PROCEEDINGS OF SCIENCE

$a_0(980)$ as a companion pole of $a_0(1450)$

Thomas Wolkanowski*
Institute for Theoretical Physics, Goethe University, D-60438 Frankfurt am Main, Germany
E-mail: wolkanowski@th.physik.uni-frankfurt.de

Francesco Giacosa
Institute for Theoretical Physics, Goethe University, D-60438 Frankfurt am Main, Germany
Institute of Physics, Jan Kochanowski University, PL-25406 Kielce, Poland
E-mail: fgiacosa@ujk.edu.pl

Light scalar hadrons are often understood as dynamically generated resonances. These arise as 'companion poles' in the propagators of $q\bar{q}$ seed states when accounting for meson-loop contributions to the self-energies of the latter. Following this idea, we demonstrate that for the scalar–isovector state $a_0(1450)$ the full one-loop propagator has two poles: a pole of the seed state $a_0(1450)$ and a companion pole corresponding to $a_0(980)$. The positions of these poles are studied by varying the relative coupling strength between the non-derivative and derivative parts of the interactions.

The 8th International Workshop on Chiral Dynamics
29 June 2015 - 03 July 2015
Pisa, Italy

*Speaker.

PoS(CD15)131

1. Introduction

Intense research during the past decades has demonstrated that the majority of mesons can be understood as being predominantly $q\bar{q}$ states [1]. However, various unconventional mesonic states such as glueballs, hybrids, and four-quark states are expected [2]. Along this line, a concept of 'dynamically generated' states was put forward in Refs. [3, 4, 5, 6]. Allthough there is not a generally accepted definition of dynamical generation [7], other versions of this idea can also be found in Refs. [8, 9, 10, 11].

The general idea is summarized in the following: Consider, for instance, a single seed state, *e.g.* a $q\bar{q}$ meson with certain quantum numbers. This state interacts with other mesons, giving rise to loop contributions in the corresponding self-energy and dressing its own full propagator. These contributions shift the corresponding pole of the seed state which moves away from the real axis and follows a certain trajectory in the appropriate unphysical Riemann sheet [12]. Moreover, new poles may appear. The latter are sometimes denoted as companion poles. If one of them happens to be situated sufficiently close to the physical region, *i.e.*, the real axis, it could correspond to a dynamically generated resonance. As a consequence, we are left with two resonances emerging from a single seed state. In this work, we aim to discuss this mechanism in the context of the resonance $a_0(980)$.

From what was found for example in Refs. [13, 14, 15, 16] the scalar resonances $f_0(1370)$, $f_0(1500)$, $K_0^*(1430)$, and $a_0(1450)$ seem to be predominantly ordinary $q\bar{q}$ states. On the other hand, the light scalar states $f_0(500)$, $f_0(980)$, $K_0^*(800)$, and $a_0(980)$ are (most likely) predominantly something different (see *e.g.* Refs. [10, 14, 17, 18, 19, 20, 21, 22] and refs. therein). In fact, we have proven in a recent publication [23] that for the heavy scalar–isovector seed state $a_0(1450)$, the coupling of this state to $\pi\eta$, $K\bar{K}$, and $\pi\eta'$ is capable to dynamically generate the light state $a_0(980)$. We will shortly outline the hadronic model that was applied there, which includes meson interactions via derivative and non-derivative couplings. We will then study the dependence of the pole structure with respect to the the relative coupling strength between the non-derivative and derivative parts of the interactions.

These proceedings are based on Ref. [23]. Our units are $\hbar = c = 1$. The metric tensor is $\eta_{\mu\nu} = diag(+,-,-,-)$.

2. Formalism

Following the earlier work of Ref. [3], some seminal studies investigated the scalar sector in a unitarized quark model by including meson-loop contributions [5, 6]. They showed that meson-loop effects may serve to explain the existence of the light scalar mesons. As can be seen from our discussion in Ref. [23], the situation is however somewhat inconclusive regarding the number and the location of propagator poles and how to assign them to physical states. After reviewing the general approach, we present our way to model the scalar–isovector sector.

The main goal is the determination of the inverse propagator of a resonance after applying a Dyson resummation of loop contributions to the self-energy:

$$\Delta^{-1}(s) = s - m_0^2 - \Pi(s) \; . \tag{2.1}$$

Here, m_0 is the bare mass of the seed state and $\Pi(s) = \sum_i \Pi_i(s)$ is the self-energy. The sum runs over the loops emerging from the coupling of the resonance to various mesons. The real part of $\Pi(s)$ on the real axis is related to the imaginary part by the dispersion relation

$$Re\Pi(s) = \frac{1}{\pi} \fint \mathrm{d}s' \, \frac{-Im\Pi(s')}{s-s'} \; . \tag{2.2}$$

The actual modeling occurs in the particular expression of the imaginary part of $\Pi_i(s)$. According to the optical theorem, it corresponds to the partial decay width of the resonance into mesons in channel i, see Sec. 3. Furthermore, a form factor is usually introduced,

$$F_i(s) = \exp[-k_i^2(s)/\Lambda^2] \; , \tag{2.3}$$

where Λ is a cutoff parameter and $k_i(s)$ is the absolute value of the three-momentum of the decay particles in the rest frame of the resonance:

$$k_i(s) = \frac{1}{2\sqrt{s}} \sqrt{s^2 + (m_{i1}^2 - m_{i2}^2)^2 - 2(m_{i1}^2 + m_{i2}^2)s} \; . \tag{2.4}$$

Here, m_{i1}, m_{i2} are the masses of the decay products, *i.e.*, in our case the pseudoscalar mesons. The function $F_i(s)$ guarantees that the imaginary part of $\Pi(s)$ vanishes sufficiently fast for $s \to \infty$.

The self-energy on the unphysical sheet(s), $\Pi^c(s)$, is obtained by analytic continuation:

$$Disc\Pi(s) \;=\; 2i \lim_{\varepsilon \to 0^+} \sum_i Im\Pi_i(s+i\varepsilon) \; , \quad s \in \mathbb{R} \; , \tag{2.5}$$

$$\Pi^c(s) \;=\; \Pi(s) + Disc\Pi(s) \; . \tag{2.6}$$

In this work only the three sheets nearest to the physical region will be regarded (in the standard notation $\pi\eta \leftrightarrow$II, $K\bar{K} \leftrightarrow$III, $\pi\eta' \leftrightarrow$VI).

3. A simple effective model with derivative interactions

3.1 Lessons from previous works

A first attempt to incorporate the mentioned mechanism of dynamical generation for the scalar states with $I = 1$ in a consistent scheme [15, 16, 24] was presented in Refs. [25, 26]. There, the seed state was assigned to be (in the mass region of) the $a_0(1450)$, and the s-dependence of the amplitudes was completely neglected (apart from the cutoff dependence), yielding a width of the seed state which is too small. Moreover, *no* additional pole for the $a_0(980)$ was dynamically generated.

It was then demonstrated in Ref. [23] that it is however possible to obtain a narrow resonance with mass around 1 GeV, the pole coordinates of which fit quite well with those of the physical $a_0(980)$ resonance, and *simultaneously* obtain a pole for the seed state in agreement with that for the $a_0(1450)$ [1]. An important requirement seems to be the inclusion of s-dependent amplitudes and derivative interaction terms in the Lagrangian, respectively.

3.2 Effective model with both non-derivative and derivative interactions

We now consider the effective model for the isovector states from Ref. [23] which contains non-derivative and derivative interactions. The Lagrangian is given by the sum of the following terms:

$$\begin{aligned} \mathscr{L}_{a_0\eta\pi} &= A_1 a_0^0 \eta \pi^0 + B_1 a_0^0 \partial_\mu \eta \partial^\mu \pi^0 \,, \\ \mathscr{L}_{a_0\eta'\pi} &= A_2 a_0^0 \eta' \pi^0 + B_2 a_0^0 \partial_\mu \eta' \partial^\mu \pi^0 \,, \\ \mathscr{L}_{a_0 K\bar{K}} &= A_3 a_0^0 (K^0 \bar{K}^0 - K^- K^+) + B_3 a_0^0 (\partial_\mu K^0 \partial^\mu \bar{K}^0 - \partial_\mu K^- \partial^\mu K^+) \,. \end{aligned} \tag{3.1}$$

Then, Eq. (3.1) gives rise to the following s-dependent amplitudes:

$$\mathscr{M}_i^{eff}(s) = \left[A_i - \frac{1}{2} B_i \left(s - m_{i1}^2 - m_{i2}^2 \right) \right] F_i(s) \,, \tag{3.2}$$

where we have already included a regularization function $F_i(s)$ as defined in Sec. 2.

The imaginary part of the one-loop self-energy[1] is computed by using the optical theorem,

$$Im\Pi_i(s) = -\sqrt{s}\,\Gamma_i^{tree}(s) = -\frac{k_i(s)}{8\pi\sqrt{s}} |{-i}\mathscr{M}_i^{eff}(s)|^2 \Theta(s - s_{th,i}) \,, \tag{3.3}$$

and the real part comes from the dispersion relation in Eq. (2.2). The step function ensures that the decay channel i contributes only when the squared energy of the resonance exceeds the threshold value $s_{th,i}$. Notice that from a careful analysis we showed the necessity to introduce subtractions that are not visible here – for a detailed presentation of this issue see Ref. [23].

Our fitting procedure was aimed to find a set of parameters $\{m_0,\ \Lambda\}$ for which (i) two poles appropriate for the $I = 1$ resonances $a_0(980)$ and $a_0(1450)$ can be found, *i.e.*, poles that lie on the second and sixth sheet, respectively, and (ii) the six coupling constants A_i, B_i $(i = 1,2,3)$ produce branching ratios of $a_0(1450)$ in good agreement with the central values of the experimental branching ratios [1]. We obtained [23]:

$$m_0 = 1.15 \text{ GeV} \,, \qquad \Lambda = 0.6 \text{ GeV} \,, \tag{3.4}$$

$$\begin{aligned} A_1 &= 2.52 \text{ GeV} \,, & B_1 &= -8.07 \text{ GeV}^{-1} \,, \\ A_2 &= 9.27 \text{ GeV} \,, & B_2 &= 9.25 \text{ GeV}^{-1} \,, \\ A_3 &= -6.56 \text{ GeV} \,, & B_3 &= -1.54 \text{ GeV}^{-1} \,. \end{aligned} \tag{3.5}$$

By using the tree-level decay widths obtained from the optical theorem (3.3) at the peak value of the spectral function above 1 GeV, $m_{a_0(1450)}^{peak} = 1.419$ GeV:

$$\frac{\Gamma^{tree}_{a_0(1450)\to\eta'\pi}}{\Gamma^{tree}_{a_0(1450)\to\eta\pi}} \simeq 0.44 \,, \quad \frac{\Gamma^{tree}_{a_0(1450)\to K\bar{K}}}{\Gamma^{tree}_{a_0(1450)\to\eta\pi}} \simeq 0.96 \,. \tag{3.6}$$

The pole corresponding to the $a_0(980)$ has coordinates[2]

$$\sqrt{s_{pole}}\Big|_{a_0(980)} = (0.970 - i0.045) \text{ GeV} \,, \tag{3.7}$$

[1] The one-loop approximation for the self-energy is quite reliable, since vertex corrections can be shown to have a negligible effect [27].

[2] We apply the usual parameterization for propagator poles, $s_{pole} = m_{pole}^2 - i m_{pole} \Gamma_{pole}$.

i.e., we find the $a_0(980)$ to have a mass of $m_{pole}^{a_0(980)} = 0.969$ GeV and a width of $\Gamma_{pole}^{a_0(980)} = 0.090$ GeV. For the resonance above 1 GeV we get

$$\sqrt{s_{pole}}\Big|_{a_0(1450)} = (1.456 - i0.134) \text{ GeV} , \tag{3.8}$$

or $m_{pole}^{a_0(1450)} = 1.450$ GeV and $\Gamma_{pole}^{a_0(1450)} = 0.270$ GeV.

4. Results

In the following, we do something different with respect to Ref. [23]. We introduce a dimensionless parameter $\delta \in [0,1]$ and replace the derivative coupling constants in Eq. (3.1) by $B_i^2 \to \delta B_i^2$. In consequence, for $\delta = 0$ the self-energy contains only non-derivative interactions, while for $\delta = 1.0$ we reproduce the poles stated before. Increasing δ from zero to one, the derivative interaction is successively increased and we can monitor in a controlled manner how the pole structure changes. The result can be seen in Fig. 1.

It turns out that it is possible to obtain two poles even for vanishing δ where the derivative interactions give no contribution. In this case the real part of the corresponding pole for $a_0(980)$ (second sheet) is maybe too small, but the imaginary part is definitely too small. On the other hand, the latter of the pole for $a_0(1450)$ (sixth sheet) is obviously too large. Driving $\delta \to 1.0$, both poles reach their final positions in different ways: The pole on the second sheet gains changes in both its real and imaginary parts; both are increased. Concerning the pole on the sixth sheet, the real and imaginary parts decrease, but the latter is more affected by a change of δ. For the spectral function see Ref. [23]

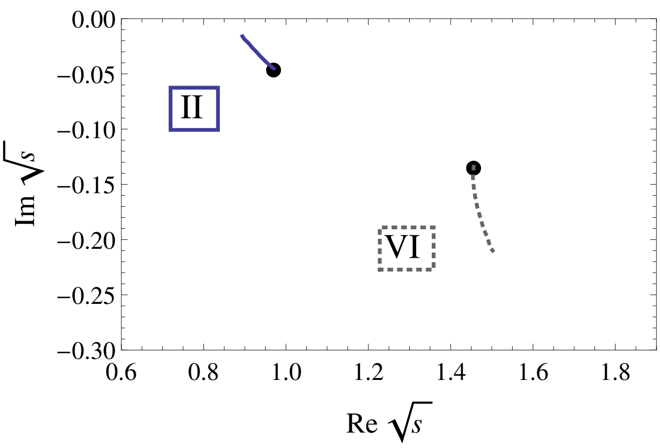

Figure 1: Pole structure of our effective model in dependence of δ. Black dots indicate the position of the poles for $\delta = 1.0$. The roman number indicates on which sheet the respective pole can be found.

5. Conclusions

Our results demonstrate that it is in fact possible to correctly describe the resonances $a_0(980)$ and $a_0(1450)$ in a unique framework, where originally only a single $q\bar{q}$ seed state is present. The

PoS(CD15)131

starting point is an effective Lagrangian that includes both derivative and non-derivative interaction terms, see Eq. (3.1). The form of the Lagrangian is inspired by the one of the extended Linear Sigma Model (eLSM). From our variation of the overall coupling strength δ we furthermore showed that both terms seem to be equally important.

The presented mechanism of dynamical generation (of light scalar mesons) may be extended in two directions: (i) One could study the isodoublet, *i.e.*, by describing the resonances $K_0^*(800)$ and $K_0^*(1430)$ in a similar unified framework (this we already started, see Ref. [28]). The pole of $K_0^*(800)$ is not yet very well known and there is need of improved analyses. (ii) Furthermore, the scalar–isoscalar sector could be investigated, where the resonances $f_0(500)$ and $f_0(980)$ should be dynamically generated, while $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ would be predominantly a non-strange quarkonium, a strange quarkonium, and a scalar glueball, respectively.

Another interesting project is the study of dynamical generation in the framework of resonances in the charmonium sector [29], see for example Ref. [30] and refs. therein. Namely, a whole class of mesons, called X, Y, and Z states, has been experimentally discovered but is so far not fully understood [31, 32]. As shown in Ref. [33] for the case of $X(3872)$, some of the X and Y states could emerge as companion poles of $q\bar{q}$ states.

Acknowledgements

The authors thank Dirk H. Rischke, M. Pennington, J. Wambach, G. Pagliara, J. Reinhardt, D. D. Dietrich, R. Kamiński, J. R. Peláez, and H. van Hees for useful discussions. T. W. acknowledges financial support from HGS-HIRe, F&E GSI/GU, and HIC for FAIR Frankfurt.

References

[1] K. A. Olive *et al.* [Particle Data Group], *Review of Particle Physics*, *Chin. Phys. C* **38** 090001 (2014).

[2] C. Amsler and N. A. Törnqvist, *Mesons beyond the naive quark model*, *Phys. Rep.* **389** (2004) 61; E. Klempt and A. Zaitsev, *Glueballs, hybrids, multiquarks: Experimental facts versus QCD inspired concepts*, *Phys. Rep.* **454** (2007) 1 [arXiv:0708.4016 [hep-ph]].

[3] E. van Beveren, T. A. Rijken, K. Metzger, C. Dullemond, G. Rupp and J. E. Ribeiro, *A low lying scalar meson nonet in a unitarized meson model*, *Z. Phys. C* **30** (1986) 615 [arXiv:0710.4067 [hep-ph]].

[4] D. Morgan and M. R. Pennington, *New data on the $K\bar{K}$ threshold region and the nature of the $f_0(S^*)$*, *Phys. Rev. D* **48** (1993) 1185.

[5] N. A. Törnqvist, *Understanding the scalar meson $q\bar{q}$ nonet*, *Z. Phys. C* **68** (1995) 647 [arXiv:hep-ph/9504372]; N. A. Törnqvist and M. Roos, *Confirmation of the Sigma Meson*, *Phys. Rev. Lett.* **76** (1996) 1575 [arXiv:hep-ph/9511210].

[6] M. Boglione and M. R. Pennington, *Dynamical generation of scalar mesons*, *Phys. Rev. D* **65** 114010 (2002) [arXiv:hep-ph/0203149].

[7] F. Giacosa, *Dynamical generation and dynamical reconstruction*, *Phys. Rev. D* **80** 074028 (2009) [arXiv:0903.4481 [hep-ph]].

[8] F. E. Close and N. A. Törnqvist, *Scalar mesons above and below* 1 *GeV*, *J. Phys. G* **28** (2002) R249 [arXiv:hep-ph/0204205].

[9] J. R. Peláez, *From controversy to precision on the sigma meson: a review on the status of the non-ordinary $f_0(500)$ resonance*, (2015) [arXiv:1510.00653 [hep-ph]]. J. R. Peláez and G. Ríos, *Nature of the $f_0(600)$ Scalar Meson from its N_c Dependence at Two Loops in Unitarized Chiral Perturbation Theory*, *Phys. Rev. Lett.* **97** 242002 (2006) [arXiv:hep-ph/0610397]; J. R. Peláez, *Nature of Light Scalar Mesons from their Large-N_c Behavior*, *Phys. Rev. Lett.* **92** 102001 (2004) [arXiv:hep-ph/0309292].

[10] J. A. Oller and E. Oset, *Chiral symmetry amplitudes in the S-wave isoscalar and isovector channels and the σ, $f_0(980)$, $a_0(980)$ scalar mesons*, *Nucl. Phys. A* **620** (1997) 438 [arXiv:hep-ph/9702314]; J. A. Oller, E. Oset and J. R. Peláez, *Nonperturbative Approach to Effective Chiral Lagrangians and Meson Interactions*, *Phys. Rev. Lett.* **80** (1998) 3452 [arXiv:hep-ph/9803242]; J. A. Oller, E. Oset and J. R. Peláez, *Meson-meson interactions in a nonperturbative chiral approach*, *Phys. Rev. D* **59** 074001 (1999) [Erratum-ibid. **D60**, 099906 (1999); Erratum-ibid. **D75**, 099903 (2007)] [arXiv:hep-ph/9804209].

[11] E. van Beveren, D. V. Bugg, F. Kleefeld and G. Rupp, *The nature of σ, κ, $a_0(980)$ and $f_0(980)$*, *Phys. Lett. B* **641** (2006) 265 [arXiv:hep-ph/0606022].

[12] R. E. Peierls, *Interpretation and properties of propagators*, Proceedings of the Glasgow Conference on Nuclear and Meson Physics, Pergamon Press, London 1954.

[13] C. Amsler and F. E. Close, *Evidence for a scalar glueball*, *Phys. Lett. B* **353** (1995) 385; W. Lee and D. Weingarten, *Scalar quarkonium masses and mixing with the lightest scalar glueball*, *Phys. Rev. D* **61** 014015 (1999) [arXiv:hep-lat/9910008]; F. E. Close and A. Kirk, *Scalar glueball-$q\bar{q}$ mixing above 1 GeV and implications for lattice QCD*, *Eur. Phys. J. B* **21** (2001) 531 [arXiv:hep-ph/0103173]; H.-Y. Cheng, C.-K. Chua and K.-F. Liu, *Scalar glueball, scalar quarkonia, and their mixing*, *Phys. Rev. D* **74** 094005 (2006) [arXiv:hep-ph/0607206]; F. Giacosa, Th. Gutsche, V. E. Lyubovitskij and A. Faessler, *Scalar nonet quarkonia and the scalar glueball: Mixing and decays in an effective chiral approach*, *Phys. Rev. D* **72** 094006 (2005) [arXiv:hep-ph/0509247]; F. Giacosa, Th. Gutsche and A. Faessler, *Covariant constituent quark-gluon model for the glueball-quarkonia content of scalar-isoscalar mesons*, *Phys. Rev. C* **71** 025202 (2005) [arXiv:hep-ph/0408085].

[14] F. Giacosa, *Mixing of scalar tetraquark and quarkonia states in a chiral approach*, *Phys. Rev. D* **75** 054007 (2007) [arXiv:hep-ph/0611388].

[15] S. Gallas, F. Giacosa and D. H. Rischke, *Vacuum phenomenology of the chiral partner of the nucleon in a linear sigma model with vector mesons*, *Phys. Rev. D* **82** 014004 (2010) [arXiv:0907.5084 [hep-ph]]; D. Parganlija, F. Giacosa and D. H. Rischke, *Vacuum properties of mesons in a linear sigma model with vector mesons and global chiral invariance*, *Phys. Rev. D* **82** 054024 (2010) [arXiv:1003.4934 [hep-ph]]; S. Janowski, D. Parganlija, F. Giacosa and D. H. Rischke, *Glueball in a chiral linear sigma model with vector mesons*, *Phys. Rev. D* **84** 054007 (2011) [arXiv:1103.3238 [hep-ph]].

[16] D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa and D. H. Rischke, *Meson vacuum phenomenology in a three-flavor linear sigma model with (axial-)vector mesons*, *Phys. Rev. D* **87** 014011 (2013) [arXiv:1208.0585 [hep-ph]].

[17] R. L. Jaffe, *Multiquark hadrons. I. Phenomenology of $Q^2\bar{Q}^2$ mesons*, *Phys. Rev. D* **15** (1977) 267; R. L. Jaffe, *Multiquark hadrons. II. Methods*, *Phys. Rev. D* **15** (1977) 281.

[18] L. Maiani, F. Piccinini , A. D. Polosa and V. Riquer, *New Look at Scalar Mesons*, *Phys. Rev. Lett.* **93** 212002 (2004) [arXiv:hep-ph/0407017].

[19] G. 't Hooft, G. Isidori, L. Maiani, A. D. Polosa and V. Riquer, *A theory of scalar mesons*, *Phys. Lett. B* **662** (2008) 424 [arXiv:0801.2288 [hep-ph]].

[20] A. H. Fariborz, *Isosinglet Scalar Mesons Below* 2 *GeV and the Scalar Glueball Mass*, *Int. J. Mod. Phys. A* **19** (2004) 2095 [arXiv:hep-ph/0302133]; A. H. Fariborz and R. Jora and J. Schechter, *Toy model for two chiral nonets*, *Phys. Rev. D* **72** 034001 (2005) [arXiv:hep-ph/0506170].

[21] M. Napsuciale and S. Rodriguez, *Chiral model for $\bar{q}q$ and $\bar{q}\bar{q}qq$ mesons*, *Phys. Rev. D* **70** 094043 (2004) [arXiv:hep-ph/0407037].

[22] O. Krehl and J. Speth, *Quasi bound states from T-channel exchange*, *Nucl. Phys. A* **623** (1997) 162c; C. M. B. Voloshin and L. B. Okun, *Hadron Molecules and Charmonium Atom*, *Pisma Zh. Eksp. Teor. Fiz.* **23** (1976) 369 [*JETP Lett.* **23**, (1976) 333]; V. Baru, J. Haidenbauer, C. Hanhart, Yu. Kalashnikova and A. E. Kudryavtsev, *Evidence that the $a_0(980)$ and $f_0(980)$ are not elementary particles*, *Phys. Lett. B* **586** (2004) 53 [arXiv:hep-ph/0308129]; T. Branz, T. Gutsche and V. E. Lyubovitskij, *Strong and radiative decays of the scalars $f_0(980)$ and $a_0(980)$ in a hadronic molecule approach*, *Phys. Rev. D* **78** 114004 (2008) [arXiv:0808.0705 [hep-ph]]; S. Krewald and R. H. Lemmer and F. P. Sassen, *Lifetime of kaonium*, *Phys. Rev. D* **69** 016003 (2004) [arXiv:hep-ph/0307288].

[23] T. Wolkanowski, Francesco Giacosa and Dirk H. Rischke, *$a_0(980)$ revisited*, *Phys. Rev. D* **93** 014002 (2016) (2015) [arXiv:1508.00372 [hep-ph]].

[24] S. Janowski, F. Giacosa and D. H. Rischke, *Is $f_0(1710)$ a glueball?*, *Phys. Rev. D* **90** 114005 (2014) [arXiv:1408.4921 [hep-ph]].

[25] T. Wolkanowski, *Dynamical Generation of Hadronic Resonances*, *Acta Phys. Polon. B Proceed. Suppl.* **8** (2015) 273 [arXiv:1410.7022 [hep-ph]].

[26] T. Wolkanowski and F. Giacosa, *The Scalar-Isovector Sector in the Extended Linear Sigma Model*, *Acta Phys. Polon. B Proceed. Suppl.* **7** (2014) 469 [arXiv:1404.5758 [hep-ph]].

[27] J. Schneitzer, T. Wolkanowski and F. Giacosa, *The role of the next-to-leading order triangle-shaped diagram in two-body hadronic decays*, *Nucl. Phys. B* **888** (2014) 287 [arXiv:1407.7414 [hep-ph]].

[28] T. Wolkanowski, M. Sołtysiak and F. Giacosa, (2016) [arXiv:1512.01071 [hep-ph]].

[29] M. R. Pennington and D. J. Wilson, *Phys. Rev. D* **76** 077502 (2007) [arXiv:0704.3384 [hep-ph]].

[30] N. Brambilla *et al.*, *Heavy quarkonium: progress, puzzles, and opportunities*, *Eur. Phys. J. C* **71** (2011) 1534 [arXiv:1010.5827 [hep-ph]].

[31] E. Braaten, C. Langmack and D. H. Smith, *Born-Oppenheimer approximation for the XYZ mesons*, *Phys. Rev. Lett.* **90** 014044 (2014) [arXiv:1402.0438 [hep-ph]]; G. T. Bodwin *et al.*, *Quarkonium at the Frontiers of High Energy Physics: A Snowmass White Paper*, 2007 [arXiv:1307.7425 [hep-ph]].

[32] L. Maiani, A. D. Polosa and V. Riquer, *Indications of a Four-Quark Structure for the $X(3872)$ and $X(3876)$ Particles from Recent Belle and BABAR Data*, *Phys. Rev. Lett.* **99** 182003 (2007) [arXiv:0707.3354 [hep-ph]].

[33] S. Coito, G. Rupp and E. van Beveren, *$X(3872)$ is not a true molecule*, *Eur. Phys. J. C* **73** (2013) 2351 [arXiv:1212.0648 [hep-ph]].