PROCEEDINGS OF SCIENCE

Noncommutative version of Borcherds' approach to quantum field theory

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Richard Borcherds proposed an elegant geometric version of renormalized perturbative quantum field theory in curved spacetimes, where Lagrangians are sections of a Hopf algebra bundle over a smooth manifold. However, this framework looses its geometric meaning when Borcherds introduces a (graded) commutative normal product. We present a fully geometric version of Borcherds' quantization where the (external) tensor product plays the role of the normal product. We construct a noncommutative many-body Hopf algebra and a module over it which contains all the terms of the perturbative expansion and we quantize it to recover the expectation values of standard quantum field theory when the Hopf algebra fiber is (graded) cocommutative. This construction enables to the second quantize any theory described by a cocommutative Hopf algebra bundle.

Frontiers of Fundamental Physics 14 - FFP14,
15-18 July 2014
Aix Marseille University (AMU) Saint-Charles Campus, Marseille

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PoS(FFP14)134

1. Introduction

In an article entitled "Renormalization and quantum field theory" [1], Richard Borcherds described a rigorous approach to renormalized perturbative quantum field theory in curved space-times. Borcherds' approach is closely related to the causal algebraic formalism [2], and it employs sheaf theory and Hopf algebras to achieve a particularly elegant and compact picture of quantum field theory (QFT). In particular, the combinatorial aspects of quantization and renormalization are completely taken care of by a Hopf algebraic structure. Moreover, Borcherds' approach has definite advantages when it comes to generalization. For example, the use of Hopf algebras is particularly powerful to deal with systems involving an initial state which is not quasi-free [3] and many of its tools (for example vector bundles and Hopf algebras) have natural noncommutative analogues that can be used to investigate noncommutative versions of quantum field theory.

In the present paper, which is a sketch of a more detailed article in preparation, we extend parts of Borcherds' approach by replacing his graded commutative normal product of classical fields by a tensor product which (i) allows us to formulate a fully geometric version of second quantization, (ii) provides a manageable topology for the many-body algebra, (iii) enables us to second quantize any cocommutative Hopf algebra bundle.

2. Hopf algebra bundles

In this section we introduce some concepts that are used in Borcherds' approach to QFT. Classical fields are sections of vector bundles over the space-time manifold M. We first reformulate Borcherds' sheaves into more familiar sections of vector bundles.

Let M be a smooth manifold and $F \xrightarrow{\pi} M$ a smooth vector bundle over M [4]. We denote by $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ the local trivializations (where V is a vector space) and by $t_{\alpha\beta}$ the transition functions such that $\phi_\alpha \circ \phi_\beta^{-1}(x,v) = (x, t_{\alpha\beta}(x)v)$, where $\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V$ and where the isomorphism $t_{\alpha\beta}(x)$ is an element of $GL(V)$. A vector bundle is an *algebra bundle* if the fiber model V is an algebra over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and if the transition functions are algebra isomorphisms: $t_{\alpha\beta}(x)(u \cdot v) = t_{\alpha\beta}(x)(u) \cdot t_{\alpha\beta}(x)(v)$. An algebra bundle is a *Hopf algebra bundle* if V is a Hopf algebra over $\mathbb{K}$ and the transition functions are Hopf algebra morphisms. In particular, the coproduct sends V to $V \otimes V$, which is the fiber of the (internal) tensor product of vector bundles $F \otimes F \xrightarrow{\pi'} M$ [4].

The space of sections $\Gamma(M,F)$ is an infinite-dimensional vector space, but it is also a module over the ring $C^\infty(M)$ of $\mathbb{K}$-valued smooth functions: as such, it admits a (locally) finite basis which allows to use simple linear algebra tools. If F is an algebra bundle, then the space of sections $\Gamma(M,F)$ is an algebra over the ring $C^\infty(M)$: if σ_1 and σ_2 are such sections with $\phi_\alpha\big(\sigma_1(x)\big) = (x, v_1)$ and $\phi_\alpha\big(\sigma_2(x)\big) = (x, v_2)$, then $\phi_\alpha\big(\sigma_1 \cdot \sigma_2(x)\big) = (x, v_1 \cdot v_2)$. Similarly, the space of sections of a Hopf algebra bundle is a Hopf algebra over the ring $C^\infty(M)$. In particular, the coproduct is now a map from $\Gamma(M,F)$ to $\Gamma(M, F \otimes F) \cong \Gamma(M,F) \hat{\otimes}_{C^\infty(M)} \Gamma(M,F)$.

Borcherds starts from a vector bundle $E \xrightarrow{\pi} M$ of finite rank whose sections are the classical fields of the model. To define Lagrangian densities as polynomials in the field and its derivatives, he considers the infinite jet bundle $JE \xrightarrow{\pi} M$ and the Hopf algebra bundle $S(JE^*) \xrightarrow{\pi} M$, which describes the polynomial functions on JE.

For example, the element $L = f + g_\mu + h_{\mu\nu} + k$, where $f \in \Gamma(M,E^*)$, $g^\mu \in \Gamma(M,J^1E^*)$, $h^{\mu\nu} \in \Gamma(M,S^2(J^1E^*))$ and $k \in \Gamma(M,S^4(E^*))$, corresponds to the Lagrangian density $L(\varphi) = \langle f,\varphi\rangle + \langle g^\mu,\varphi_\mu\rangle + \langle h^{\mu\nu},\varphi_\mu\varphi_\nu\rangle + \langle k,\varphi\varphi\varphi\varphi\rangle$, where φ is a field, φ_μ its derivatives and $\langle\cdot,\cdot\rangle$ is the duality pairing between $\Gamma(M,S(JE^*))$ and $\Gamma(M,S(JE))$ induced by the duality pairing between JE^* and JE. This Hopf algebra is commutative and cocommutative. Note that the topological properties of this algebra must be carefully taken into account because JE^* is an infinite-dimensional Fréchet manifold.

In the next section, we shall consider a general Hopf algebra bundle $F \xrightarrow{\pi} M$ whose sections play the role of Lagrangian densities, where $F = S(JE^*)$ in Borcherds' case.

3. The Fock Hopf algebra of classical fields

Second quantization starts from the construction of an algebra containing classical fields defined on any number of spacetime points. The commutative product of this many-body algebra is called the normal product, and it will be deformed to define a quantum field algebra. In Borcherds' paper, the algebra corresponding to the normal product of QFT is the symmetric algebra $S_{\mathbb{K}}(\Gamma(M,F))$ on the space of sections, which is too big to have a reasonable topology and which is no longer geometric, in the sense that $S_{\mathbb{K}}(\Gamma(M,F))$ is not the space of sections of a bundle over a manifold. This is because this manifold should be the quotient of M^n by the action of the symmetric group on n elements, which is generally not a topological manifold [5].

To solve that problem, note that for any bundle $F \xrightarrow{\pi} M$ there exists an external tensor product of bundles $F \boxtimes F \xrightarrow{\pi\times\pi} M \times M$ whose space of sections describes the (completed) tensor product of sections (over $\mathbb{K}$), $\Gamma(M\times M, F\boxtimes F) \cong \Gamma(M,F)\hat{\otimes}_{\mathbb{K}}\Gamma(M,F)$, that is, $\sigma(x_1,x_2) = \sum \sigma_1(x_1)\otimes\sigma_2(x_2)$. Moreover, since $\Gamma(M,F)$ is a Hopf algebra over $C^\infty(M)$, then $\Gamma(M\times M, F\boxtimes F)$ is a Hopf algebra over $C^\infty(M^2)$. Similarly,

Definition 1. *If $F \xrightarrow{\pi} M$ is a Hopf algebra bundle, the normal product of classical fields over n spacetime points is described by the* normal product algebra $\Gamma(M^n, F^{\boxtimes n})$, *which is a Hopf algebra over $C^\infty(M^n)$.*

Therefore, our normal product is encoded in the tensor product of sections, corresponding to the external tensor product of bundles. From a physical point of view, if $F = S(JE^*)$ is the bundle of polynomial Lagrangians of E-valued fields, the external tensor product $\boxtimes$ describes exactly the normal product of field polynomials at 2 points of M: e.g. the normal product$\varphi^4(x_1)\partial_\mu\varphi(x_2)\partial^\mu\varphi(x_2)$ corresponds to the section $\sigma(x_1,x_2) = \big((x_1,x_2),\varphi^4\otimes\partial_\mu\varphi\partial^\mu\varphi\big)$ of the bundle $F\boxtimes F$ over the point $(x_1,x_2)\in M\times M$. The exterior tensor product can be performed on any number n of copies of the bundle F, giving the Hopf bundle $F^{\boxtimes n} \xrightarrow{\pi^n} M^n$.

To describe QFT, then, we need to define a single algebra which contains all numbers of points. The difficulty is that the algebras $\Gamma(M^n, F^{\boxtimes n})$ are defined over different rings $C^\infty(M^n)$, one for each n. It turns out that this problem was solved a long time ago by Bourbaki. The first step is to build a ring $R = \varinjlim C^\infty(M^n)$ [6], which is the inductive limit of the rings $C^\infty(M^n)$ corresponding to the map $\phi_{mn} : C^\infty(M^m) \to C^\infty(M^n)$, with $m \le n$, defined by $\phi_{mn}(f)(x_1,\dots,x_n) = f(x_1,\dots,x_m)$. The inductive limit of algebras over different rings is also defined by Bourbaki [6] and its extension to

Hopf algebras is straightforward. Thus, we obtain a Hopf algebra which is reminiscent of the Fock space in the sense that it contains any number of points.

Definition 2. *If* $F \xrightarrow{\pi} M$ *is a Hopf algebra bundle, the* Fock Hopf algebra *is the inductive limit of Hopf algebras* $H_{\text{Fock}} = \varinjlim \Gamma(M^n, F^{\boxtimes n})$*, which is a Hopf algebra over the ring* $R_{\text{Fock}} = \varinjlim C^\infty(M^n)$.

Note that the Fock Hopf algebra is commutative iff F is commutative. The Hopf algebra structure on the Fock algebra is used to perform its deformation quantization.

We can now wonder whether the Fock Hopf algebra is a space of sections of a bundle over some infinite dimensional manifold. When M can be described by a single chart to $\mathbb{R}^d$, then the answer is yes and the manifold is $\varprojlim M^n$, which is a Fréchet manifold built on $\varprojlim(\mathbb{R}^d)^n$. If M needs several charts, then the projective limit topology is not compatible with the structure of a Fréchet manifold and we need more general concepts of infinite-dimensional manifolds. We can also wonder whether the definition of ϕ_{mn} is not too arbitrary. Instead of picking up the m first points of $(x_1,\dots,x_n)$, we can define an inductive limit corresponding to any subset of m elements, but by doing so we recover exactly H_{Fock} and R_{Fock} (because the family of sets $\{1,\dots,n\}$ is cofinal in the family of subsets of $\mathbb{N}$ [7]) so we stick to the simpler definition because countable inductive limits have better properties than uncountable ones.

4. Deformation quantization of H_{Fock}

It remains to quantize the Fock Hopf algebra to recover the operator product of standard quantum field theory as a special case. A convenient method to do so is to use quantum groups, that Drinfeld created as a quantization of algebras [8]. His foundation paper even cites the quantization method of Berezin, Vey, Lichnerowicz, Flato and Sternheimer (i.e. deformation quantization or star product). However, the quantization of fields does not use Drinfeld's quasitriangular structure but its dual, the *Laplace pairing*, which was first defined by Lyubashenko [9]. Rota and Stein called it a Laplace pairing because, for anticommuting variables, its definition is equivalent to the Laplace identity of determinants [10]. Borcherds calls it a *bicharacter*.

4.1 Laplace pairing

The problem is now that the Fock Hopf algebra is made of products of polynomials of smooth sections and their derivatives, whereas the quantum field amplitudes are distributions. Therefore, we need to introduce the space $\mathscr{D}'_{\text{Fock}} = \varinjlim \mathscr{D}'(M^n)$, which is the inductive limit of the spaces of distributions on M^n. The Laplace pairing is an R_{Fock}-linear map $(\cdot|\cdot) : H_{\text{Fock}} \otimes_{R_{\text{Fock}}} H_{\text{Fock}} \to \mathscr{D}'_{\text{Fock}}$, such that, for a, b and c in H_{Fock}, $(1|a) = (a|1) = \varepsilon(a)$ and $(a|bc) = \sum (a_{(1)}|b)(a_{(2)}|c)$ and $(ab|c) = \sum (a|c_{(1)})(b|c_{(2)})$. Since the terms $(a_{(1)}|b)(a_{(2)}|c)$ and $(a|c_{(1)})(b|c_{(2)})$ involve distributions, the product is only done when wavefront set conditions are satisfied [11].

In the case of standard quantum field theory, where H_{Fock} is built from the fiber $F = S(JE^*)$, the Laplace pairing is determined for f and g in $\Gamma(M,E^*)$ by $(f \otimes 1|1 \otimes g) = \langle f \otimes g, D_+\rangle$, where $D_+ \in \mathscr{D}'(M^2, E^{\boxtimes 2})$ is the Wightman propagator. This can also be written in a more physical way as $(\varphi \otimes 1|1 \otimes \varphi) = D_+$ or in a non-rigorous way $(\varphi(x)|\varphi(y)) = D_+(x,y)$ in the fiber over (x,y). This definition is extended to derivatives of fields by $(\partial^\alpha \varphi \otimes 1|1 \otimes \partial^\beta \varphi) = \partial^\alpha \partial^\beta D_+$, where α and β

are multi-indices. This pairing is well defined because of the structural theorem [12]

$$\begin{aligned}\mathscr{D}'(M^2,E^{\boxtimes 2}) &= \Big(\Gamma_c(M^2,(E^*)^{\boxtimes 2})\Big)' \cong \mathscr{D}'(M^2)\otimes_{C^\infty(M^2)}\Gamma(M^2,E^{\boxtimes 2})\\ &\cong \mathscr{L}_{C^\infty(M^2)}\big(\Gamma(M^2,(E^*)^{\boxtimes 2}),\mathscr{D}'(M^2)\big).\end{aligned}$$

4.2 Star product

Quantum group quantization was first defined by Rota and Stein [10], then developed by Fauser and coworkers [13, 14, 15]. Its equivalence with the star product was proved by Hirshfeld [16]. Borcherds does not define this product.

Definition 3. *Let $F \xrightarrow{\pi} M$ be a Hopf algebra bundle and H_{Fock} the corresponding Fock Hopf algebra. Then, $C_{\mathrm{Fock}} = \varinjlim \mathscr{D}'(M^n)\otimes_{C^\infty(M^n)}\Gamma(M^n,F^{\boxtimes n})$ is a H_{Fock}-Hopf module where the coaction β is defined on $c = u\otimes h$ by $\beta c = \sum c'\otimes c'' = \sum (u\otimes h_{(1)})\otimes h_{(2)}$. The* star product *on C_{Fock} is defined by*

$$c\star d \;=\; \sum c'd'(c''|d''), \tag{4.1}$$

where $(c''|d'')$ is identified with $(c''|d'')\otimes 1$. If the Hopf algebra is cocommutative, the star product is associative.

If we consider the example $c = u\otimes h$ and $d = v\otimes k$ we find $c\star d = \sum uv(h_{(2)}|k_{(2)})\otimes h_{(1)}k_{(1)}$. The product $ab(h_{(1)}|k_{(2)})$ is a product of three distributions which is well-defined by the wavefront set condition [11] for standard quantum field theory [2]. Note that C_{Fock} equipped with the star product is a sort of generalized Frobenius algebra, in the sense that $(c\star d|e) = (c|d\star e)$ [15].

For example if $c = (1\otimes 1)\otimes(\varphi\otimes 1)$ and $d = (1\otimes 1)\otimes(1\otimes\varphi)$, then $c\star d = (1\otimes 1)\otimes(\varphi\otimes\varphi) + D_+\otimes(1\otimes 1)$ and we recover Wick's theorem usually written $\varphi(x)\star\varphi(y) = {:}\varphi(x)\varphi(y){:} + D_+(x,y)$ in QFT textbooks. This completes the quantization of the Fock Hopf algebra, i.e. the second quantization of the Hopf algebra bundle F.

4.3 The time-ordered product

The last step to obtain Green functions of QFT is to define time-ordered products. We do this by following the causal approach developed by Stueckelberg, Bogoliubov, Epstein, Glaser [18] and finally Brunetti and Fredenhagen [2]. Then, the time-ordered product becomes a comodule morphism $T : C_{\mathrm{Fock}} \to C_{\mathrm{Fock}}$ and the Wick expansion of time-ordered products takes the simple form $T(c) = \sum t(c')c$", where $t(c) = (1\otimes\varepsilon)(T(c))$ [15]. The time-ordered product is defined recursively by the *causality relation*[1] saying that $T(cd) = T(c)\star T(d)$ if the spacetime support of c is not earlier than the spacetime support of d. By Stora's lemma[2], the causality relation and the partial order imply that T is defined recursively except on the diagonals, where the distributions have to be extended [2]. The ambiguity of this extension is organized by the renormalization group.

[1]Borcherds' Gaussian property is a consequence of the causality relation [18].

[2]It can easily be inferred from a remark by Bergbauer [17] that Stora's lemma only requires a (closed) partial order on M, which is taken to be the causal order in applications to Lorentzian manifolds.

5. Conclusion

A second quantization method was described for any theory whose Lagrangian density is an element of a cocommutative Hopf algebra bundle. Fermions can be taken into account by using a graded cocommutative Hopf algebra [14]. Since we do not require the Hopf algebra to be commutative, we expect this approach to play a role in the second quantization of noncommutative geometry.

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