Effective Actions with the First Order Form of Gauge Theories

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Introduction

In the Einstein-Hilbert action

$$S = \int d^d x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{1}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right) \tag{2}$$

and

$$R_{\mu\nu}(\Gamma) = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \tag{3}$$

it is possible to treat both $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ as being independent [1]; the equation of motion for $\Gamma^\lambda_{\mu\nu}$ in this first order action yields Eq. (2).

Employing the first order Einstein-Hilbert (1EH) has the advantage over the usual second order form of the action that the interaction vertices are greatly simplified [2, 3]. This approach has been employed in the case of the Yang-Mills theory, where it is easy to show that the Green's functions derived from the 1YM and 2YM actions are equivalent. However, it is not readily apparent that the 1EH and 2EH actions lead to the same Green's functions [3]. More recently, we have demonstrated that in fact they are the same [4]. In doing so, we arrive at a set of Feynman rules from the 1EH action that are much simpler than those that follow from the 2EH action as there are but three three-point vertices and two propagators. We also consider some explicit calculations both at zero and finite temperature which demonstrate the equivalence of the two formulations.

Action and Feynman rules

It is convenient to use $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and $G^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}\left(\delta^\lambda_\mu \Gamma^\sigma_{\nu\sigma} + \delta^\lambda_\nu \Gamma^\sigma_{\mu\sigma}\right)$ as independent fields, so that Eq. (1) yields

$$\mathcal{L}_{1EH} = h^{\mu\nu}\left(G^\lambda_{\mu\nu\,,\lambda} + \frac{1}{d-1} G^\lambda_{\mu\lambda} G^\sigma_{\nu\sigma} - G^\lambda_{\mu\sigma} G^\sigma_{\nu\lambda} \right). \tag{4}$$

Employing the Faddeev-Popov path integral [5]

$$Z_{1EH} = \int \mathcal{D}h^{\mu\nu} \mathcal{D}G^\lambda_{\mu\nu} \Delta_{FP}(h) \exp i \int d^d x \left[\mathcal{L}_{1EH} + \mathcal{L}_{gf} \right] \tag{5}$$

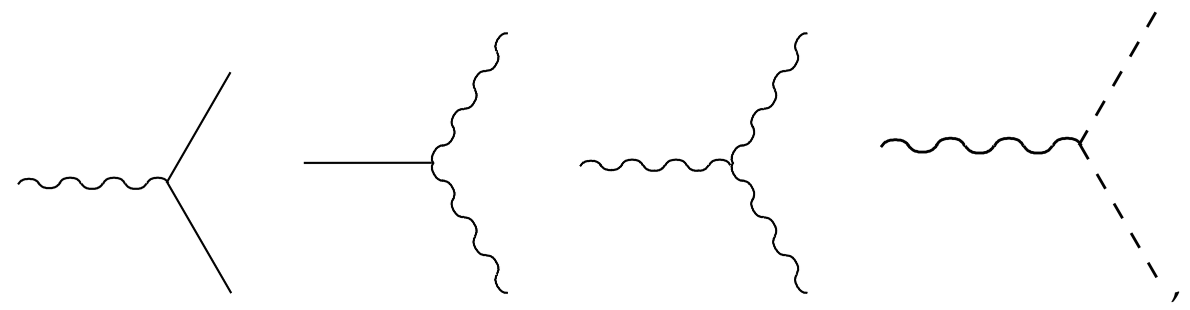
and using $h^{\mu\nu}(x) = \eta^{\mu\nu} + \phi^{\mu\nu}(x)$, the propagators $\langle\phi\phi\rangle$, $\langle GG\rangle$, $\langle\phi G\rangle$ and the vertex $\langle\phi GG\rangle$ have been obtained in ref. [3]. Subsequently we have proved the equivalence with the usual second order formalism and we have also redefined the fields in such a way that only simple propagators (not mixed) arises [4]. The corresponding generating functional obtained from Eq. (5) is given by

$$\begin{aligned} Z_{1EH} = & \int \mathcal{D}h^{\mu\nu} \mathcal{D}G^\lambda_{\mu\nu} \Delta_{FP}(h) \exp i \int d^d x \left[\frac{1}{2} G^\lambda_{\mu\nu} M_{\lambda\ \ \sigma}^{\mu\nu\ \pi\tau}(\eta) G^\sigma_{\pi\tau} - \frac{1}{2} \phi^{\mu\nu}_{,\lambda} M^{-1\lambda\ \ \sigma}_{\ \ \mu\nu\ \pi\tau}(\eta) \phi^{\pi\tau}_{,\sigma} \right. \\ & \left. + \frac{1}{2}\left(G^\lambda_{\mu\nu} + \phi^{\alpha\beta}_{,\rho} \left(M^{-1}\right)^{\rho\ \ \lambda}_{\ \alpha\beta\ \mu\nu}(\eta) \right) \left(M_{\lambda\ \ \sigma}^{\mu\nu\ \pi\tau}(\phi) \right) \left(G^\sigma_{\pi\tau} + \left(M^{-1}\right)^{\sigma\ \ \xi}_{\ \pi\tau\ \gamma\delta}(\eta) \phi^{\gamma\delta}_{,\xi} \right) + \mathcal{L}_{gf} \right], \end{aligned} \tag{6}$$

where

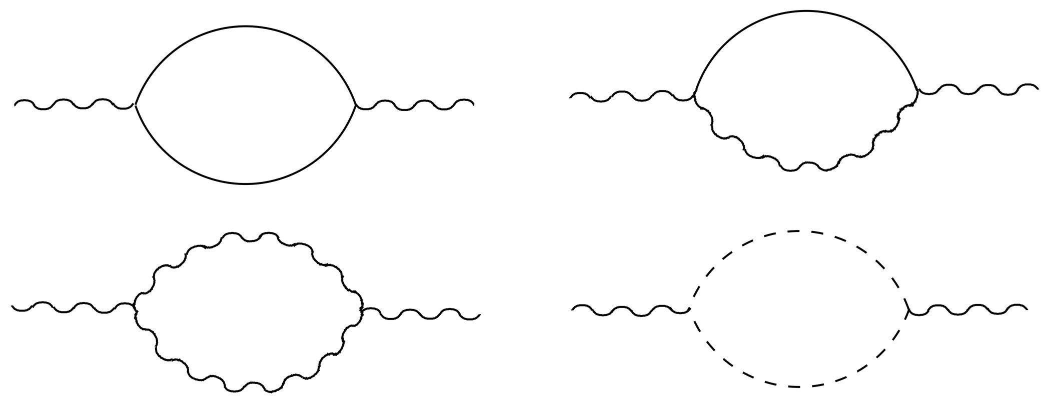
$$\begin{aligned} M_{\lambda\ \ \sigma}^{\mu\nu\ \pi\tau}(h) = & \frac{1}{2}\left[\frac{1}{d-1}\left(\delta^\nu_\lambda \delta^\tau_\sigma h^{\mu\pi} + \delta^\mu_\lambda \delta^\tau_\sigma h^{\nu\pi} + \delta^\nu_\lambda \delta^\pi_\sigma h^{\mu\tau} + \delta^\mu_\lambda \delta^\pi_\sigma h^{\nu\tau} \right) \right. \\ & \left. - \left(\delta^\tau_\lambda \delta^\nu_\sigma h^{\mu\pi} + \delta^\tau_\lambda \delta^\mu_\sigma h^{\nu\pi} + \delta^\pi_\lambda \delta^\nu_\sigma h^{\mu\tau} + \delta^\pi_\lambda \delta^\mu_\sigma h^{\nu\tau} \right) \right] \end{aligned} \tag{7}$$

(d is the space-time dimension), so that there is two simple propagators $\langle\phi\phi\rangle$ and $\langle GG\rangle$, the usual ghost propagator and the following interaction vertices [4].



where the wavy, solid and dashed lines represent the graviton, the G and the ghost fields, respectively. *This is a quite remarkable result when compared with the infinite number of interaction vertices which arises in the usual second order formalism.* Furthermore, these Feynman rules are expressed in a much more compact form (see Eqs. (3.25) of ref. [4]) in terms of $M_{\lambda\ \ \sigma}^{\mu\nu\ \pi\tau}$ and its inverse.

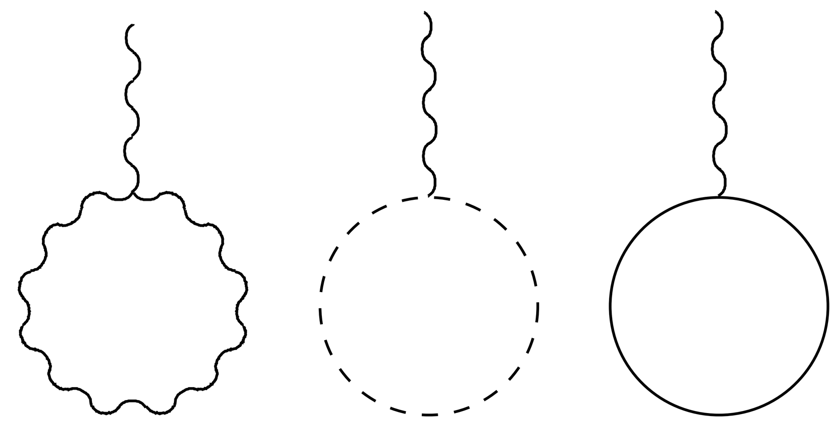
As a simple example we have computed the zero-temperature one-loop graviton self-energy in a general covariant gauge which is given by the following diagrams:



The known result in the DeDonder gauge [6] is in agreement with ours (see Eqs. (3.32), (3.33) and (3.34) of ref. [4]).

Thermal effective action

At one-loop order, the one-graviton contribution to the thermal effective action can be obtained from the following diagrams



Using the Feynman rules in the first order formalism and the thermal field theory imaginary time formalism one can show that only the first two diagrams contribute (the third one has no loop momentum dependence). After performing the Matsubara sum and momentum integration we obtain the following result for the thermal one-graviton function

$$\Gamma^{\rm th}_{\mu\nu} = \frac{d(d-3)}{2} \frac{\zeta(d)\Gamma(d)}{2(d-1)} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \frac{T^d}{(2\pi)^{d-1}} \left(\eta_{\mu\nu} - d u_\mu u_\nu\right), \tag{8}$$

where ζ is the Riemann zeta function and Γ is the gamma function. The factor $d(d-3)/2$ counts the number of degrees of freedom of the graviton field in d dimensions. For $d = 4$

$$\left.\Gamma^{\rm th}_{\mu\nu}\right|_{d=4} = \frac{\pi^2 T^4}{90}\left(\eta_{\mu\nu} - 4 u_\mu u_\nu\right), \tag{9}$$

which is in agreement with the known result obtained using the second order formalism [11].

Discussion

We have shown that the first and second order forms are equivalent. This equivalence holds provided it is possible to discard tadpole diagrams (which are regulated to zero when using dimensional regularization.)

We have also shown that by rewriting the 1EH action judiciously, it is possible to have just two propagating fields and three three-point functions. This may prove to be an advantage when considering higher order diagrams in the loop expansion in (super-)gravity.

It is quite straightforward to adopt the methods of refs. [7, 8, 9, 10], involving the use of geodesic coordinates in conjunction with a background field for $\phi^{\mu\nu}$, to determine counter terms while working with the 1EH Lagrangian. It would be interesting to investigate if renormalization maintains the set of Feynman rules finite, or rather an infinite set of counter-terms would arise.

It would also be interesting to compute the one loop correction to the two-point function $\langle\phi\phi\rangle$ using the transverse-traceless (TT) gauge of ref. [12].

The calculation of higher order contributions to the thermal effective action, which includes contributions from all the n-graviton thermal Green functions, is work in progress. Using the first order formalism, it may be possible to sum all the contributions in a closed form, as has been done in the case of the Yang-Mills theory [13]. We expect that this may be possible, since we have a finite number of interaction vertices in the first order formalism of gravity, as is the case in the Yang-Mills theory.

Acknowledgments

We would like to thank CNPq and Fapesp (Brazil) for financial support.

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[1] This form of the action is often credited to Palatini, though Einstein first considered it [1]