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A rephasing invariant study of neutrino mixing

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A set of renormalization group equations (RGE) for Dirac neutrinos can be derived using a rephasing invariant parametrization. Under flavor permutation, the symmetric properties of these equations further facilitate the derivation of some exact and approximate RGE invariants. We also provide a numerical example that illustrates the evolution of neutrino mixing parameters.

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1. Introduction

The bridging over high energy models and measured neutrino parameters at low energies will be more reasonable if the renormalization effects of neutrino parameters are properly considered. As the energy scale changes, one expects the formulation of theory to evolve according to the renormalization group equations (RGE). Intensive effort has thus been devoted to the study of RGE effect of neutrino parameters (see, e.g., [1, 2, 3, 4, 5, 6]). The formulation is in general rather complicated under the framework of standard parametrization. While it may appear that the choice of parametrization is not important since the resultant physical quantities are independent of the parametrization and must be expressed as rephasing invariant combinations of the parameters. However, for instance, when one considers the relations between parameters such as in the RGE for neutrino mixing which depends on the energy scale, a particular choice of parametrization may be advantageous over others.

Whether neutrinos are Dirac or Majorana particles has been one of the most important open questions of neutrino physics. Both scenarios are well motivated from the theoretical perspectives (see, e.g., [7, 8]). Although many favor the well known see-saw mechanism as a support for Majorana neutrinos, there exists other models that provide theoretical grounds for the possibility of Dirac neutrinos (see, e.g., Ref.[4] and the references therein). In addition to further experimental evidences, investigating the theoretical implication for both scenarios may also provide indispensable hints at a more fundamental level.

It is the purpose of this work to study the RGE evolution of Dirac neutrinos using a new set of rephasing invariant parameters. It is found that the evolution equations for many of the parameters take similar matrix forms, which may facilitate both analytical and numerical studies of the RGE evolution, and may provide an alternative view to the underlying physics. We discuss some intriguing properties of the parameters and derive RGE invariants which are combinations of neutrino masses and mixing. For illustration purpose, we also show examples of numerical solutions.

2. A rephasing invariant parametrization

The rephasing invariant combinations of elements V_{ij} for the neutrino mixing matrix V can be constructed by the product [9, 10, 11, 12, 13]

$$\Gamma_{ijk} = V_{1i}V_{2j}V_{3k} = R_{ijk} - iJ, \tag{2.1}$$

where the common imaginary part is identified with the Jarlskog invariant [14], and the real parts are defined as

$$(R_{123}, R_{231}, R_{312}; R_{132}, R_{213}, R_{321}) = (x_1, x_2, x_3; y_1, y_2, y_3). \tag{2.2}$$

The (x_i, y_j) variables are constrained by two conditions:

$$detV = (x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1, \tag{2.3}$$

$$(x_1x_2 + x_2x_3 + x_3x_1) - (y_1y_2 + y_2y_3 + y_3y_1) = 0, \tag{2.4}$$

and they are related to the Jarlskog invariant,

$$J^2 = x_1x_2x_3 - y_1y_2y_3. \tag{2.5}$$

In addition, the (x_i, y_j) variables are bounded by ± 1: $-1 \geq (x_i, y_j) \geq +1$, with $x_i \geq y_j$ for any pair of (i, j).

It is convenient to write $|V_{ij}|^2$ in a matrix form with elements $x_i - y_j$:

$$W = [|V_{\alpha i}|^2] = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix} \tag{2.6}$$

The matrix of the cofactors of W, denoted as w with $w^T W = (detW)I$, is given by

$$w = [|V_{\alpha i}|^2] = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ x_3 + y_2 & x_1 + y_3 & x_2 + y_1 \\ x_2 + y_3 & x_3 + y_1 & x_1 + y_2 \end{pmatrix} \tag{2.7}$$

The elements of w are also bounded, $-1 \geq w_{\alpha i} \geq +1$.

One may further obtain useful expressions of the rephasing invariant combination formed by products of four mixing elements,

$$\pi^{\alpha\beta}_{ij} = V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}, \tag{2.8}$$

which can be reduced to

$$\begin{aligned} \pi^{\alpha\beta}_{ij} &= |V_{\alpha i}|^2|V_{\beta j}|^2 - \sum_{\gamma k} \varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}V_{\alpha i}V_{\beta j}V_{\gamma k} \\ &= |V_{\alpha j}|^2|V_{\beta i}|^2 + \sum_{\gamma k} \varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}V^*_{\alpha j}V^*_{\beta i}V^*_{\gamma k}, \end{aligned} \tag{2.9}$$

where the second term in either expression is one of the Γ's (Γ^*'s) defined in Eq. (2.1).

3. RGE for the neutrino parameters

The RGE for the Hermitian matrix $M \equiv Y^\dagger_\nu Y_\nu$, where Y_ν is the neutrino Yukawa coupling matrix, is given by

$$16\pi^2 \frac{dM}{dt} = \alpha M + P^\dagger M + MP \tag{3.1}$$

at the one-loop level. Here α is real and model-independent, $P = C^e_\nu Y^\dagger_l Y_l + C^\nu_\nu Y^\dagger_\nu Y_\nu$, with model-dependent coefficients C^l_ν and C^ν_ν. The Yukawa coupling matrices for neutrinos and charged leptons are denoted as Y_ν and Y_l, respectively. One may choose the flavor basis where the charged-lepton Yukawa coupling matrix Y_l is diagonal, and the matrix M can be diagonalized by the matrix V:

$$M = V[diag(h^2_1, h^2_2, h^2_3)]V^\dagger, \tag{3.2}$$

where h^2_i are the eigenvalues of M.

$[A_i]$	$[A'_i]$
$[A_1]=\begin{pmatrix} 2x_1y_1 & x_1x_2+y_2y_3 & x_1x_3+y_2y_3 \\ x_1x_3+y_1y_2 & 2x_1y_3 & x_1x_2+y_1y_2 \\ x_1x_2+y_1y_3 & x_1x_3+y_1y_3 & 2x_1y_2 \end{pmatrix},$	$[A'_1]=\begin{pmatrix} 2x_1y_1 & x_2x_3+y_1y_2 & x_2x_3+y_1y_3 \\ x_1x_3+y_1y_2 & x_1x_3+y_1y_3 & 2x_2y_1 \\ x_1x_2+y_1y_3 & 2x_3y_1 & x_1x_2+y_1y_2 \end{pmatrix}$
$[A_2]=\begin{pmatrix} x_1x_2+y_1y_3 & 2x_2y_2 & x_2x_3+y_1y_3 \\ x_2x_3+y_2y_3 & x_1x_2+y_2y_3 & 2x_2y_1 \\ 2x_2y_3 & x_2x_3+y_1y_2 & x_1x_2+y_1y_2 \end{pmatrix},$	$[A'_2]=\begin{pmatrix} x_1x_3+y_1y_2 & 2x_2y_2 & x_1x_3+y_2y_3 \\ 2x_3y_2 & x_1x_2+y_2y_3 & x_1x_2+y_1y_2 \\ x_2x_3+y_2y_3 & x_2x_3+y_1y_2 & 2x_1y_2 \end{pmatrix}$
$[A_3]=\begin{pmatrix} x_1x_3+y_1y_2 & x_2x_3+y_1y_2 & 2x_3y_3 \\ 2x_3y_2 & x_1x_3+y_1y_3 & x_2x_3+y_1y_3 \\ x_2x_3+y_2y_3 & 2x_3y_1 & x_1x_3+y_2y_3 \end{pmatrix},$	$[A'_3]=\begin{pmatrix} x_1x_2+y_1y_3 & x_1x_2+y_2y_3 & 2x_3y_3 \\ x_2x_3+y_2y_3 & 2x_1y_3 & x_2x_3+y_1y_3 \\ 2x_2y_3 & x_1x_3+y_1y_3 & x_1x_3+y_2y_3 \end{pmatrix}$

Table 1: The explicit expressions of the matrices $[A_i]$ and $[A'_i]$.

The evolution of the mixing matrix V satisfies the relation,

$$dV/dt = VT, \tag{3.3}$$

here the matrix T is anti-Hermitian. One may define $\mathscr{D} = 16\pi^2 \frac{d}{dt}$ with $t = \ln(\mu/M_W)$ and compare the diagonal elements of $\mathscr{D}(V[diag(h_1^2, h_2^2, h_3^2)]V^\dagger)$ with that of $\mathscr{D}M$ to obtain

$$\mathscr{D}h_i^2 = h_i^2[\alpha + 2C_\nu^l(|V_{1i}|^2 f_1^2 + |V_{2i}|^2 f_2^2 + |V_{3i}|^2) f_3^2], \tag{3.4}$$

where f_i^2 are the eigenvalues of the matrix $Y_l^\dagger Y_l$, and the C_ν^ν terms have been ignored. From the off-diagonal elements, we obtain the expression of T_{ij}:

$$T_{ij} = -H_{ij}P'_{ij}/(16\pi^2), \tag{3.5}$$

with $P' = V^\dagger P V$ and

$$H_{ij} = \frac{h_i^2 + h_j^2}{h_i^2 - h_j^2}. \tag{3.6}$$

To derive the RGE for neutrino mixing parameters, we may use Eq. (3.3), Eq. (3.5), and Eq. (3.6) to obtain

$$\mathscr{D}\Gamma_{ijk} = -[(\sum_{l\neq i} V_{1l}H_{li}P'_{li})V_{2j}V_{3k} + V_{1i}(\sum_{l\neq j} V_{2l}H_{lj}P'_{lj})V_{3k} + V_{1i}V_{2j}(\sum_{l\neq k} V_{3l}H_{lk}P'_{lk})]. \tag{3.7}$$

The real part of $\mathscr{D}\Gamma_{ijk}$ gives rise to $\mathscr{D}x_i$ and $\mathscr{D}y_i$ in the following matrix forms:

$$\mathscr{D}x_i = -2C_\nu^l[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][A_i][H_{23}, H_{31}, H_{12}]^T, \tag{3.8}$$

$$\mathscr{D}y_i = -2C_\nu^l[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][A'_i][H_{23}, H_{31}, H_{12}]^T, \tag{3.9}$$

here $\Delta f_{ij}^2 = f_i^2 - f_j^2$, and the matrices $[A_i]$ and $[A_i']$ are given by Table I. In addition, the imaginary part leads to

$$\mathscr{D}J^2 = -2C_\nu^l[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][w][H_{23}, H_{31}, H_{12}]^T \tag{3.10}$$

The evolution equations for $W_{ij} = |V_{ij}|^2$ can be derived directly from $\mathscr{D}x_i$ and $\mathscr{D}y_i$:

$$\mathscr{D}W_{ij} = -2C_\nu^l[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][S_{ij}][H_{23}, H_{31}, H_{12}]^T, \tag{3.11}$$

where the matrices $[S_{ij}]$ are obtained by using suitable combinations of $[A_i]$ and $[A_i']$. For instance, $[S_{23}] = [A_2] - [A_1']$, etc. In addition, we denote the real part of $\pi_{\gamma k}$ as $\Lambda_{\gamma k}$,

$$\pi_{ij}^{\alpha\beta} \equiv \pi_{\gamma k} = \Lambda_{\gamma k} + iJ. \tag{3.12}$$

Since $Re(\pi_{ij}^{\alpha\beta})$ takes the following forms,

$$Re(\pi_{ij}^{\alpha\beta}) = |V_{\alpha i}|^2|V_{\beta j}|^2 - x_a = |V_{\beta i}|^2|V_{\alpha j}|^2 + y_b, \tag{3.13}$$

the average of the two different forms is thus equal to $\Lambda_{\gamma k}$,

$$\Lambda_{\gamma k} = \frac{1}{2}(|V_{\alpha i}|^2|V_{\beta j}|^2 + |V_{\alpha j}|^2|V_{\beta i}|^2 - |V_{\gamma k}|^2), \tag{3.14}$$

where (α, β, γ) and (i, j, k) are cyclic permutations. Furthermore, we may write down the elements of the matrix $[w]$ as

$$w_{\gamma k} = |V_{\alpha i}|^2|V_{\beta j}|^2 - |V_{\alpha j}|^2|V_{\beta i}|^2. \tag{3.15}$$

The evolution equation for the elements of $[w]$ can then be shown to take the form:

$$\mathscr{D}w_{ij} = -2C_\nu^l[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][G_{ij}][H_{23}, H_{31}, H_{12}]^T \tag{3.16}$$

where the matrices $[G_{ij}]$ are obtained by using suitable combinations of $[A_i]$ and $[A_i']$. For instance, $[G_{21}] = [A_3] - [A_2']$, etc.

4. The RGE invariants

The study of RGE invariants has drawn certain attention in recent literature (see, e.g., [15, 16, 17]). The invariants are formed by combinations of physical observables which remain the same under the RGE running of energy scale. To search for neutrino RGE invariants using our parametrization, we first define the neutrino mass ratio $r_{ij} = m_i/m_j$, where $h_i = \sqrt{2}m_i/v$ in the SM and $h_i = \sqrt{2}m_i/(v\cos\beta)$ in the MSSM, with $v \simeq 246$ GeV and $\tan\beta$ is given by the ratio of the two Higgs VEVs in MSSM. By calculating $\mathscr{D}(\sinh^2\ln r_{ij})$ and $\mathscr{D}[\ln(\sinh^2\ln r_{ij})]$, we obtain

$$\Sigma_{ij}\mathscr{D}[\ln(\sinh^2\ln r_{ij})] = 2C_\nu^l[\Delta f_{23}, \Delta f_{31}, \Delta f_{12}][w][H_{23}, H_{31}, H_{12}]^T. \tag{4.1}$$

Combining this result with the expression of $\mathscr{D}J^2$, we find that

$$\mathscr{D}[\ln(J^2 \cdot \sinh^2\ln r_{12} \cdot \sinh^2\ln r_{23} \cdot \sinh^2\ln r_{31})] = 0, \tag{4.2}$$

i.e., $J^2(\Pi_{ij}\sinh^2\ln r_{ij})$ is a RGE invariant.

Furthermore, one notes that any quantity that is identical to J^2 also gives rise to an invariant when multiplied by $\Pi_{ij}\sinh^2\ln r_{ij}$ in Eq. (4.2). With the expression of $\Lambda_{\gamma k}$ in Eq. (3.14), it is straightforward to write down nine different forms of $J^2 = \pi^2_{\gamma k} - \Lambda^2_{\gamma k}$, which correspond to nine different combinations of (γ, k),

$$\begin{aligned} J^2 &= \pi^2_{\gamma k} - \Lambda^2_{\gamma k} \\ &= |V_{\alpha i}|^2|V_{\beta j}|^2|V_{\alpha j}|^2|V_{\beta i}|^2 - \Lambda^2_{\gamma k}. \end{aligned} \tag{4.3}$$

This leads to nine RGE invariants which consist of $|V_{ij}|^2$ and the mass ratios $\ln r^2_{ij}$:

$$[|V_{\alpha i}|^2|V_{\beta j}|^2|V_{\alpha j}|^2|V_{\beta i}|^2 - \Lambda^2_{\gamma k}](\sinh\ln r^2_{21})(\sinh\ln r^2_{32})(\sinh\ln r^2_{13}) = constant \tag{4.4}$$

One may also find approximate invariants under certain limiting cases. If $f^2_\alpha \gg f^2_\beta, f^2_\gamma$, the evolution equations for $\mathscr{D}[\ln(J^2)]$ and $\mathscr{D}|V_{ij}|^2$ lead directly to

$$J^2/|V_{\alpha 1}|^2|V_{\alpha 2}|^2|V_{\alpha 3}|^2 \simeq constant. \tag{4.5}$$

Furthermore, if the neutrino masses satisfy the hierarchical condition: $H_{ij} \gg H_{jk}, H_{ki}$, then $(d\ln J^2/dt) + (d\ln(\sinh^2\ln r_{ij})/dt) \approx 0$ and the approximate invariant follows:

$$J^2(\sinh^2\ln r_{ij}) = constant. \tag{4.6}$$

5. Numerical examples

For the numerical calculation of this work, we adopt the required quark RGE in the (x, y) parametrization from Ref.[18]. Note that to assess the general nature of the RGE, it is appropriate to start from a point with fast evolution, so that most changes may be accomplished in its neighborhood, with minor corrections afterwards. It is pointed out that the low energy physics values are close to a fixed point of the RGE, and it is crucial to study how they are approached from the high energy values.

For the purpose of illustration, we adopt the following initial input parameters at high energy. (i) The quark masses at $t = 30$ are taken to be near degenerate with $m_q \sim 173$ GeV. (ii) The mixing parameters for quarks are assumed to be $[x_1, x_2; y_1, y_2] = [(1/6)+\varepsilon, (1/6)-\varepsilon; (-1/6)+\varepsilon, (-1/6)+\varepsilon]$, with $\varepsilon = 0.01$. (iii) The neutrino masses at $t = 30$ are also taken to be near degenerate, $m'_3 \sim m'_2 \sim m'_1 \sim 0.05$ eV. (iv) As for the neutrino mixing parameters, it is found that (x_i, y_j) for neutrinos only evolve slightly from their respective initial values. We thus adopt $[x_1, x_2; y_1, y_2] = [(1/3)-\varepsilon, (1/6)-\varepsilon; (-1/3)+\varepsilon, (-1/6)+\varepsilon]$ as the input at $t = 30$ so as to yield reasonable mixing parameters at low energy.

We show in Figure 1 a numerical example that illustrates the evolution of $|V_{1i}|^2(t)$ for both quarks and neutrinos in the unit of the respective initial value at $t = 30$, $|V_{1i}|^2(t)(30)$. It is seen that the quark mixing can evolve quite significantly from high to low t. However, since the neutrino mixing parameters (x_i and y_i) only evolve slightly, their elements $|V_{ij}|^2$ do not evolve much. This general behavior is in agreement with the expectation of recent study (see, e.g., Ref[19]). Despite the tiny evolution of neutrino mixing, the SM and MSSM models suggest opposite trends of evolution for $|V_{ij}|^2$ due to opposite signs of C^l_ν in the models. It is seen that $|V_{ij}|^2$ decreases under one model while increases under the other.

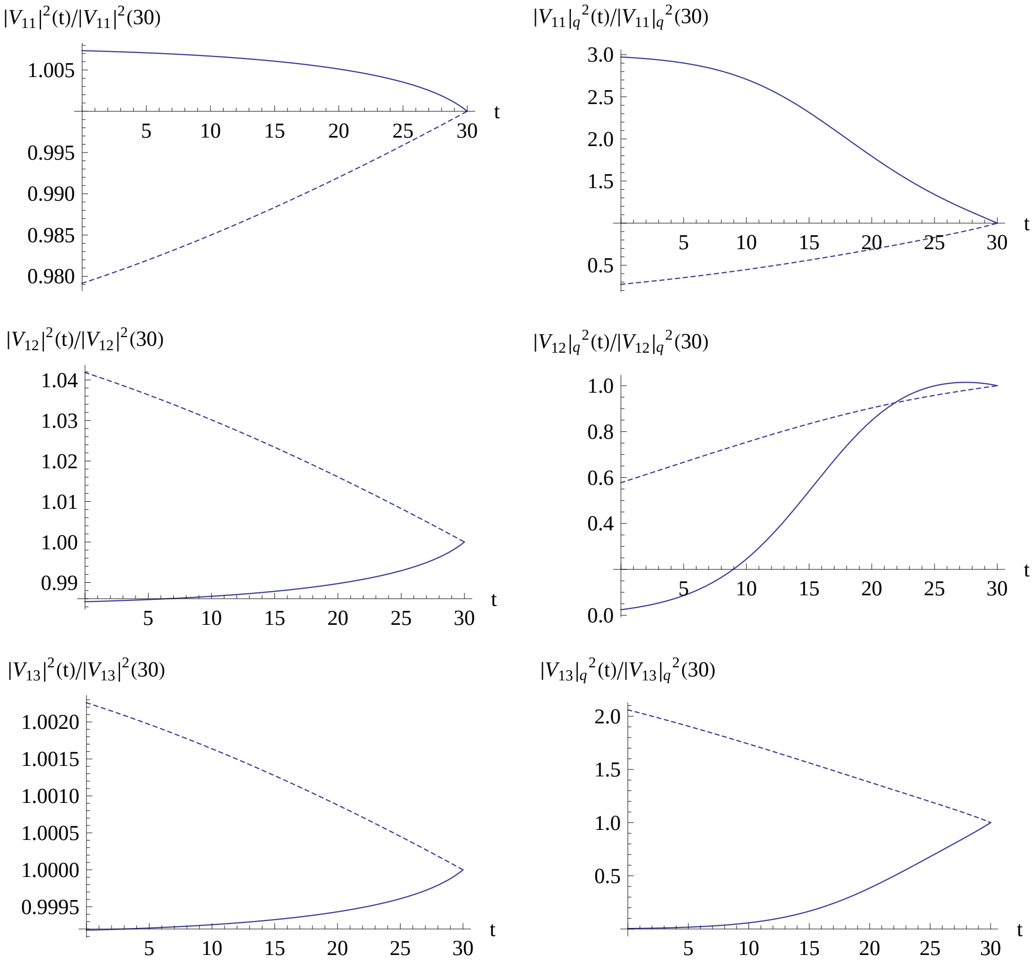

Figure 1: The evolution of $|V_{1i}|^2(t)/|V_{1i}|^2(t=30)$ for neutrinos (left column) and $|V_{1i}|_q^2(t)/|V_{1i}|_q^2(t=30)$ for quarks (right column) under SM (solid) and MSSM (dashed).

6. conclusion

We have been able to derive and analyze the RGE for Dirac neutrinos using a newly introduced set of parameters. Specifically, we show the analytical expressions for $\mathscr{D}h_i^2$ (or $\mathscr{D}m_i^2$), $\mathscr{D}x_i$, $\mathscr{D}y_i$, $\mathscr{D}J^2$, $\mathscr{D}|V_{ij}|^2$ (or $\mathscr{D}W_{ij}$), and $\mathscr{D}w_{ij}$ in a matrix form. The matrix equations are shown to be highly symmetric among various parameters, and certain properties become transparent under this parametrization framework. The results facilitate both the theoretical and numerical study of the RGE effects for the neutrinos.

Measurable physical quantities are independent of the framework of parametrization, and one should be able to formulate these observales in an invariant form. Based on the RGE of the parameters, we derive several RGE invariants which are combinations of the mass ratios, r_{ij}, and the mixing elements of neutrinos, $|V_{ij}|^2$. In particular, such invariants are shown in Eq. (4.2) and Eq. (4.4). Approximate invariants are also presented. We also provide a numerical example that illustrate the RGE effects of the mixing elements for quarks and neutrinos. It should be pointed out that certain RGE equations may be sensitive to the input parameters and that fine-tuning may be required to obtain reasonable values for all the parameters at low energy.

This rephasing invariant parametrization may be further applied to the formulation of RGE for the Majorana neutrinos, which requires two more phases (or parameters). We shall present the details of related investigation in a future work.

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References

[1] J. A. Casas, J. R. Espinosa, A. Ibarra and I. Navarro, Nucl. Phys. B **573**, 652 (2000).

[2] P. H. Chankowski, W. Krolikowski and S. Pokorski, Phys. Lett. B **473**, 109 (2000)

[3] S. Antusch, J. Kersten, M. Lindner and M. Ratz, Nucl. Phys. B **674**, 401 (2003).

[4] M. Lindner, M. Ratz and M. A. Schmidt, JHEP **0509**, 081 (2005).

[5] S. Luo and Z. z. Xing, Phys. Lett. B **632**, 341 (2006)

[6] S. Ray, Int. J. Mod. Phys. A **25**, 4339 (2010)

[7] R. N. Mohapatra *et al.*, hep-ph/0412099.

[8] A. Y. Smirnov, hep-ph/0411194.

[9] T. K. Kuo and T.-H. Lee, Phys. Rev. D **71**, 093011 (2005).

[10] S. H. Chiu, T. K. Kuo, and Lu-Xin Liu, Phys. Lett. B **687**, 184 (2010).

[11] S. H. Chiu and T. K. Kuo, JHEP **11**, 080 (2010), Erratum-*ibid*. **01**, 147 (2011).

[12] S. H. Chiu and T. K. Kuo, Phys. Rev. D **84**, 013001 (2011).

[13] S. H. Chiu and T. K. Kuo, Eur. Phys. J. C **73**, 2579 (2013)

[14] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[15] D. A. Demir, JHEP **0511**, 003 (2005)

[16] P. F. Harrison, R. Krishnan and W. G. Scott, Phys. Rev. D **82**, 096004 (2010)

[17] T. Feldmann, T. Mannel and S. Schwertfeger, arXiv:1507.00328 [hep-ph].

[18] S. H. Chiu, T. K. Kuo, T. H. Lee and C. Xiong, Phys. Rev. D **79**, 013012 (2009)

[19] T. Ohlsson and S. Zhou, Nature Commun. 5 (2014) 5153

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