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Status of Chiral-Scale Perturbation Theory

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Chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$ has been proposed as an alternative to chiral $SU(3)_L \times SU(3)_R$ perturbation theory which explains the $\Delta I = 1/2$ rule for kaon decays. It is based on a low-energy expansion about an infrared fixed point in three-flavor QCD. In $\chi\mathrm{PT}_\sigma$, quark condensation $\langle \bar{q}q \rangle_{\mathrm{vac}} \neq 0$ induces nine Nambu-Goldstone bosons: π, K, η and a QCD dilaton σ which we identify with the $f_0(500)$ resonance. Partial conservation of the dilatation and chiral currents constrains low-energy constants which enter the effective Lagrangian of $\chi\mathrm{PT}_\sigma$. These constraints allow us to obtain new phenomenological bounds on the dilaton decay constant via the coupling of σ/f_0 to pions, whose value is known precisely from dispersive analyses of $\pi\pi$ scattering. Improved predictions for $\sigma \to \gamma\gamma$ and the σNN coupling are also noted. To test $\chi\mathrm{PT}_\sigma$ for kaon decays, we revive a 1985 proposal for lattice methods to be applied to $K \to \pi$ *on-shell*.

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1. Approximate Scale Invariance in Low-Energy QCD

In the low-energy regime of QCD with heavy quarks t, b, c decoupled, the relevance of scale (dilatation) invariance is determined by the trace anomaly [1]–[4] of the resulting 3-flavor theory:[1]

$$\theta^\mu_\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu} G^{a\mu\nu} + \big(1+\gamma_m(\alpha_s)\big) \sum_{q=u,d,s} m_q \bar{q} q \,. \tag{1.1}$$

Depending on the infrared behaviour of β, there are only two realistic scenarios (Fig. 1 (A)):

1. If β remains negative and non-zero, possibly diverging linearly at large α_s, scale invariance is explicitly broken by θ^μ_μ being large *as an operator*. There is *no hint* of approximate scale invariance: quantities such as the nucleon mass $M_N = \langle N|\theta^\mu_\mu|N\rangle$ are generated almost entirely by the gluonic term in (1.1). Then conventional chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT$_3$ is the appropriate low-energy effective theory for QCD amplitudes expanded in powers of $O(m_K)$ external momenta and light quark masses $m_{u,d,s} = O(m_K^2)$.

2. If β vanishes when α_s runs non-perturbatively to an infrared fixed point α_{IR}, the gluonic term $\sim G^a_{\mu\nu} G^{a\mu\nu}$ in (1.1) is absent and the dilatation current $D_\mu = x^\nu \theta_{\mu\nu}$ becomes conserved in the limit of vanishing quark masses:

$$\begin{aligned} \partial^\mu D_\mu\big|_{\alpha_s=\alpha_{\mathrm{IR}}} = \theta^\mu_\mu\big|_{\alpha_s=\alpha_{\mathrm{IR}}} &= \big(1+\gamma_m(\alpha_{\mathrm{IR}})\big) \sum_{q=u,d,s} m_q \bar{q} q \\ &\to 0 \,,\ SU(3)_L \times SU(3)_R \text{ limit}. \end{aligned} \tag{1.2}$$

   Although the Hamiltonian preserves dilatations in this limit, *the vacuum state is not scale invariant* due to the formation of a quark condensate $\langle \bar{q}q\rangle_{\mathrm{vac}} \neq 0$. As a result, both chiral $SU(3)_L \times SU(3)_R$ and scale symmetry are realized in the Nambu-Goldstone (NG) mode and the spectrum contains nine massless bosons: π, K, η and a 0^{++} QCD dilaton σ. Non-NG bosons remain massive *despite the vanishing of* θ^μ_μ and have their scale set by $\langle \bar{q}q\rangle_{\mathrm{vac}}$. The relevant low-energy expansion involves a combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{\mathrm{IR}} \,, \tag{1.3}$$

   and leads to a new effective theory χPT$_\sigma$ of approximate chiral-scale symmetry [5, 6]. In this scenario, the dilaton mass is set by m_s, so the natural candidate for σ is the $f_0(500)$ resonance, a broad 0^{++} state whose complex pole mass has real part $\lesssim m_K$ [7, 8, 9].

Until now, scenario 1 has been the generally accepted view, but we have observed [5, 6] that χPT$_\sigma$ offers several advantages over χPT$_3$: it explains the mass and width of $f_0(500)$, produces convergent chiral expansions as a result of σ/f_0 being promoted to the NG sector, and most importantly, explains the $\Delta I = 1/2$ rule for non-leptonic K decays (Fig. 1 (B)).

Because approximate scale symmetry is included, the effective Lagrangian for χPT$_\sigma$ (Sec. 2) contains several new low-energy constants (LECs) yet to be determined precisely from data. Of particular interest is the dilaton decay constant F_σ given by $m_\sigma^2 F_\sigma = -\langle \sigma|\theta^\mu_\mu|\mathrm{vac}\rangle$. If F_σ is roughly 100

[1] Here, $G^a_{\mu\nu}$ is the gluon field strength, $\alpha_s = g_s^2/4\pi$ is the strong running coupling, and $\beta = \mu^2 \partial\alpha_s/\partial\mu^2$ and $\gamma_m = \mu^2 \partial \ln m_q/\partial\mu^2$ refer to a mass-independent renormalization scheme with scale μ.

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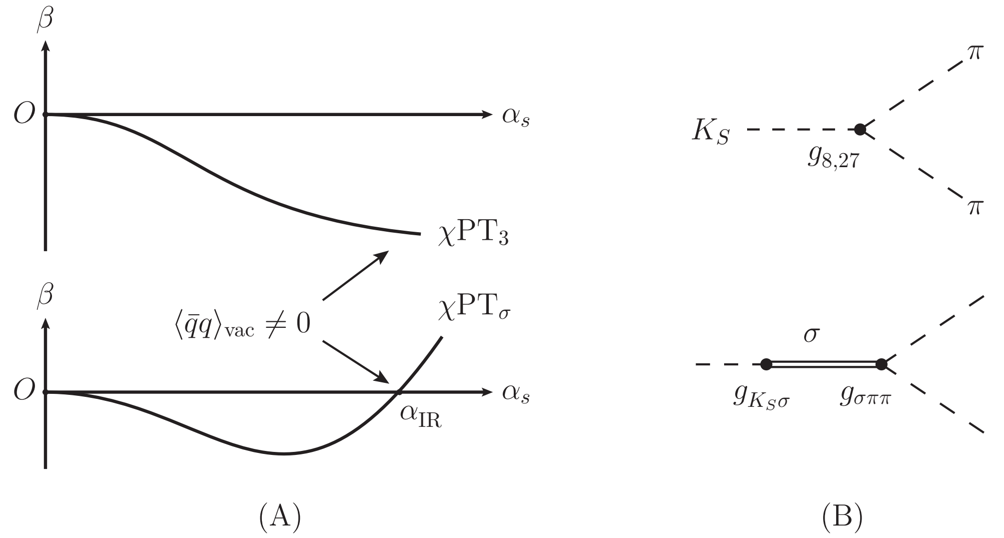

Figure 1: (A) Scenarios for the β function in three-flavor QCD, with corresponding low-energy expansions. In the absence of an infrared fixed point α_{IR} (top diagram), there is no approximate scale invariance and chiral $SU(3)_L \times SU(3)_R$ perturbation theory $\chi\mathrm{PT}_3$ is relevant at low-energies. If α_{IR} exists (bottom diagram), quark condensation $\langle\bar{q}q\rangle_{\mathrm{vac}} \neq 0$ implies that the NG spectrum contains a QCD dilaton σ, and $\chi\mathrm{PT}_3$ *must* be replaced by chiral-scale perturbation theory $\chi\mathrm{PT}_\sigma$. (B) Diagrams for $K \to \pi\pi$ decay in lowest-order $\chi\mathrm{PT}_\sigma$. The dilaton pole diagram is responsible for the dominant $\Delta I = 1/2$ amplitude.

MeV, scale breaking by the vacuum can generate large masses such as $m_N \approx F_\sigma g_{\sigma NN}$ (Goldberger-Treiman relation for dilatons [10]) for m_σ small. The imprecise value of F_σ in our previous work [5, 6] arose from large uncertainties in the phenomenological value of $g_{\sigma NN}$ [11, 12].

We circumvent this difficulty in Secs. 3 and 4. First, we find new constraints on LECs in the $\chi\mathrm{PT}_\sigma$ effective Lagrangian by requiring full consistency with the dilatation and chiral currents being conserved in the limit (1.2). These constraints allow us to determine F_σ from the $\sigma\pi\pi$ coupling, whose value is known to remarkable precision from dispersive analyses [7, 8, 9] of $\pi\pi$ scattering. Then we obtain improved predictions for the non-perturbative Drell-Yan ratio

$$R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-) \quad \text{at } \alpha_{\mathrm{IR}}\,, \tag{1.4}$$

as well as the σNN coupling.

In Sec. 5, we resurrect an old proposal [13] to apply lattice QCD for $K \to \pi$ *on-shell* to determine the couplings $g_{8,27}$ in Fig. 1 (B). Comments on the validity of $\chi\mathrm{PT}_\sigma$ are reviewed in Sec. 6.

2. Chiral-Scale Lagrangian

For strong interactions, the most general effective Lagrangian of $\chi\mathrm{PT}_\sigma$ is of the form

$$\mathscr{L}_{\chi\mathrm{PT}_\sigma} = :\mathscr{L}_{\mathrm{inv}}^{d=4} + \mathscr{L}_{\mathrm{anom}}^{d>4} + \mathscr{L}_{\mathrm{mass}}^{d<4}:, \tag{2.1}$$

where

$$d_{\mathrm{anom}} = 4 + \gamma_{G^2}(\alpha_s) \qquad \text{and} \qquad d_{\mathrm{mass}} = 3 - \gamma_m(\alpha_s) \tag{2.2}$$

are the respective scaling dimensions of $G^a_{\mu\nu}G^{a\mu\nu}$ and $\bar{q}q$. In lowest order (LO) of the chiral-scale expansion, we have $\gamma_m = \gamma_m(\alpha_{\text{IR}})$ and

$$\gamma_{G^2}(\alpha_s) \equiv \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s = \beta'(\alpha_{\text{IR}}) + O(\alpha_s - \alpha_{\text{IR}})\,, \tag{2.3}$$

so the resulting terms in (2.1) are

$$\begin{aligned}
\mathscr{L}^{d=4}_{\text{inv,LO}} &= \{c_1\mathscr{K} + c_2\mathscr{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\}e^{2\sigma/F_\sigma}\,, \\
\mathscr{L}^{d>4}_{\text{anom,LO}} &= \{(1-c_1)\mathscr{K} + (1-c_2)\mathscr{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\}e^{(2+\beta')\sigma/F_\sigma}\,, \\
\mathscr{L}^{d<4}_{\text{mass,LO}} &= \text{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,,
\end{aligned} \tag{2.4}$$

where

$$\mathscr{K} = \tfrac{1}{4}F_\pi^2\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \qquad \text{and} \qquad \mathscr{K}_\sigma = \tfrac{1}{2}(\partial_\mu\sigma)^2\,. \tag{2.5}$$

As $\alpha_s \to \alpha_{\text{IR}}$, the gluonic anomaly vanishes, so $\mathscr{L}_{\text{anom}} = O(\partial^2, M)$ and we must set $c_4 = O(M)$. Vacuum stability in the σ direction about $\sigma = 0$ (no tadpoles) implies

$$\begin{aligned}
4c_3 + (4+\beta')c_4 &= -(3-\gamma_m)\big\langle\text{Tr}(MU^\dagger + UM^\dagger)\big\rangle_{\text{vac}} \\
&= -(3-\gamma_m)F_\pi^2\big(m_K^2 + \tfrac{1}{2}m_\pi^2\big)\,,
\end{aligned} \tag{2.6}$$

so c_3 is also $O(M)$. Expanding (2.4) about $\sigma = 0$ and $U = I$ yields the $\sigma\pi\pi$ coupling

$$\mathscr{L}_{\sigma\pi\pi} = \big\{\big[2 + (1-c_1)\beta'\big]|\partial\boldsymbol{\pi}|^2 - (3-\gamma_m)m_\pi^2|\boldsymbol{\pi}|^2\big\}\sigma/(2F_\sigma)\,, \tag{2.7}$$

while the corresponding $\sigma\pi\pi$ vertex for an on-shell dilaton is

$$g_{\sigma\pi\pi} = -\frac{1}{2F_\sigma}\Big\{\big[2 + (1-c_1)\beta'\big]m_\sigma^2 + 2\big[1 - \gamma_m - (1-c_1)\beta'\big]m_\pi^2\Big\}\,. \tag{2.8}$$

3. Effective Energy-Momentum Tensor and its Trace

In any field theory, the energy-momentum tensor can be identified by adding a gravitational source field $g_{\mu\nu}(x)$ coupled to matter fields in a generally covariant fashion. In χPT_σ, this amounts to the substitution

$$\mathscr{L}_{\chi\text{PT}_\sigma}[U, U^\dagger, \sigma] \to \mathscr{L}_{\chi\text{PT}_\sigma}[U, U^\dagger, \sigma, g_{\mu\nu}]\,, \tag{3.1}$$

where the new effective Lagrangian must be constructed in terms of generally covariant operators. Then the energy-momentum tensor is defined via the variation

$$\theta_{\mu\nu}(x) = 2\left[\frac{\delta}{\delta g^{\mu\nu}(x)}\sqrt{-g}\mathscr{L}[U, U^\dagger, \sigma, g_{\mu\nu}]\right]_{g_{\mu\nu} = \eta_{\mu\nu}}\,, \tag{3.2}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and $\eta_{\mu\nu}$ is the flat Minkowski metric. Generalising Donoghue and Leutwyler [14], we obtain the lowest order result

$$\begin{aligned}
\theta_{\mu\nu} = &\big[\tfrac{1}{2}F_\pi^2\text{Tr}\big(\partial_\mu U \partial_\nu U^\dagger\big) - g_{\mu\nu}\mathscr{K}\big]\big[c_1 e^{2\sigma/F_\sigma} + (1-c_1)e^{(2+\beta')\sigma/F_\sigma}\big] \\
&+ \big(\partial_\mu\sigma\partial_\nu\sigma - g_{\mu\nu}\mathscr{K}_\sigma\big)\big[c_2 e^{2\sigma/F_\sigma} + (1-c_2)e^{(2+\beta')\sigma/F_\sigma}\big] \\
&- g_{\mu\nu}\text{Tr}\big(MU^\dagger + UM^\dagger\big)e^{(3-\gamma_m)\sigma/F_\sigma} - g_{\mu\nu}e^{4\sigma/F_\sigma}\big(c_3 + c_4 e^{\beta'\sigma/F_\sigma}\big)\,.
\end{aligned} \tag{3.3}$$

The trace of (3.3) involves *scale invariant* operators like $\mathrm{Tr}\big(\partial_\mu U \partial^\mu U^\dagger\big)e^{2\sigma/F_\sigma}$ which obscure the connection between the scale invariance and a conserved dilatation current D_μ. To remedy this, we "improve" $\theta_{\mu\nu}$ [15] by adding a term

$$I_{\mu\nu} = \frac{F_\sigma^2}{6}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\left[c_2 e^{2\sigma/F_\sigma} + \frac{2(1-c_2)}{2+\beta'}e^{(2+\beta')\sigma/F_\sigma}\right], \tag{3.4}$$

such that the trace of

$$\theta_{\mu\nu}\big|_{\mathrm{eff}} = \theta_{\mu\nu} + I_{\mu\nu}\,, \tag{3.5}$$

is given entirely in terms of explicit scale-breaking operators $\mathscr{L}_d$ of scale dimension d:

$$\partial^\mu D_\mu|_{\mathrm{eff}} = \theta^\mu_\mu\big|_{\mathrm{eff}} = \sum_d (d-4)\mathscr{L}_d\,. \tag{3.6}$$

Explicitly, the improved trace is

$$\begin{aligned}\theta^\mu_\mu\big|_{\mathrm{eff}} &= \beta'\mathscr{L}^{d>4}_{\mathrm{anom}} - (1+\gamma_m)\mathscr{L}^{d<4}_{\mathrm{mass}} \\ &= \beta'\big\{(1-c_1)\mathscr{K} + (1-c_2)\mathscr{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\big\}e^{(2+\beta')\sigma/F_\sigma} \\ &\quad - (1+\gamma_m)\mathrm{Tr}(MU^\dagger + UM^\dagger)e^{(3-\gamma_m)\sigma/F_\sigma}\,. \end{aligned}\tag{3.7}$$

It vanishes in the chiral-scale limit (1.2) only if the low-energy constants associated with $d>4$ operators satisfy

$$c_1 = c_2 = 1\,, \qquad \text{for } m_{u,d,s} \to 0 \text{ and } \alpha_s \to \alpha_{\mathrm{IR}}\,, \tag{3.8}$$

in addition to the condition $c_4 = O(M)$ required by tadpole cancellation (2.6). Note that the condition $c_1 \to 1$ in (3.8) ensures that chiral currents have vanishing anomalous dimensions. We can summarise these LO conditions by writing

$$c_i = 1 + O(M)\,, \qquad i = 1,2\,, \tag{3.9}$$

where the $O(M)$ term is a linear superposition of $O(p^2, M)$ operators and associated LECs.

4. Improved Predictions

An immediate consequence of the constraint (3.9) is that the $\sigma\pi\pi$ coupling for an on-shell dilaton (2.8) takes a particularly simple form

$$g_{\sigma\pi\pi} = -\frac{1}{F_\sigma}\big[m_\sigma^2 + (1-\gamma_m)m_\pi^2\big]\,, \qquad \text{where } -1 \le 1-\gamma_m < 2\,. \tag{4.1}$$

Since the narrow-width approximation is valid in lowest order $\chi\mathrm{PT}_\sigma$ [6], we have

$$\Gamma_{\sigma\pi\pi} = \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma}\sqrt{1 - 4m_\pi^2/m_\sigma^2}\,, \tag{4.2}$$

and this allows us to obtain bounds on F_σ from dispersive analyses of $\pi\pi$ scattering based on the Roy equations. For example, the f_0/σ's mass and width from [7]

$$m_\sigma = 441^{+16}_{-8}\ \mathrm{MeV}\,, \qquad \Gamma_{\sigma\pi\pi} = 544^{+18}_{-25}\ \mathrm{MeV}\,, \tag{4.3}$$

constrain F_σ to lie within the interval 44 MeV $\leq F_\sigma \leq$ 61 MeV, where we have allowed $1-\gamma_m$ to vary according to (4.1). For the moment, we assume that NLO corrections are not a problem.

With F_σ fixed in this manner, we can now use the Golberger-Treiman relation for dilatons [10] to *predict* the value for the σNN coupling. We find $16 \leq g_{\sigma NN} \leq 21$, which is somewhat larger than previous phenomenological determinations [11, 12]. Another important application concerns $\sigma \to \gamma\gamma$, where an analysis [5, 6] of the electromagnetic trace anomaly in $\chi\mathrm{PT}_\sigma$ relates the $\sigma\gamma\gamma$ coupling to (1.4):

$$g_{\sigma\gamma\gamma} = \frac{2\alpha}{3\pi F_\sigma}\Big(R_{\mathrm{IR}} - \tfrac{1}{2}\Big)\,. \tag{4.4}$$

By fixing $g_{\sigma\gamma\gamma}$ from the di-photon width $\Gamma_{\sigma\gamma\gamma} = 2.0 \pm 0.2$ keV [16], we find $2.4 \leq R_{\mathrm{IR}} \leq 3.1$, which is to be compared with our previous estimate $R_{\mathrm{IR}} \approx 5$ [5, 6].

5. Proposal to test $K \to \pi$ on the Lattice

The key idea [13] is to keep both K and π on shell and allow $O(m_K)$ momentum transfers.

The lowest-order diagrams for the decay $K \to \pi\pi$ in Fig. 1 (B) are derived from an effective weak $\chi\mathrm{PT}_\sigma$ Lagrangian [5, 6]

$$\mathscr{L}_{\mathrm{weak}} = Q_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27} Q_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} + Q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma} + \mathrm{h.c.} \tag{5.1}$$

which reduces to the standard $\chi\mathrm{PT}_3$ Lagrangian

$$\mathscr{L}_{\mathrm{weak}}\big|_{\sigma=0} = g_8 Q_8 + g_{27} Q_{27} + Q_{mw} + \mathrm{h.c.} \tag{5.2}$$

in the limit $\sigma \to 0$. Eqs. (5.1) and (5.2) contain an octet operator [17]

$$Q_8 = J^\mu_{13} J_{\mu 21} - J^\mu_{23} J_{\mu 11}\ , \quad J^\mu_{ij} = (U\partial^\mu U^\dagger)_{ij} \tag{5.3}$$

the U-spin triplet component [13, 18] of a **27** operator

$$Q_{27} = J^\mu_{13} J_{\mu 21} + \tfrac{3}{2} J^\mu_{23} J_{\mu 11} \tag{5.4}$$

and a weak mass operator [19]

$$Q_{mw} = \mathrm{Tr}(\lambda_6 - i\lambda_7)\big(g_M M U^\dagger + \bar{g}_M U M^\dagger\big)\,. \tag{5.5}$$

Powers of e^{σ/F_σ} are used to adjust the operator dimensions of Q_8, Q_{27}, and Q_{mw} in (5.1), with octet quark-gluon operators allowed to have differing dimensions at α_{IR}.

In 1985, it was observed [13] that the isospin-$\frac{1}{2}$ term Q_{mw} in Eq. (5.2), when combined with the strong mass term, would be removed by vacuum realignment and therefore could not help solve the $\Delta I = 1/2$ puzzle. In $\chi\mathrm{PT}_\sigma$, the outcome is different [5, 6] due to the σ dependence of the Q_{mw} term in Eq. (5.1). Provided there is a mismatch between the weak mass operator's dimension $(3-\gamma_{mw})$ and the dimension $(3-\gamma_m)$ of $\mathscr{L}_{\mathrm{mass}}$, the σ dependence of $Q_{mw} e^{(3-\gamma_{mw})/F_\sigma}$ cannot be eliminated by a chiral rotation. As a result, there is a residual interaction $\mathscr{L}_{K_S\sigma} = g_{K_S\sigma} K_S \sigma$ which mixes K_S and σ in *lowest* $O(p^2)$ order[2]

$$g_{K_S\sigma} = (\gamma_m - \gamma_{mw})\mathrm{Re}\{(2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M\} F_\pi / F_\sigma \tag{5.6}$$

[2] We have corrected a factor of 2 in the formula for the $K_S\sigma$ coupling in our original papers [5, 6].

and produces the $\Delta I = 1/2$ σ-pole amplitude of Fig. 1 (B).

The χPT$_3$ analysis of 1985 [13] included a suggestion that kaon decays be tested by applying lattice QCD to the weak process $K \to \pi$, with *both* K and π on shell. It was made at a time when low-lying scalar resonances ($\varepsilon(700)$ before 1974, $f_0(500)$ since 1996) were thought not to exist.

This proposal now needs to be taken seriously because:

- Lattice calculations are much easier with only two particles on shell instead of the three in $K \to \pi\pi$ (all on shell) being analysed by the RBC/UKQCD collaborations [20, 21].

- The 1985 analysis is easily extended to χPT$_\sigma$ by including σ/f_0 pole amplitudes in chiral Ward identities connecting on-shell $K \to \pi\pi$ to $K \to \pi$ on shell. The no-tadpoles theorem

$$\langle K|\mathscr{H}_{\text{weak}}|\text{vac}\rangle = O\big(m_s^2 - m_d^2\big)\,,\ K \text{ on shell}\,, \tag{5.7}$$

  remains valid.

- The lattice result for $K \to \pi\pi$ on-shell will not distinguish $\Delta I = 1/2$ contributions from the g_8 contact diagram and the σ/f_0 pole diagram in Fig. 1 (B). A lattice calculation of $K \to \pi$ on shell would measure g_8 (and g_{27}) directly, with no interference from σ/f_0 poles. Then we would *finally* learn whether g_8 is unnaturally large or not.

A key feature of the proposal is that the operator in the on-shell amplitude $\langle\pi|[F_5, \mathscr{H}_{\text{weak}}]|K\rangle$ necessarily carries *non-zero* momentum $q^\mu = O(m_K)$. For either χPT$_\sigma$ or χPT$_3$, the $K \to \pi$ amplitude can be evaluated in the range

$$-m_K^2 \lesssim q^2 \leqslant \big(m_K - m_\pi\big)^2\,. \tag{5.8}$$

We highlight the point $q^\mu \neq 0$ because since 1985, there has been a widespread misconception in the literature[3] that the analysis [13] involved setting $q^\mu = 0$ as in [19], with the pion in $K \to \pi$ sent off shell via an interpolating operator. There was and is no reason for this. For example, when writing a soft meson theorem for $\Sigma \to p\pi$, it is not necessary to force one of the baryons off shell.

6. Issues

When considering the validity of χPT$_\sigma$, it is important to avoid any presumption that dimensional transmutation necessarily implies that θ^μ_μ is large and $\neq 0$. Implicit in this intuition is a prejudice that scale invariance cannot be strongly broken via the vacuum when $\theta^\mu_\mu \to 0$. If the dilaton is a true NG boson, i.e. $m_\sigma \to 0$ with $F_\sigma \neq 0$ for $\theta^\mu_\mu \to 0$, it can couple to mass insertion terms in Callan-Symanzik equations and cause them to be *non-zero* in the zero-mass limit. Then Green's functions do not exhibit the power-law scaling expected for manifestly scale-invariant field theories.

This point is illustrated for the quark condensate in Fig. 1 (A). In scenario 1 (top diagram), the running of α_s is driven by the presence of quantities like $\langle\bar{q}q\rangle_{\text{vac}}$ (a mechanism often cited in papers on walking gauge theories [22]). In scenario 2 (bottom diagram), the running coupling freezes at α_{IR}, where the condensate is a *scale-breaking property of the vacuum*.

[3] We thank the final referee of our long paper [6] for drawing our attention to this.

Lattice investigations of IR fixed points inside the conformal window $8 \lesssim N_f \leqslant 16$ all depend on naive scaling of Green's functions [22], so they correspond to *scale-invariant vacua*. A recent lattice study [23] of the running of α_s for two flavors with *no* naive scaling suggests that it freezes: the fixed point realises scale invariance in NG mode, i.e. with a scale-breaking vacuum. That is what $\chi\mathrm{PT}_\sigma$ assumes for three flavors.

The term "dilaton" often refers to a spin-0 particle or resonance which couples to $\theta_{\mu\nu}$ and acquires its mass "spontaneously" due to self interactions. Originally, this idea concerned a scalar component of gravity [24], but now it is a key ingredient of dynamical electroweak symmetry breaking (pp. 198 and 1622-3, PDG tables [9]). This approximates theories with *scale-invariant vacua*, as is evident in walking technicolor. Therefore it has *nothing* to do with our dilaton [25].

It is well known that a resonance cannot be represented by a local interpolating operator, so is the fact that $\sigma/f_0(500)$ has a finite width a problem for $\chi\mathrm{PT}_\sigma$? The answer is "no" because $\chi\mathrm{PT}_\sigma$ is an expansion in powers and logarithms of $m_{\pi,K,\eta,\sigma}$ with coefficients determined in the *exact* chiral-scale limit (1.2) where σ has zero width [6]. In any perturbation theory, decay rates are calculated that way.

A related remark concerns what is current best practice for scenario 1. The resonance $f_0(500)$ is treated as a member of the non-NG sector with an accidentally small mass. It causes $\chi\mathrm{PT}_3$ to produce divergent expansions for amplitudes involving $f_0(500)$ poles: the radius of convergence is too small. Instead, these amplitudes are approximated dispersively via contributions from the dominant $f_0(500)$ poles with corrections from nearby thresholds, subject to exact chiral $SU(3)\times SU(3)$ constraints such as Adler zeros. One would certainly not use local fields in this framework.

However $\chi\mathrm{PT}_\sigma$ is a more ambitious theory. Having promoted σ/f_0 to the NG sector, we expect convergent asymptotic expansions for *all* mesonic amplitudes (scenario 2). The NLO corrections are still being worked out, but a first guess is to set all multi-dilaton vertices to zero. That is equivalent to adding the simplest dilaton diagrams to all $\chi\mathrm{PT}_3$ diagrams. It seems to produce amplitudes very similar to those of the dispersive approximations of scenario 1.

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