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Novel parametrization for the leptonic mixing matrix and CP violation

David Emmanuel-Costa[a], **Nuno Rosa Agostinho**[*a], **J.I. Silva-Marcos**[a], **Daniel Wegman**[a,b]

a Departamento de Física and CFTP, Instituto Superior Técnico (IST), Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

b IFPA, Université de Liège, Bât B5 , Sart Tilman B-4000 Liège 1, Belgium

E-mail: david.costa@tecnico.ulisboa.pt
nuno.agostinho@tecnico.ulisboa.pt
juca@cftp.tecnico.ulisboa.pt
dwegman@ulg.ac.be

We study leptonic CP violation from a new perspective. For Majorana neutrinos, a new parametrization for leptonic mixing of the form $V = O_{23}\, O_{12}\, K^i_a \cdot O$ reveals interesting aspects that are less clear in the standard parametrization. We identify several important scenario-cases with mixing angles in agreement with experiment and leading to large leptonic CP violation. If neutrinos happen to be quasi-degenerate, this new parametrization might be very useful, e.g., in reducing the number of relevant parameters of models. [†]

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[*]Speaker.
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1. Introduction

Observations of neutrino oscillations have solidly established the massiveness of the neutrinos and the existence of leptonic mixing, which means that new physics beyond the Standard Model are required. During the last decades, several attempts were made in order to overcome questions as the origin of the leptonic flavor structure or why leptonic mixing differs tremendously from the observed quark mixing. In particular, one may impose family symmetries forbidding certain couplings and at the same time explaining successfully the observed structure of masses and mixings, as well as predicting some other observables [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Although the structure of leptonic mixing is predicted in such models, the mass spectrum turns out to be unconstrained by such symmetries. The absolute neutrino mass scale is still missing, one does not know whether neutrinos are Majorana or Dirac particles, and the nature of leptonic CP violation is still open (for a recent review see Ref. [16]). From the analysis of neutrino oscillation experiments one can extract bounds for the light neutrino mass square differences $\Delta m^2_{21} \equiv m_2^2 - m_1^2$ and $\Delta m^2_{31} \equiv m_3^2 - m_1^2$. Recent cosmological observations, based on model depend analysis, have constrained the sum of neutrino masses [17], which then imply an upper bound of the lightest neutrino mass. All knowledge on the light neutrino mixing is encoded in the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS) [18, 19, 20]. In order to further analyze the leptonic flavour structure, it is essential to parametrize all the entries of the full PMNS matrix in terms of six independent parameters. It is clear that the choice of a parametrization does not impose any constraints on the physical observables.

Some underlying aspects, such as symmetries or relations that the experimental data may suggest can be understood by means of parametrizations. Although parametrizations are equivalent among themselves, some of them may be much appropriate to describe symmetries and patterns. Recall the usefulness of the Wolfenstein parametrization [21] in the quark sector to describe the hierarchical character of the quark families.

Among many parametrization proposed in the literature, the standard parametrization is the most widely used, and the six parameters are three mixing angles, namely θ_{12}, θ_{13}, $\theta_{23} \in [0, \pi/2]$, one Dirac-type phase δ and two Majorana phases α_1, α_2 in following form:

$$V^{SP} = K \cdot O_{23} \cdot K_D \cdot O_{13} \cdot O_{12} \cdot K_M \,, \tag{1.1}$$

where the real orthogonal matrices O_{12}, O_{13} and O_{23} are the usual rotational matrices in the $(1,2)-$, $(1,3)-$, and $(2,3)-$sector, respectively. The diagonal unitary matrices K_D and K_M are given by $K_D \equiv \mathrm{diag}(1, 1, e^{i\alpha_D})$ and $K_M \equiv \mathrm{diag}(1, e^{i\alpha_1^M}, e^{i\alpha_2^M})$. Within the standard parametrization, one may recall that the consistent values for the neutrino mixing angles θ_{12} and θ_{23} together with the smallness of θ_{13} suggest that the neutrino mixing is rather close to the tribimaximal mixing (TBM) [22]. It is important to stress that this parametrization is (modulo irrelevant phases), the same as the one used for the quark sector, despite the fact of leptonic mixing being quite different. The nature of leptonic CP violation is still an open question, and it is, thus, not yet clear what, is the most adequate form to express or parametrize this phenomenon. In this paper we study leptonic CP Violation in the context of a new parametrization for leptonic mixing of the form

$$V = O_{23}\, O_{12}\, K^i_\alpha \cdot O \,, \tag{1.2}$$

where $K_\alpha^i = \text{diag}(1, i, e^{i\alpha})$ and O is a real orthogonal matrix parametrized with three mixing angles. We then have a total of six parameters, namely five mixing angles and one complex phase α, which is the required number of independent parameters for describing the PMNS matrix.

The paper is organized as follows. In the next section, we present in detail the new parametrization stated in Eq. (1.2). In Sec. 3, we motivate the use of this new parametrization in the limit of degenerate or quasi-degenerate neutrino spectrum. Then in Sec. 4, we present an alternative view of large leptonic CP violation and interesting aspects that are less clear in the standard parametrization, using the new parametrization for leptonic mixing, discuss its usefulness and identify five scenario-cases that lead to large Dirac-CP violation, and which have mixing angles in agreement with experimental data. Results are shown for mixing and CP violation. In Sec. 5, we give a numerical analysis of the scenarios described in the previous section, and, for the quasi-degenerate Majorana neutrinos, a numerical analysis of their stability. Finally, in Sec. 6, we present our conclusions.

2. A novel parametrization

In this section, we present the new parametrization for the lepton mixing matrix:

$$V = K_S \; O_{23} \, O_{12} \; K_\alpha^i \cdot O \,, \tag{2.1}$$

where $K_S = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$[1] is a pure phase unitary diagonal matrix, O_{23}, O_{12} are two elementary orthogonal rotations in the (23)- and (12)-planes, $K_\alpha^i = \text{diag}(1, i, e^{i\alpha})$ has just one complex phase α (apart from the imaginary unit i), and O is a general orthogonal real matrix described by 3 angles. The proof is given in Ref. [1].

It is clear that, as with the standard parametrization in Eq. (1.1), this parametrization has also six physical parameters, but, some are now of a different nature: two angles in O_{23} and O_{12}, three other angles in O, but just one complex phase α in K_α^i. From now on, we use explicitly the following full notation

$$V \;=\; O_{23}^L \, O_{12}^L \cdot K_\alpha^i \cdot O = O_{23}^L \, O_{12}^L \cdot K_\alpha^i \cdot O_{23}^R \, O_{13}^R \, O_{12}^R \,, \tag{2.2}$$

where we have identified each of the elementary orthogonal rotations, either on the left or on the right of the CP-violating pure phase matrix K_α^i, with a notation superscript L, R.

2.1 CP violation

In the Standard Parametrization (SP), we may distinguish two types of CP-violating phases: Dirac and Majorana CP violation phases. The Dirac-type phases are determined by the four independent arguments of the quartets $\arg(V_{1i} V_{kj} V_{1j}^* V_{ki}^*)$, with $i \neq j \neq k$ and the Majorana-type phases are given by the six independent arguments of the bilinears $\arg(V_{ij} V_{ik}^*)$, with $j \neq k$. In the SP, these phases are the minimal CP-violation quantities when neutrinos are Majorana particles [23, 24, 25, 26, 27, 28, 29].

[1] The diagonal matrix K_S has no physical meaning, since it only rephases the PMNS matrix V on the left. This can be clearly seen in the weak basis where the charged lepton mass matrix is diagonal and through a weak basis transformation the phases in K_S can be absorbed by the redefinition of the right-handed charged lepton fields.

Since the nature of leptonic CP violation is still open, it is not clear what could be the most adequate form to express or parametrize this phenomenon. Here, as an alternative, we choose to express CP violation in a different way, namely, as combinations of mixing angles and the unique complex phase α. It is worth to note that, in our new parametrization, even when $\alpha = 0$ or π, we still have CP violation due to the presence of an imaginary unit in the diagonal matrix K_i^{α}. In particular, setting $\alpha = 0$ the Dirac CP violation invariant $I_{CP} \equiv \mathrm{Im}(V_{12}V_{23}V_{22}^{*}V_{13}^{*})$ yields:

$$
\begin{aligned}
I_{CP} = \frac{1}{32}&(\sin 2\theta_{23}^{L}\cos 2\theta_{23}^{R}(\sin^2\theta_{12}^{L}\cos\theta_{12}^{L}\sin 2\theta_{12}^{R}(3\sin 3\theta_{13}^{R} - 5\sin\theta_{13}^{R}) \\
&+ 8\sin^2\theta_{12}^{L}\cos\theta_{12}^{L}\cos 2\theta_{12}^{R}\cos 2\theta_{13}^{R}\sin 2\theta_{23}^{R} + (7\cos\theta_{12}^{L} + \cos 3\theta_{12}^{L})\sin 2\theta_{12}^{R}\sin\theta_{13}^{R}\cos^2\theta_{13}^{R}) \\
&+ 2\sin 2\theta_{12}^{L}\cos 2\theta_{23}^{R}\cos\theta_{23}^{R}\left(\sin 2\theta_{12}^{R}\cos\theta_{13}^{R}(\cos 2\theta_{13}^{R} - 3)\cos 2\theta_{23}^{R} + 2\cos^2\theta_{13}^{R}\right) \\
&- 2\cos 2\theta_{12}^{R}\sin 2\theta_{13}^{R}\sin 2\theta_{23}^{R}))\,,
\end{aligned}
\tag{2.3}
$$

which vanishes when $\theta_{12}^{L} = \theta_{23}^{L} = 0$ (i.e, omitting the left orthogonal matrices in Eq. (2.2)) and when $\theta_{12}^{L} = \theta_{23}^{R} = 0$.

2.2 Usefulness

There are several motivations to consider a new parametrization. We still do not know the exact nature of neutrinos: if they are Majorana or Dirac, as well if neutrinos mass spectrum is hierarchical or quasi-degenerate. It turns out that if the mass spectrum is quasi-degenerate, the new parametrization will be very useful and may reflect some specific nature of neutrinos, i.e. if they are Majorana and quasi-degenerate. The left part $O_{23}^{L} \cdot O_{12}^{L} \cdot K_{\alpha}^{i}$ of Eq. (2.2) suggests some major intrinsic Majorana character of neutrino mixing and CP violation, while the right part $O_{23}^{R} \cdot O_{13}^{R} \cdot O_{12}^{R}$, with three angles, reflects that there are 3 neutrino families and results in small mixing, comparable to the quark sector, of the order of the Cabibbo angle. Indeed, we consider that our parametrization incorporates well diverse fixed structures for the lepton mixing [22, 30, 31, 32] in the limit $V_{13} = 0$, and in particular the case of TBM which e.g. in [33, 34] occurs as the result of a family symmetry. If such a family symmetry exists, once it is broken at the electroweak scale, the reactor angle gets a small contribution of the order of the Cabibbo angle, possibly related to the small neutrino mass differences. Thus, from this point of view, the dominant contribution for large neutrino mixing must come from the Majorana character of neutrinos. Other interesting aspect that arrives with this new parametrization is the fact that the Dirac and Majorana CP violation quantities are related to just one complex phase α present in K_{α}^{i}.

In the next sections, we shall the discuss how this new parametrization enables to view large leptonic mixing from a new perspective.

3. Degenerate and Quasi-degenerate Majorana Neutrinos

3.1 Quasi-degenerate neutrino masses

In the weak basis where the charged lepton mass matrix is diagonal and real-positive, the matrix S_o has a special meaning in the limit of exact neutrino mass degeneracy [35, 36]. In this limit the neutrino mass matrix M_o assumes the following form:

$$M_o = \mu\, S_o = \mu\, U_o^* U_o^\dagger\,, \tag{3.1}$$

where μ is the common neutrino mass. The matrix U_o accounts for the leptonic mixing. In the limit of exact degenerate neutrinos, the orthogonal matrix O on the right of the new parametrization in Eq. (2.2), has no physical meaning, since it can be absorbed in the degenerate neutrino fields. This has motivated our proposal for the use of the new parametrization.

The usefulness of the new parametrization is particularly interesting if neutrinos are quasi-degenerate. When the degeneracy is lifted, i.e. for quasi-degenerate neutrinos, the full neutrino mass matrix becomes slightly different from the exact limit in Eq. (3.1):

$$M = \mu\ (S_o + Q^\varepsilon)\,, \tag{3.2}$$

where Q^ε is some small perturbation. In general, this perturbation may significantly modify the mixing result for the exact case in Eq. (3.1). In view of our new parametrization, now the full lepton mixing matrix diagonalizing M is described by

$$V = U_o' \cdot O\,, \tag{3.3}$$

where U_o' is of the same form as U_o. As a consequence of the perturbation, the U_o' matrix will differ slightly from the unitary matrix U_o that diagonalizes the degenerate limit part S_o, when $Q^\varepsilon \to 0$. The same happens to the matrix O, which can be either small or possibly some large general orthogonal matrix. In Sec. 5, we shall quantify this more explicitly, using numerical simulations.

3.2 CP Violation of Quasi-degenerate Neutrinos

It was pointed out in Ref. [35], that if neutrinos are quasi-degenerate (or even exact degenerate) CP violation continues to be relevant. This can be understood if one defines Weak-Basis invariant quantities sensitive to CP violation. An important invariant quantity, in this case, is

$$G_m \equiv \left| Tr\left([M_\nu\, H_l\, M_\nu^*, H_l^*]^3 \right) \right|\,, \tag{3.4}$$

where $H_l = M_l M_l^\dagger$ is the squared charged lepton mass matrix. Contrary to the usual quantity $I = Tr\left([M_\nu^\dagger M_\nu, H_l]^3\right)$ which is proportional to the Dirac CP violation quantity I_{CP}, we find that the quantity G_m signals CP violation even if neutrinos are exact degenerate. In fact, we obtain in this limit

$$G \equiv \frac{G_m}{\Delta_m} = \frac{3}{4}\left|\sin 2\theta_{12}^L \sin 4\theta_{12}^L \sin^2 2\theta_{23}^L \sin 2\alpha\right|\,,\ \text{where}\ \Delta_m \equiv \mu^6 (m_\tau^2 - m_\mu^2)^2 (m_\tau^2 - m_e^2)^2 (m_\mu^2 - m_e^2)^2\,, \tag{3.5}$$

with μ the common neutrino mass. θ_{12}^L and θ_{23}^L are, respectively, the angles of O_{12}^L and O_{23}^L in Eq. (2.2)), and α is the complex phase of $K_\alpha^i = \mathrm{diag}(1, i, e^{i\alpha})$. Obviously, with the new parametrization for the lepton mixing in Eq. (2.2), this invariant takes on a new and relevant meaning. It is a curious fact that G is so specifically (and in such a clean way) dependent on only, what we have called, the left part of Eq. (2.2) and on $\sin 2\alpha$.

4. Leptonic CP violation from a New Perspective

In the case of the Standard Parametrization, if neutrinos are of the Dirac type, maximum CP violation occurs by choosing $\alpha_D = \pi/2$. This is the only phase responsible for generating large CP violation. On the other-hand, if neutrinos are Majorana there are two more CP violating phases α_1^M and α_2^M. Thus, one finds that large CP violation is limited to consider these two facts.

It turns out that if we switch to the new parametrization, one gets a much richer structure for large CP violation, particularly if neutrinos are quasi-degenerate. In the next subsection, we shall present different limit cases where it is possible to generate large CP violation. We explore all combinations of the O_{23}'s and O_{12}'s that result in both solar and atmospheric mixing angles near experimental result, while the other O_{ij}'s are kept small in order to obtain mixing matrices near to $V_{13} = 0$, i.e. with a small reactor angle. Our choice to fix some parameters, assuming a preexisting model and symmetry, is justified for the closeness of the TBM with the experimental data:

$$|V_{13}|^2 = 0, \;\; sin^2\theta_{atm} = 1/2, \;\; sin^2\theta_{sol} = 1/3. \tag{4.1}$$

Thus, in zeroth order, we are able to reproduce the TBM [2]. The free parameters, which are treated as perturbation parameters, will be responsible for approximate our initial ansatz to the current mixing angles, always with large leptonic CP violation. These values are close to the experimental results at one-sigma level [37],

$$0.439 < \sin^2\theta_{23} < 0.599\,, \quad 0.0214 < \sin^2\theta_{13} < 0.0254\,, \quad 0.307 < \sin^2\theta_{12} < 0.339\,. \tag{4.2}$$

given in terms of the Standard Parametrization angles. Is easy to observe that 1/3 is an allowed value for $\sin^2\theta_{12}$, but values slightly lower are better. The central value for $\sin^2\theta_{23}$ is above 1/2, but values both below and above are preferred.

The experimentally measured mixing angles are given by the parameters of the new parametrization as:

$$|V_{13}|^2 = s^2_{\theta^L_{12}} c^2_{\theta^R_{13}} s^2_{\theta^R_{23}} + c^2_{\theta^L_{12}} s^2_{\theta^R_{13}}\,, \tag{4.3a}$$

$$\sin^2\theta_{sol} = \frac{s^2_{\theta^L_{12}}\left(c_{\theta^R_{12}}c_{\theta^R_{23}} - s_{\theta^R_{12}}s_{\theta^R_{13}}s_{\theta^R_{23}}\right)^2 + c^2_{\theta^L_{12}}s^2_{\theta^R_{12}}c^2_{\theta^R_{13}}}{1-|V_{13}|^2}\,, \tag{4.3b}$$

$$\sin^2\theta_{atm} = \frac{c^2_\alpha c^2_{\theta^R_{13}} c^2_{\theta^R_{23}} s^2_{\theta^L_{23}} - 2c_\alpha c_{\theta^L_{23}} c_{\theta^R_{13}} c_{\theta^R_{23}} s_{\theta^L_{12}} s_{\theta^L_{23}} s_{\theta^R_{13}} + c^2_{\theta^L_{23}} s^2_{\theta^L_{12}} s^2_{\theta^R_{13}} + c^2_{\theta^R_{13}}\left(s_\alpha c_{\theta^R_{23}} s_{\theta^L_{23}} + c_{\theta^L_{12}} c_{\theta^L_{23}} s_{\theta^R_{23}}\right)^2}{1-|V_{13}|^2}\,. \tag{4.3c}$$

where we have used the identification $c_X = \cos X$ and $s_X = \sin X$. In table 1 we present the values of the parameters for each case.

[2] We could use other schemes with $V_{13} = 0$, instead of the TBM scheme: the hexagonal mixing [30], [32] or the golden ratio mixing of type I [31]

Table 1: Values of the parameters for each case.

	O_{23}^L	O_{12}^L	O_{23}^R	O_{13}^R	O_{12}^R
I-A	$-\pi/4$	$\sin^{-1}(1/\sqrt{3})$	$\varepsilon\, t_{23}^R$	$\varepsilon\, t_{13}^R$	$\varepsilon\, t_{12}^R$
I-B	$-\pi/4$	$\varepsilon\, t_{12}^L$	$\varepsilon\, t_{23}^R$	$\varepsilon\, t_{13}^R$	$\sin^{-1}(1/\sqrt{3})$
I-C	$-\pi/4$	$\sin^{-1}(1/2)$	$\varepsilon\, t_{23}^R$	$\varepsilon\, t_{13}^R$	$\sin^{-1}(1/\sqrt{6})$
II-A	$\varepsilon\, t_{23}^L$	$\varepsilon\, t_{12}^L$	$-\pi/4$	$\varepsilon\, t_{13}^R$	$\sin^{-1}(1/\sqrt{3})$
II-B	$\sin^{-1}(1/\sqrt{3})$	$\varepsilon\, t_{12}^L$	$-\pi/4$	$\varepsilon\, t_{13}^R$	$\sin^{-1}(1/\sqrt{3})$

4.1 Limit case I

In this limit, we consider that the combination $O_{23}^R \cdot O_{13}^R$ has small rotation angles, depending on some small parameter ε typically of the order of Cabibbo angle or smaller:

$$V = O_{23}^L \cdot O_{12}^L \cdot K_\alpha^i \cdot O_{23}^{\varepsilon R} \cdot O_{13}^{\varepsilon R} \cdot O_{12}^R. \tag{4.4}$$

Then, using a notation where the angle θ_{23}^L refers to O_{23}^L, θ_{12}^L and θ_{12}^R refer to O_{12}^L and O_{12}^R, respectively, while the small angles coming from $O_{23}^{\varepsilon R}$ and $O_{13}^{\varepsilon R}$ are denoted as $\varepsilon\, t_{23}^R$ and $\varepsilon\, t_{13}^R$, respectively, we can compute the large mixing angles, the element $|V_{13}|^2$ and the Dirac CP violation quantity I_{CP}. Within limit case I, we may distinguish two opposite scenarios and some extra intermediate scenario, denoted by I-A, I-B and I-C.

4.1.1 Scenario I-A

This scenario corresponds to having O_{12}^L large with $\cos 2\theta_{12}^L = \frac{1}{3}$, and O_{12}^R with $\cos 2\theta_{12}^R = \varepsilon\, t_{12}^R$. It was also analyzed in Ref. [36]. Taking $t_{13} = 0$ and using the TBM scheme where $\theta_{23}^L = -\frac{\pi}{4}$, we obtain:

$$\begin{aligned}
\sin^2(\theta_{atm}) &= \frac{1}{2} + \sqrt{\frac{2}{3}}\varepsilon\, t_{23}^R \sin\alpha, \\
\sin^2(\theta_{sol}) &= \frac{1}{3} + \frac{\varepsilon^2}{9}\left(3\left(t_{12}^R\right)^2 - 2\left(t_{23}^R\right)^2\right), \\
|V_{13}|^2 &= \frac{\varepsilon^2\left(t_{23}^R\right)^2}{3}, \\
I_{CP} &= \frac{\sqrt{2}}{6\sqrt{3}}\left|\varepsilon\, t_{23}^R \cos\alpha\right| = \frac{\sqrt{2}}{6}\left|V_{13}\cos\alpha\right|.
\end{aligned} \tag{4.5}$$

The mixing angles and V_{13} can be fit with just the phase α, and the small parameter $\varepsilon\, t_{23}$ of the order of the Cabibbo angle. Within this scenario, it was also possible obtain a large value for I_{CP}. A non zero value for α is necessary to have a value of $\sin\theta_{atm} \neq 1/2$. As for the Majorana CP violating phases, we obtain in leading order large values ($\sim \pi/2$) for both, since t_{13} is zero:

$$\tan\alpha_1^M = \frac{\sqrt{2}}{\varepsilon\, t_{12}^R}, \quad \alpha_2^M = \frac{\pi}{2}, \tag{4.6}$$

It is also interesting to compute the form of the neutrino mass mass for this scenario and the CP violating quantity G for the TBM scheme:

$$M_\nu = \frac{\mu}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & \frac{-1+3e^{-2i\alpha}}{2} & \frac{1+3e^{-2i\alpha}}{2} \\ -2 & \frac{1+3e^{-2i\alpha}}{2} & \frac{-1+3e^{-2i\alpha}}{2} \end{pmatrix}, \quad G = \frac{4}{9}\left|\sin(2\alpha)\right|. \tag{4.7}$$

4.1.2 Scenario I-B

In this scenario, considering the TBM, we have O_{12}^L small with $\cos 2\theta_{12}^L \approx 1$, O_{12}^R large with $\cos 2\theta_{12}^R \approx \frac{1}{3}$ and $\theta_{23}^L = -\frac{\pi}{4}$. This choice will provide to be crucial in order fit the experimental results on mixing. Taking a small $\theta_{12}^L = \varepsilon\, t_{12}^L$, we find:

$$\begin{aligned} \sin^2(\theta_{atm}) &= \frac{1}{2} + \varepsilon\, t_{23}^R \sin\alpha, \\ \sin^2(\theta_{sol}) &= \frac{1}{3} + \frac{\left(t_{12}^L\right)^2}{3}\varepsilon^2, \\ |V_{13}|^2 &= \varepsilon^2\left(t_{13}^R\right), \\ I_{CP} &= \frac{\varepsilon}{3\sqrt{2}}\left|t_{13}^R \cos\alpha\right|. \end{aligned} \tag{4.8}$$

We need at least three parameters $\varepsilon\, t_{13}^R$, $\varepsilon\, t_{23}^R$, and α to fit the experimental result and generate large CP violation. The central value for the solar angle can not be achieved, not even with the use of all parameters. The Majorana phases in leading order are small violating CP phases:

$$\tan\alpha_1^M = \frac{3}{\sqrt{2}}\varepsilon\, t_{12}^L, \quad \tan\alpha_2^M = \frac{\varepsilon\, t_{12}^L}{\sqrt{2}}\left(1 + \frac{\sqrt{2}t_{23}^R}{t_{13}^R}\right). \tag{4.9}$$

As for the neutrino mass matrix and CP violating quantity G, we obtain in leading order for the TBM scheme:

$$M_\nu = \frac{\mu}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\alpha & \cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \end{pmatrix}, \quad G = 0. \tag{4.10}$$

4.1.3 Scenario I-C

In this intermediate scenario, we take both O_{12}^L and O_{12}^R large with $\sin\theta_{12}^R \approx \frac{1}{\sqrt{6}}$ and $\sin\theta_{12}^L = \frac{1}{2}$. Also choosing $\theta_{23}^R = -\frac{\pi}{4}$, we guarantee that the mixing is close to the TMB. We obtain in leading order:

$$\begin{aligned} \sin^2(\theta_{atm}) &= \frac{1}{2}\left(1 - \varepsilon\, t_{13}^R \cos\alpha - \sqrt{3}\,\varepsilon\, t_{23}^R \sin\alpha\right), \\ \sin^2(\theta_{sol}) &= \frac{1}{3} + \frac{\varepsilon^2}{24}\left(3\left(t_{13}^R\right)^2 - 2\sqrt{5}t_{13}^R t_{23}^R - 3\left(t_{23}^R\right)^2\right), \\ |V_{13}|^2 &= \frac{\varepsilon^2}{4}\left(3\left(t_{13}^R\right)^2 + \left(t_{23}^R\right)^2\right), \\ I_{CP} &= \frac{\varepsilon}{8}\left|t_{13}^R\left(-\sin\alpha + \sqrt{\frac{5}{3}}\cos\alpha\right) + \frac{1}{\sqrt{3}}t_{23}^R\left(\sqrt{\frac{5}{3}}\sin\alpha + \cos\alpha\right)\right|. \end{aligned} \tag{4.11}$$

The Majorana CP violating phases, in leading order, are large:

$$\tan\alpha_1^M = 3\sqrt{\frac{3}{5}}, \quad \tan\alpha_2^M = \frac{\sqrt{3}t_{13}^R + \sqrt{15}t_{23}^R}{3\sqrt{5}t_{13}^R - t_{23}^R}. \tag{4.12}$$

The Majorana quasi-degenerate mass matrix and G quantity are in leading order for the TBM scheme:

$$M_\nu = \frac{\mu}{2}\begin{pmatrix} 1 & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} & \frac{2e^{-2i\alpha}-1}{2} & \frac{2e^{-2i\alpha}+1}{2} \\ \sqrt{\frac{3}{2}} & \frac{1+2e^{-2i\alpha}}{2} & \frac{2e^{-2i\alpha}-1}{2} \end{pmatrix}, \quad G = \frac{9}{16}\left|\sin 2\alpha\right|. \tag{4.13}$$

For the above results, we only need 2 parameters to fit the experimental results on mixing and generate large CP violation: either the combination of the small parameter εt_{13}^R with the phase α or the combination of the small parameter εt_{23}^R with the phase α. We must remember that this is entirely dependent on the choice of the two large angles of O_{12}^L and O_{12}^R.

4.2 Limit case II

In this limit, we consider that the combination $O_{23}^L \cdot K_\alpha^i \cdot O_{23}^R \cdot O_{12}^R$ gives the large contribution whereas the other O_{12}^L and O_{13}^R have small angles:

$$V = O_{23}^L \cdot O_{12}^{\varepsilon L} \cdot K_\alpha^i \cdot O_{23}^R \cdot O_{13}^{\varepsilon R} \cdot O_{12}^R. \tag{4.14}$$

For this case we use the following notation: θ_{23}^L and θ_{23}^R are the angles of the orthogonal matrices O_{23}^L and O_{23}^R, respectively; θ_{12}^R refers to O_{12}^R; the smaller angles εt_{12}^L and εt_{13}^R are the rotation angles of $O_{12}^{\varepsilon L}$ and $O_{13}^{\varepsilon R}$, respectively. As in the limit case I, we may construct two opposite scenarios: a scenario where O_{23}^L is large or a scenario where O_{23}^R is large. The scenario where O_{23}^L is large, whereas O_{23}^R is small, is already contained in the scenario I-A of limit case I (modulo some slight modifications which produce equivalent results). Therefore, it is sufficient to focus on a scenario where O_{23}^L is small and O_{23}^R is large, or exceptionally on a scenario between, where both are large.

4.2.1 Scenario II-A

In this scenario O_{23}^L is small and O_{23}^R is large. The angle of O_{23}^L is given by $\theta_{23}^L = \varepsilon t_{23}^L$, $\sin^2\theta_{12}^R = \frac{1}{3}$ and $\theta_{23}^R = -\frac{\pi}{4}$. Then, we obtain:

$$\begin{aligned} \sin^2(\theta_{atm}) &= \frac{1}{2} + \varepsilon t_{23}^L \sin\alpha, \\ \sin^2(\theta_{sol}) &= \frac{1}{3} + \frac{\left(t_{12}^L\right)^2}{6}\varepsilon^2, \\ |V_{13}|^2 &= \frac{\varepsilon^2}{2}\left(2\left(t_{13}^R\right)^2 + \left(t_{12}^L\right)^2\right), \\ I_{CP} &= \frac{\varepsilon}{6}\left|t_{12}^L\right|. \end{aligned} \tag{4.15}$$

It is clear that we need at least three parameters to fit the experimental results and a large value for Dirac-CP violation: the phase α plus the two small parameters εt_{23}^L and εt_{12}^L. We cannot achieve a central value for the solar angle and using $\varepsilon t_{12}^L \neq 0$, to guarantee that there is large I_{CP}, will only increase the value of the solar angle above $1\,\sigma$. As for the Majorana-CP violating phases, only the second one can be large:

$$\tan\alpha_1^M = \frac{3}{2}\varepsilon t_{12}^L, \quad \tan\alpha_2^M = \frac{t_{12}^L}{\sqrt{2}t_{13}^R}, \tag{4.16}$$

The neutrino mass matrix and the quantity G are (in leading order) for the TBM scheme:

$$M_\nu = \mu I_{3\times 3}, \quad G = 0. \tag{4.17}$$

4.2.2 Scenario II-B

This is an intermediate scenario where both angles of O_{23}^L and O_{23}^R are large, such that $\sin^2(\theta_{23}^L) = \frac{1}{3}$, $\sin^2(\theta_{23}^R) = \frac{1}{2}$ and $\sin^2(\theta_{12}^R) = \frac{1}{3}$. In leading order, one finds:

$$\begin{aligned}
\sin^2(\theta_{atm}) &= \frac{1}{2} + \frac{\sqrt{2}}{3}\sin\alpha - \frac{\varepsilon^2}{12}\left(\left(t_{12}^L\right)^2 - 8t_{12}^L t_{13}^R\cos\alpha\right), \\
\sin^2(\theta_{sol}) &= \frac{1}{3} + \frac{\left(t_{12}^L\right)^2}{6}\varepsilon^2, \\
|V_{13}|^2 &= \frac{\varepsilon^2}{2}\left(2\left(t_{13}^R\right)^2 + \left(t_{12}^L\right)^2\right), \\
I_{CP} &= \frac{\varepsilon}{18}\left|4t_{13}^R\cos\alpha + t_{12}^L\right|.
\end{aligned} \tag{4.18}$$

If we take t_{12}^L much smaller than t_{13}^R and α small, we can make the prediction of $I_{CP} = \frac{2}{9}|V_{13}|$. The Majorana-CP violating phases are obtained, in leading order,

$$\tan\alpha_1^M = \frac{3}{2}\varepsilon t_{12}^L, \quad \tan\alpha_2^M = \frac{t_{12}^L}{\sqrt{2}t_{13}^R}, \tag{4.19}$$

where the second one can be large. For the mass matrix of the quasi-degenerate Majorana neutrinos, as well for the G quantity, one obtains, in leading order, for the TBM scheme:

$$M_\nu = \mu\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{e^{-2i\alpha}-2}{3} & \frac{\sqrt{2}\left(e^{-2i\alpha}+1\right)}{3} \\ 0 & \frac{\sqrt{2}\left(e^{-2i\alpha}+1\right)}{3} & \frac{2e^{-2i\alpha}-1}{3}\end{pmatrix}, \quad G = 0. \tag{4.20}$$

In matter of fact, with the TBM mixing scheme, we only need a small parameter εt_{13}^R and α to fit the experimental mixing angles and large Dirac-CP violation. By inspection of the expression for $\sin^2(\theta_{atm})$, we conclude that the phase α needs to be of the order of the Cabibbo angle. We emphasize that this only was accomplished with the previous adjustment made to the two large angles of O_{23}^L and O_{23}^R.

4.3 Standard Parametrization vs New Parametrization

To justify that the new parametrization is an added value to better understand neutrino physics, in particular quasi-degenerate Majorana neutrinos, we choose the scenario I-A and reproduce it in the standard parametrization. Scenario I-A seems to be the most appealing since we only need two extra parameters to fit the experimental results on leptonic mixing. The other scenarios need more parameters, or need more adjustment. This scenario also provides large Dirac-CP violation and large Majorana phases, which can also be seen from a different perspective using the combined CP violation quantity G: it is one of two scenarios that have $G \neq 0$, in leading order, and with an appropriate choice of α, this quantity can be large.

In the standard parametrization, the scenario I-A is obtained with:

$$V_{SP} = O_{23}^{\pi/4} \cdot K_D \cdot O_{13}^{\theta} \cdot O_{12}^{\phi_0}, \tag{4.21}$$

with $\sin\phi_0 = \frac{1}{\sqrt{3}}$ and $\theta = 0$. For simplicity, we leave out the Majorana phases. In order to have $|V_{13}| \neq 0$, we have to switch on the angle O_{13}. However, irrespective to the value that we choose for θ, it is impossible to change the atmospheric angle, since $|V_{23}| = |V_{33}|$. To avoid this problematic situation, we must choose from the start another angle for O_{23} different from $\pi/4$, or change the TBM limit with some additional contribution, afterwards. With this, we show that in the Standard Parametrization, it is impossible to adjust the TBM and correct the atmospheric mixing angle using the remaining parameters.

This is in clear contrast with what one obtains in the context of the new parametrization. In scenario I-A, with the suitable choice for the parameter εt_{23}^R, it is possible to adjust the atmospheric mixing angle and generate a small $|V_{13}|$, simultaneously, with large values for CP violation.

5. Numerical Simulation and Stability

For completeness, we give a numerical analysis of some of the scenarios described in the previous section. We choose a fixed scheme, the TBM scheme constructed with the 5 different scenarios. More precisely, we test

$$\begin{array}{lll} \textbf{I-A}: \ \mathbf{V_o = O_{23}^{\pi/4} O_{12}^{\phi_o} K_{\alpha_o}^{i}} & \textbf{I-B}: \ \mathbf{V_o = O_{23}^{\pi/4} K_{\alpha_o}^{i} O_{12}^{\phi_o}} & \textbf{I-C}: \ \mathbf{V_o = O_{23}^{\pi/4} O_{12}^{\phi_1} K_{\alpha_o}^{i} O_{12}^{\phi_2}} \\ \textbf{II-A}: \ \mathbf{V_o = O_{23}^{\pi/4} O_{12}^{\phi_o}} & \textbf{II-B}: \ \mathbf{V_o = O_{23}^{\theta_o} K_{\alpha_o}^{i} O_{23}^{\pi/4} O_{12}^{\phi_o}} & \end{array} \tag{5.1}$$

where $\sin\phi_o = \sin\theta_o = \frac{1}{\sqrt{3}}$, $\sin\phi_1 = \frac{1}{2}$, $\sin\phi_2 = \frac{1}{\sqrt{6}}$. We define the U_o as the matrix on the left of, together with the $K_{\alpha_o}^i$. In the II-A case, this is the identity matrix. We define also the O_o as the matrix on the right of the $K_{\alpha_o}^i$. In the II-A case, this is the whole matrix V_o. For case II-B, $\alpha_o = 0$ as pointed out in the previous section. For the other cases, we assume for α_o, diverse fixed values.

We illustrate in Fig. 1 the correlations among the observables for our best scenario I-A. The figures plot $\sin^2\theta_{atm}$ and I_{CP} as a function of $|V_{13}|^2$ and I_{CP} as a function of $|V_{13}|^2$, for particular values of the parameters left unconstrained in the definition of each scenario according to Table 1. The other scenarios are presented in Ref. [1]. A numerical analysis of Scenario I-A, was also done in Ref. [36]. We can conclude from Fig. 1 that a large CP invariant I_{CP} can be obtained in agreement with the allowed experimental range of the observed parameters.

In addition, we also give a numerical analysis of the stability of the different scenarios. The full neutrino mass matrix M is composed of an exact degenerate part in the form of a symmetric unitary matrix $S_o = U_o^* \cdot U_o^\dagger$, related to one of these TBM scenario schemes, and a part composed of a random perturbation Q^ε. The matrix U_o corresponds to the left part of Eq.(5.1), including the

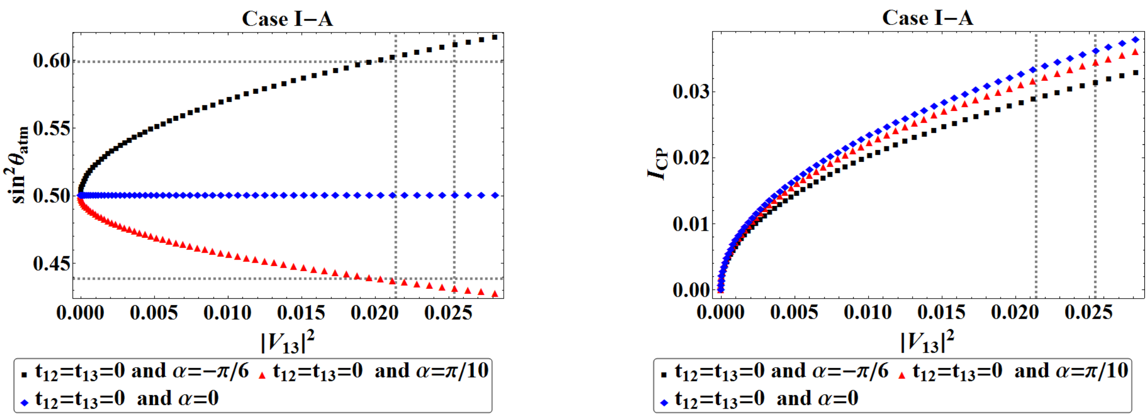

Figure 1: Plotting $\sin^2\theta_{atm}$ and the CP invariant I_{cp} as a function of $|V_{13}^2|$ for scenario I-A.

phase matrix K_α^i and will be different for each case, e.g. for case II-A, U_o is the identity matrix. The right part of Eq.(5.1) is denoted by O_o. In the case II-A, O_0 is the whole matrix V_0.

Thus, the full quasi-degenerate neutrino mass matrix is as in Eq. (3.2):

$$M = \mu\,(S_o + Q^\varepsilon)\,, \tag{5.2}$$

where Q^ε is some small complex symmetric random perturbation:

$$Q^\varepsilon \equiv \varepsilon^2\, Q\,, \quad \varepsilon^2 = \frac{\left(\Delta m_{31}^2\right)^{\exp}}{2\mu^2}\,.$$

We test the stability of lepton mixing of the different scenarios. We do not worry about the exact mass differences, with two (reasonable) exceptions: we take for ε^2 a fixed value. Taking into account the upper bound on the sum of neutrino masses as suggested by the Planck collaboration [17], obtained in a model dependent analysis, i.e., $\sum_i m_i < 0.23$ eV, one gets for the common neutrino mass $\mu \lesssim 0.08\,\mathrm{eV}$. However, if one relaxes this assumption and takes a somewhat larger value for $\mu = 0.14$, together with $\left(\Delta m_{31}^2\right)^{\exp} = 2.5\times 10^{-3}\,\mathrm{eV}^2$, we obtain $\varepsilon^2 \lesssim 0.064$, which makes ε of the order of the Cabibbo angle. These values make sure that we are in a mass range where the computed output $\Delta m_{31}^2 = O(1)\times 10^{-3}\,\mathrm{eV}^2$. We discard cases generated by the perturbation where $\left|\Delta m_{31}^2\right| < \left|\Delta m_{21}^2\right|$. Further, we do not impose any other restrictions on the random perturbation Q other than $Re(Q_{ij})$ and $Im(Q_{ij})$ to be real numbers between -1 and 1. For this analysis we do not impose any other restriction to mass differences, e.g. the experimental results for mass differences: further restrictions do not change significantly any of the plots. We are more concerned with the stability of each scenario when the degeneracy is lifted.

Our numerical analysis consists in computing the full lepton mixing matrix V, for each different mixing scenario and random $Q's$ in M, such that $V^T\cdot M\cdot V = D$ is real and positive. The lepton mixing matrix can be decomposed in the new parametrization, which allows us to compare the new $U \equiv O_{23}\cdot O_{12}\cdot K_\alpha^i$ from the perturbation, with the original U_o in the degeneracy limit, for each case in Eq.(5.1). To measure how much U differ from U_o, as well the differences between O and O_o, we

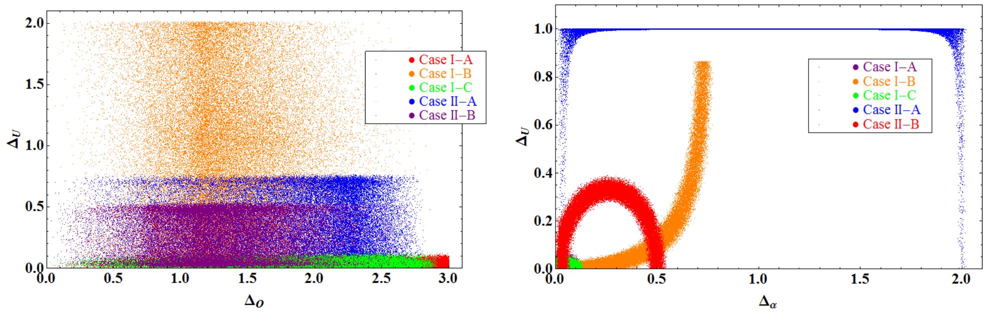

Figure 2: Plotting Δ_O versus Δ_U for the five cases identified in Eq. (5.1) with $\alpha_o = \pi/3$ (left plot). Plotting Δ_α versus Δ_U for the five cases identified in Eq. (5.1) (right plot).

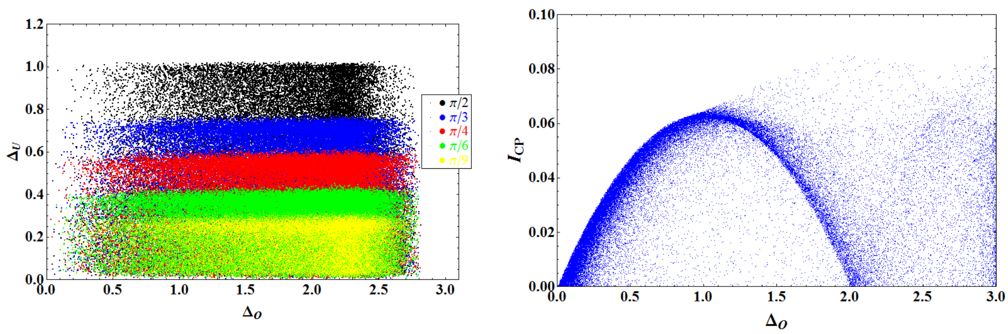

Figure 3: Plotting Δ_O versus Δ_U for the case I-B varying $\alpha_o = \pi/2$, $\pi/3$, $\pi/4$, $\pi/6$ and $\pi/9$ (left plot). Plotting the CP invariant I_{CP} as a function of Δ_O considering restricted perturbations for Q (right plot).

evaluate a quantity Δ_U and Δ_O defined by:

$$\Delta_U = \frac{1}{2}\sum\left|\left|U_{ij}\right| - \left|(U_o)_{ij}\right|\right|, \quad \Delta_O = \frac{1}{2}\sum\left|\left|O_{ij}\right| - \left|(O_o)_{ij}\right|\right|.^3 \tag{5.3}$$

Notice that this definition does not "see" the phase factors of the K^i_α of U, or of the U_o. For this, we evaluate the changing on the phases α by defining the quantity

$$\Delta_\alpha = \left||\sin\alpha| - |\sin\alpha_o|\right|, \tag{5.4}$$

that compares the phase α of the K^i_α of U, with the phase α_o of the $K^i_{\alpha_o}$ of U_o and discarding differences of π. The $II - A$ case, has no α_o phases.

In Fig. 2 we plot Δ_U as a function of Δ_O and Δ_α as a function of Δ_U, respectively, for the five scenarios. From Fig. 2 we find that the Δ_U and Δ_α of Scenarios I-A and I-C hardly suffer any change with the perturbations. This means also that these quantities do hardly depend on the parameter ε, and subsequently on the common neutrino mass as given in Eq. (5.2), which is proportional to the perturbations. We should emphasize that case I-C is an intermediate situation, somewhat artificial, because to be near of the TBM, the two phases must have very specific values. With regard to case I-B, from Fig. 3, it is interesting to note that small α leads to more stability. Case I-A is not

[3] The $1/2$ in front of Δ_O and Δ_U is a suitable normalization factor, chosen such that, e.g. in a case where the original $O_o = \mathbb{1}$ and the new O is such that $O = O_{12}$ (or any other elementary rotation) with an angle $\sin\theta_{12} = 0.2$, then also $\Delta_O \approx 0.2$, of the same order of the Cabibbo angle.

shown, since there is no apparent change of these quantities by varying α. Therefore, we focus on Case I-A. As shown in the previous section, generically, Scenario I-A has also the largest Majorana phases.

However, the results for O are quite different. From Fig. 2, we see that the perturbations generate large Δ_O contributions for all cases and in particular for scenario I-A. This situation may be improved by imposing certain restriction to Q, e.g. with some kind of symmetries. In Fig. 3, we give an example where the perturbations Q are restricted: certain elements are taken to be zero, while the imaginary part and the diagonal real part are taken to be 0.1 smaller than the others:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{x_{22}}{10} & x_{23} \\ 0 & x_{23} & \frac{x_{33}}{10} \end{pmatrix} + \frac{i}{10} \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix} \tag{5.5}$$

where the x's, y's are random real numbers varying between -1 and 1. For the initial phase α_o, we take $\alpha_o = \pi/9$. We see that most of the deviations Δ_O (from the original $O_o = \mathbb{1}$), are now around 0.2 of the order of the Cabibbo angle, and this does not affect having large values for I_{CP}.

6. Conclusions

We proposed a new parametrization for leptonic mixing of the form $V = O_{23}\, O_{12}\, K^i_\alpha \cdot O$ and we have identified several-limit cases with mixing angles in agreement with experimental results and leading to large CP violation. It turns out that if neutrinos are quasi-degenerate and Majorana, this parametrization is very useful. It may reflect some specific nature of neutrinos, suggesting that there is some major intrinsic Majorana character of neutrino mixing and CP violation in the left part of the parametrization, while the right part O may reflect that there are three neutrino families with small mass differences, resulting in small mixing comparable to the quark sector, of the order of the Cabibbo angle. Thus, from this point of view, the dominant contribution for large neutrino mixing must come from the Majorana character of neutrinos. This new parametrization enables an alternative perspective of large leptonic CP violation and shows interesting aspects that were less clear in the standard parametrization. From the limit cases studied, the scenario I-A was the most appealing. It only needs 2 extra parameters to fit the experimental results on lepton mixing and provides large Dirac-CP violation and large values for the Majorana-CP violating phases. These results are derived explicitly from the form of the new parametrization.

Furthermore, we also studied the stability of each scenario. We analyzed how much U_o and O_o, i.e. the left and the right part of the parametrization, in the limit of exact degeneracy, differ from U and O after introducing a random perturbation. We concluded that the left part of the parametrization behaves quite differently for each scenario. It turns out that, with regard to U_o, the scenario I-A is the most stable. As for the right part O of the parametrization, the perturbations generate large contributions for all cases. However, we have shown how to improve this situation, by imposing certain restrictions on the allowed perturbations.

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