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Dynamics with Histories

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We develop a general formulation of Dynamics, based on the notion of *history* (defined as a *possible*, or *kinematical*, evolution of a dynamical system), rather than evolution or phase space varaibles. It applies in particular to field theories, allowing explicitely covariant Lagrangian and Hamiltonian approaches: we give to space-time the same status than time in usual dynamics, excepted for it dimensionallity. We develop a differential calculus in the infinite-dimensional space of histories. This allows us to derive " historical versions " of the usual notions of dynamics, which remain always covariant; evolution and conservation equations, a generalized symplectic form which is the historical counterpart of the multisymplectic form... Our treatment applies to the case where field components are not scalar functions but forms, in particular to electromagnetism and first orger general relativity.
Our covariant formalism offers a synthesis between the multisymplectic geometry, the " covariant phase space" and the work of Crnkovic and Witten.

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PoS(FFP14)196

1. Introduction

We develop a general formulation of dynamics, based on the notion of *history*. An history is a *possible* (or *kinematical*) evolution of a dynamical system. When it obeys the dynamical equations, an history becomes *physical* (or dynamical), and represents a particular solution.

This formalism gives a covariant description of field theories (hereafter, FT's) exactly similar to time dynamics (hereafter, tD): space-time (in FTs) is treated on the same footing than time (in tD); only the dimensionallity differs and this never appears in the formalism. The treatment remains entirely covariant and does not involve any notion of time, or any splitting of space-time.

An history (a field) is not required to be a *function*, like, e.g., for a scalar field. It may be a *r-form* on space-time, like the Faraday 2-form for electromagnetism; or the cotetrad and connection one-forms for general relativity in the first order formalism. More generally, it is a section of a fiber bundle whose basis is called the *evolution domain*.

This requires to define a differential calculus in $\mathscr{S}$, the infinite-dimensional space of *histories*. Fist, we generalize functions to *historical-maps* (Hmaps), which form the algebra $\Omega^0(\mathscr{S})$, generalizing that of functions. Then we define derivations (generalized vector-fields) and generalized [differential] forms on $\mathscr{S}$. This allows us to perform variational calculus, and to develop both Lagrangian and Hamiltonian formalisms in that space. They admit very simple and synthetic expressions and remain entirely covariant. They provide natural *historical* generalizations of most mathematical entities appearing in usual dynamics; in particular the (canonical) *Lagrangian forms*, *Cartan forms*, *symplectic forms*, and obeying general conservation laws. Our covariant formalism offers a synthesis between the multisymplectic geometry (see, e.g., [2]), the " covariant phase space" approaches (see, e.g., [2]), the canonical approach and the geometry of the space of solutions. Its Hamiltonian version links the multisymplectic formalism with the work of [11].

The section 1 introduces the notion of histories, with their lifts to *velocity-histories* involved in the Lagrangian dynamics. Section 2 recalls the mathematical formalism [5], including the definition of *historical maps*. Section 3 shortly presents the Lagrangian version of Dynamics given in [5], with an historical expression of Noether theorem. Section 4 presents the Hamiltonian formalism and the *historical symplectic form*. More details are given in [5] and [6].

1.1 Histories

As our general definition, **an history is an r-form on an *evolution domain*** $\mathscr{D}$, taking its values in a *configuration space* Q. Occasionally, we call it a " r-history ". The *evolution domain* $\mathscr{D}$ is the time line for tD, space-time for FT's (no space + time splitting required). For general relativity, this is a differential manifold without defined metric. Very generally, $\mathscr{D}$ is a n-dimensional manifold (possibly with a metric), that we treat as some kind of " n-dimensional timeline ", w.r.t. which the evolution is expressed without any specific parameter. We occasionally use [local] coordinates x^μ on $\mathscr{D}$, which disappear in our final results written in covariant form. They generate adapted [local] coordinates in the various fiber bundles we consider. We write Vol the volume form defined from these coordinates, and $\star$ the corresponding Hodge duality (those are non covariant entities).

The configuration space Q represents the degrees of freedom of the system. A *composite* history involves many of them. However it can be expanded in components, and each of them will be be treated as a scalar–valued history, so that we restrict our treatment to the case $Q = \mathbb{R}$, without loss of generality (composite examples are treated in [5, 6]). The electromagnetic potential A, for instance, is not a composite history, but as a 1–history, i.e., a *scalar–valued* one form on space-time. The scalar field is a zero–history, a (zero-form on space-time). For first order general relativity, $\mathscr{D}$ is a differentiable manifold without prior metric.

The r-histories are sections of the *configuration bundle* $\mathbf{Q} \to \mathscr{D}$. For scalar–valued histories (that we consider here), it identifies with [a subbundle of] the bundle of r-forms $\bigwedge^r \mathrm{T}^*\mathscr{D} \subset \bigwedge \mathrm{T}^*\mathscr{D}$ (including *functions* as the case $r = 0$). They form the infinite–dimensional space of r-histories,

$$\mathscr{S} \subseteq \Omega^r_D \overset{def}{=} \mathrm{Sect}(\bigwedge^r \mathrm{T}^*\mathscr{D}) \quad \subset \quad \Omega_D \overset{def}{=} \mathrm{Sect}(\bigwedge \mathrm{T}^*\mathscr{D})\ .$$

The lift of a section c of $\mathbf{Q}$, to its first jet bundle $J^1\mathbf{Q}$, gives its *first jet extension* $C \overset{def}{=} (c, \mathrm{d}c)$. We call it the corresponding *velocity-history*, and it is nothing but a more convenient way to express c (d is the exterior derivative in $\mathscr{D}$). We call $\mathscr{S}_V$ the space of velocity-histories. In [5], we have derived the Lagrangian dynamics in $\mathscr{S}_V$, introducing the notion of *Historical Maps*.

2. Historical Maps and Differential calculus

In all formulas, the wedge product in $\mathscr{D}$ is implicitly expressed by simple juxtapostion.

An *historical map* (Hmap) is a generalization of the notion of *functional*. We define it as a map (see [5] for details)

$$F:\ \mathscr{M} \overset{def}{=} (\Omega_D)^2 \to \Omega_D:\ (c,\gamma) \to F(c,\gamma).$$

We call it a Hmap of type [0;R] when $F(c,\gamma)$ is a R–form.

The wedge product in $\mathscr{D}$ defines the product of the Hmaps (see [5]). Thus they form an algebra $\mathscr{F} = \Omega^0(\mathscr{M})$, that we treat like an algebra of functions over $\mathscr{M}$, seen itself as an infinite dimensional manifold. The differential d on $\mathscr{D}$ is easily lifted to $\mathscr{F}$, improving the grade from [0;R] to [0;R+1]. We call it occasionally " horizontal derivative " but it does not allow variational calculus and we have introduced an second " vertical " external exterior derivative D, different fromd and commuting with it (see also [4]).

Here c and γ play the role of " coordinates " in $\mathscr{M}$. We have defined the two basic *partial derivative operators* $\partial_c = \frac{\partial}{\partial c}$ and $\partial_\gamma = \frac{\partial}{\partial \gamma}$. The *general vector-field* $V = V^c\ \frac{\partial}{\partial c} + V^\gamma\ \frac{\partial}{\partial \gamma}$, with components V^c and $V^\gamma \in \mathscr{F}$, acts as a derivation on the Hmaps. We write $\mathscr{X}(\mathscr{M})$ for their set. The generalized multi-vector-field is defined through antisymmetric tensor product.

The *basis one-forms* Dc and Dγ are defined through their actions on vector-fields. The general one-form $\alpha = \alpha_c\ \mathrm{D}c + \alpha_\gamma\ \mathrm{D}\gamma$ has components α_c and $\alpha_\gamma \in \mathscr{F}$. We write $\Omega^1(\mathscr{M})$ for their set. The (vertical) exterior derivative of an arbitrary [0;R]–Hmap F, $\mathrm{D}F \overset{def}{=} \mathrm{D}c\ \frac{\partial F}{\partial c} + \mathrm{D}\gamma\ \frac{\partial F}{\partial \gamma}$, is called of type [1;R]. The wedge product of forms, $\wedge$ is the antisymmetrized tensor product, as usual (not to be confused with the wedge product in $\mathscr{D}$, always implicit here). This defines $\Omega(\mathscr{M})$.

3. Lagrangian Dynamics

The details may be found in [5]. The *action* involves the *Lagrangian functional* $\mathscr{L}$. This is a [0;n]–Hmap, with arguments a pair $C = (c, \mathrm{d}c)$, an r-history and its exterior derivative $\gamma = \mathrm{d}c$.

It returns the n-form $\mathscr{L}(C)$ on $\mathscr{D}$, whose integral on $\mathscr{D}$ gives the action $\mathscr{A}(C)$. The *historical momentum* is the type [0;n-r-1]–Hmap $P \stackrel{def}{=} \frac{\partial \mathscr{L}}{\partial \mathrm{d}c}$, with same arguments. Then

$$\mathrm{D}\mathscr{L} = \mathrm{D}c\,\frac{\partial \mathscr{L}}{\partial c} + \mathrm{D}(\mathrm{d}c)\,P = \mathrm{D}c\,(\frac{\delta^{EL}\mathscr{L}}{\delta c}) - \mathrm{d}\Theta, \tag{3.1}$$

where we defined the Euler-Lagrange derivative

$$\frac{\delta^{EL}\cdot}{\delta c} \stackrel{def}{=} \frac{\partial\cdot}{\partial c} - (-1)^{|c|}\,\mathrm{d}\frac{\partial\cdot}{\partial(\mathrm{d}c)}, \tag{3.2}$$

with $|c|$ =grade of c; and the [1; n-1]–Hform

$$\Theta \stackrel{def}{=} -\mathrm{D}c\,P = \mathrm{D}c\,\frac{\partial \mathscr{L}}{\partial(\mathrm{d}c)}. \tag{3.3}$$

Its (vertical) exterior derivation gives the closed [2; n-1]–Hform $\omega \stackrel{def}{=} \mathrm{D}\Theta = \mathrm{D}P \wedge \mathrm{D}c$. These two *historical Lagrangian forms* become the historical symplectic potential and form in the non–degenerate case.

An arbitrary variation of an history is seen as the action of a vector-field $\delta \in \mathscr{X}(\mathscr{M})$, such that $\delta^{\mathrm{d}c} = \mathrm{d}\delta^{c}$. Stationarity of the action implies the Euler-Lagrange equation

$$\frac{\delta^{EL}\mathscr{L}}{\delta c} = 0. \tag{3.4}$$

It expresses in an explicitely covariant and condensed form any FT, including the case where the c is a r-history (a form) rather than a function.

A vector-field δ is a *symmetry generator* when it does not modifies the action. This means that it modifies $\mathscr{L}$ by an exact form $\mathrm{d}X$ only:

$$\delta c\,(\frac{\delta^{EL}\mathscr{L}}{\delta c}) - \mathrm{d}(\delta c\,P) = \mathrm{d}X.$$

The *Noether current* (three-form) $j \stackrel{def}{=} X + \delta c\,P$, is conserved on shell: $\mathrm{d}j \simeq 0$. This defines locally the *Noether charge density*.

4. Hamiltonian dynamics

We call the affine dual of the first jet bundle $j_1\mathbf{Q}$, the *multiphase space bundle* [1] $\mathbf{Y}$. Its bundle manifold — that we write also $\mathbf{Y}$ — is the *De Donder-Weyl multisymplectic manifold*. We will simply refer to it as to the *phase space*.

The (usual) Legendre transform transports the dynamics from $\mathbf{V}$ to $\mathbf{Y}$. The *historical Legendre map*,

$$\mathscr{T}_L:\ \mathscr{S}_V \rightsquigarrow \mathscr{S}_Y \stackrel{def}{=} Sect(\mathbf{Y}):\ C = (c,\mathrm{d}c) \rightsquigarrow Y = (c,P),$$

[1] Different authors use various appellations for $\mathbf{Y}$, or for its associated bundle manifold: the covariant phase space bundle, the doubly extended phase space, the extended dual bundle [9], orthe extended multimomentum bundle [7] ...

is the corresponding mapping of their sections [2]: velocity-histories to Hamiltonian histories. To any history c, it associates canonically the *Hamiltonian history* $Y \overset{def}{=} (c,P)$, with P the historical momentum defined above. We work now in the space $\mathscr{S}_Y \overset{def}{=} Sect(\mathbf{Y})$ of Hamiltonian histories, were c and P play the role of coordinates.
We define the (historical) *Hamiltonian functional* (hereafter " historical " means defined in $\mathscr{S}_Y$)

$$\mathscr{H} = \Lambda^i \, \Gamma_i + P \, \mathrm{d}c - \mathscr{L}.$$

We have assumed possible constraints Γ_i, with associated Lagrange multipliers Λ^i; both being Hmaps on $\mathscr{S}_Y$. In this expression, dc and $\mathscr{L}$ must be expressed as functionals of c and P a far as allowed by inversion of the Legendre map. We concentrate on the case without constraints (constraints are treated in [5, 6]). The [0;n]–Hmap $\mathscr{H}$ admits arguments c and P, and returns an n-form. We apply the differential calculus defined in [5].

We call the canonical [1;n-1] type form $\Theta \overset{def}{=} P \, \mathrm{d}c$ the *historical Poincaré-Cartan form* (or symplectic potential). Its exterior derivative is a (exact) [2;n-1]-form on $\mathscr{S}_Y$,

$$\overline{\omega} = \mathrm{D}P \wedge \mathrm{D}c = (\mathrm{D}P^{\mu} \wedge \mathrm{D}c) \; \mathrm{Vol}_{\mu},$$

the *historical symplectic form*. It is the historical counterpart of the multisymplectic form with three main differences:
- it is always a two-form and not a $(n-1)$–form;
- it is defined in the space of (Hamltonian) histories, not in phase space;
- its values are $(n-1)$–forms in $\mathscr{D}$, not scalar functions.

The condition for a Hamiltonian history $Y = (c,P)$ being a solution is expressed by *generalized Hamilton equations*

$$\mathrm{d}c = \frac{\partial \mathscr{H}}{\partial P}; \quad \mathrm{d}P = -\frac{\partial \mathscr{H}}{\partial c}. \tag{4.1}$$

This " historical " version of the Hamilton–De Donder–Weyl equations is covariant. It applies to tD as well to FT, it includes the case where the history is a form rather than a map. In [6], we apply to electromagnetism and to first order general relativity.

Interestingly, 4.1 implies, *on shell*,

$$\mathrm{D}\mathscr{H} = \frac{\partial \mathscr{H}}{\partial c} \, \mathrm{D}c + \frac{\partial \mathscr{H}}{\partial P} \, \mathrm{D}P \simeq \mathrm{d}c \; \mathrm{D}P - \mathrm{d}P \; \mathrm{D}c.$$

After derivation, $\mathrm{D}\mathrm{D}\mathscr{H} = 0 = \mathrm{D}\mathrm{d}c \; \mathrm{D}P - \mathrm{D}\mathrm{d}P \; Dc = \mathrm{d}\overline{\omega}$: the historical symplectic form is conserved on shell. This is the covariant version of the conservation of the symplectic current.

Integration of the [2;n-1]–form $\overline{\omega}$ along any $(n-1)$-dimensional submanifold of $\mathscr{D}$, provides a [2;0] symplectic form (scalar-valued) on $\mathscr{M}$. The on–shell conservation of $\overline{\omega}$ implies that it does not depend on the choice of the hypersurface (assumed Cauchy for FTs). This canonical symplectic form on the space of solutions identifies with that introduced by [11]. Our result may be seen as a generalization of that work.

[2]thanks to the fact that a fiber-preserving map between fiber bundles induces a map between their spaces of sections

PoS(FFP14)196

Since c and P play the role of the canonical " variables ", the formula above suggests to define the Poisson-like bracket of two Hmaps as

$$\{f,g\} \stackrel{def}{=} \frac{\partial f}{\partial c}\frac{\partial g}{\partial P} - \frac{\partial g}{\partial c}\frac{\partial f}{\partial P} = X_f \lrcorner \mathrm{D}g = -X_g \lrcorner \mathrm{D}f,$$

where we have defined [6] the historical multisymplectic gradient as the historical vector field $X_f \in \mathscr{X}(\mathscr{M})$ obeying $X_f \lrcorner \overline{\varpi} = \mathrm{D}f$. This requires that the quantities involved are well defined: that both f and g have grades of greater or equal to those of c and P, namely r and $n-r-1$; or, alternatively, that they do not depend on the " canonical variables ". To illustrate, the identities

$$\{P,c\} = 1; \ \{\mathscr{H},c\} = \frac{\partial \mathscr{H}}{\partial P} = \mathrm{d}c; \ \{\mathscr{H},P\} = -\frac{\partial \mathscr{H}}{\partial c} = \mathrm{d}P$$

validate the definition; our bracket appears as a generalization of that proposed by [3]. It applies to Hmaps (not to functions) and this suggests to consider form-valued, rather than scalar-valued, observables; they provide scalar values after integration over a submanifold of convenient dimension. This appears as a convenient point of view; it corresponds for instance to what is done in Loop Quantum Gravity through the introduction of the *Holonomy-Flux algebra*.

By construction, an observable depends on histories, not on configuration – or phase–space variables. Thus, even when reformulated as a function on phase space it depends on a whole history only, which means that it remains constant during the evolution and commutes with the Hamiltonian and with the constraints. Thus observables as defined here are *Dirac observables*, or *complete observables* in the sense of [8, 1] (see also[10]).

References

[1] B. Dittrich, *Partial and Complete Observables for Hamiltonian Constrained Systems*, http://arxiv.org/abs/gr-qc/0411013v1

[2] Frédéric Hélein, *Multisymplectic formalism and the covariant phase*, http://arxiv.org/abs/1106.2086

[3] Igor V. Kanatchikov, Precanonical Quantum Gravity: quantization without the space-time decomposition, http://arxiv.org/abs/gr-qc/0012074v2

[4] Igor Khavkine, Covariant phase space, constraints, gauge and the Peierls formula, http://arxiv.org/abs/1402.1282v2

[5] Marc Lachièze-Rey, Historical Lagrangian Dynamics, http://arxiv.org/abs/1411.4291

[6] Marc Lachièze-Rey, Historical Hamiltonian Dynamics, in preparation

[7] Narciso Román-Roy, *Multisymplectic Lagrangian and Hamiltonian Formalisms of Classical Field Theories*, http://arxiv.org/abs/math-ph/0506022v3

[8] Carlo Rovelli, *Partial observables*, http://arxiv.org/abs/gr-qc/0110035

[9] W. Sarleta and G. Waeyaert, *Lifting geometric objects to the dual of the first jet bundle of a bundle fibred over R*, http://arxiv.org/abs/1308.3086v1

[10] Hans Westman and Sebastiano Sonego, *Events and Observables in generally invariant Spacetime Theories*, http://arxiv.org/abs/0708.1825v2

[11] Cedomir Crnkovic and Edward Witten, *Covariant description of canonical formalism theories in geometrical theories*, in *Three hundred Years of Gravitation*, S Hawking and W Israel ed., CUP