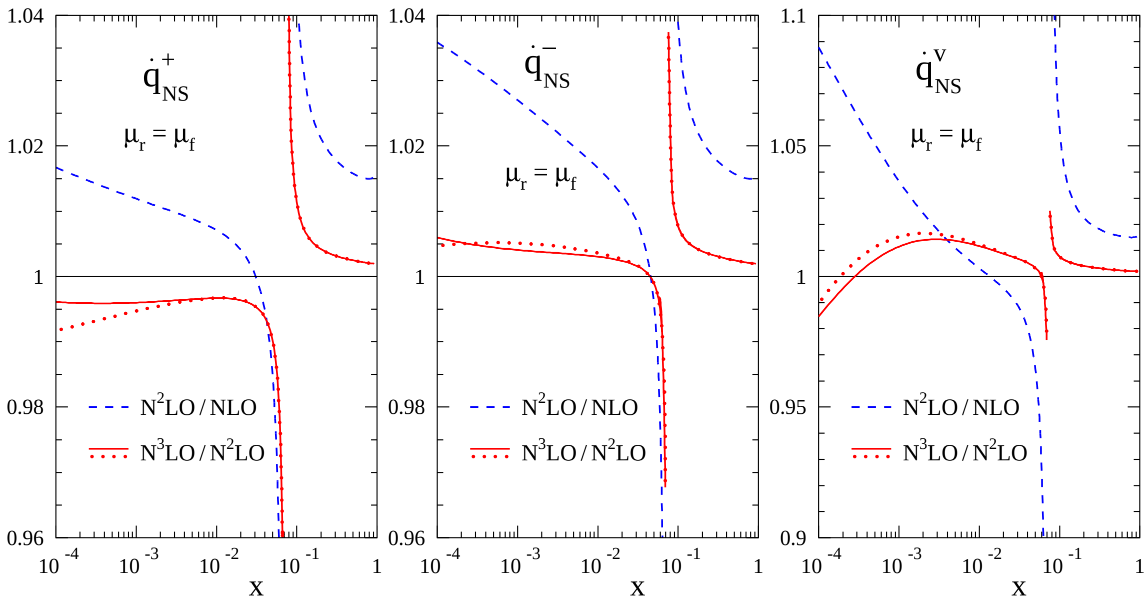

Figure 3: The relative N^{2}LO and N^{3}LO corrections to the logarithmic scale derivative of the non-singlet combinations $q_{\rm ns}^a$ of quark PDFs for the schematic order-independent input (7.1) for $n_f = 4$ at $\mu_r = \mu_f$.

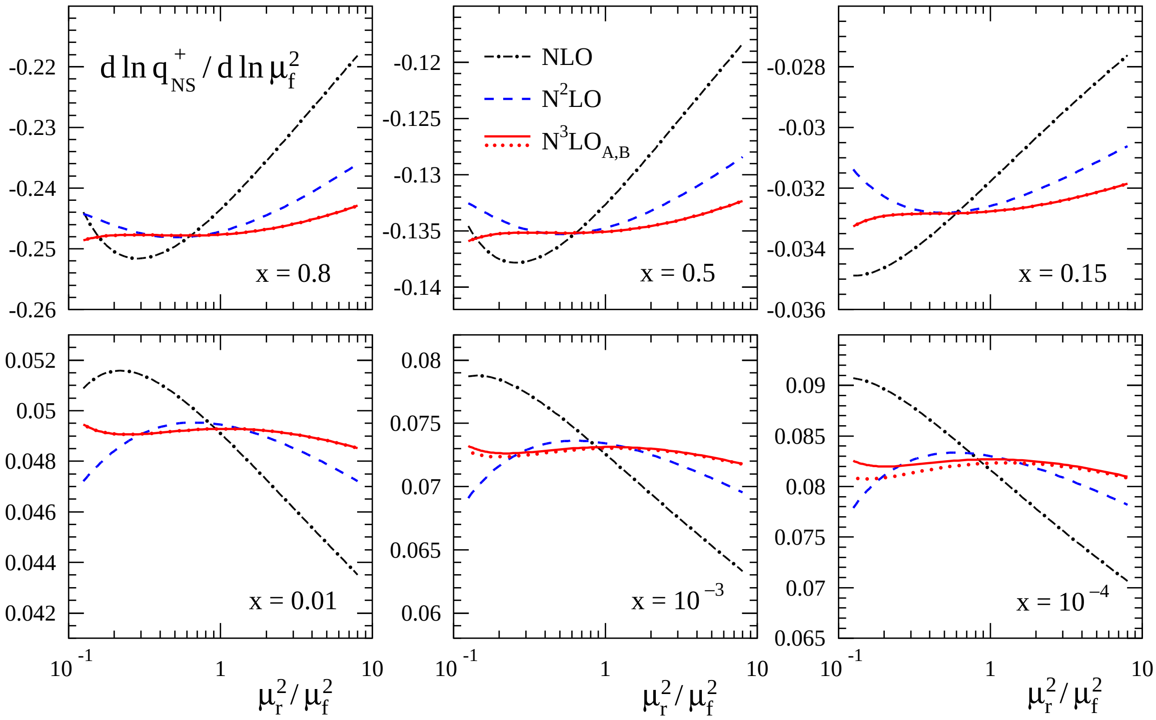

Figure 4: The dependence of the NLO, N^{2}LO and N^{3}LO results for $\dot{q}^+_{\rm ns} \equiv d\ln q^+_{\rm ns}/d\ln\mu_f^2$ on the renormalization scale μ_r at six typical values of x for the initial conditions (7.1) and $n_f = 4$ flavours. The remaining uncertainty of the four-loop splitting function $P^{+(3)}_{\rm ns}(x)$ leads to the difference of the solid and dotted curves.

8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N^{3}LO) of perturbative QCD. The quantities $P^{\pm(3)}_{\rm ns}$ are now known exactly in the limit of a large number of colours n_c. Present results for the large-n_c suppressed contributions with n_f^0 and n_f^1 are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from `arXiv.org`.

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms n_f^0 and n_f^1 parts of $P^{\pm(3)}_{\rm ns}$. So far, only their contributions proportional to the values ζ_4 and ζ_5 of the Riemann ζ-function have been completely determined, together with the (unpublished) ζ_3 part of the n_f^1 contributions. The ζ_4 parts are particularly simple; in fact, it turns out that they (and other π^2 terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The ζ_5 part of $P^{\pm(3)}_{\rm ns}$, presented in appendix D of ref. [8], includes a (non large-n_c) contribution

$$-\,\frac{128}{3}\left\{3C_F^2C_A^2 - 2C_FC_A^3 + 12\,d_F^{abcd}d_A^{abcd}/N_R\right\}\,5\zeta_5\,[S_1(N)]^2\,. \qquad (8.1)$$

The resulting $\ln^2 N$ large-N behaviour needs to be compensated by non-ζ_5 terms. Eq. (8.1) looks exactly like the ζ_5-'tail' of the so-called wrapping correction in the anomalous dimensions in $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory, see refs. [51,52].

Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions $P_{ij}^{(3)}(x)$, $i,j=q,g$. At present, it appears computationally too hard to obtain moments of all four functions beyond $N=6$ using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

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